<u>Summary</u>. $T \in B(X,Y)$ then dual T' by $(T'f)(x) = f(Tx)$ is in $B(Y', X')$ and $\|T'\| = \|T\|$. If T^{-1} exists, then $(T')^{-1} = (T^{-1})'$.

- $\ker T = {}^{\circ}(\operatorname{ran} T')$, $\ker T' = (\operatorname{ran} T)^{\circ}$ (annihilators).

- T (isometric) isomorphism $\Rightarrow$ T' (isometric) isomorphism.

- $X \simeq Y$ & X refl $\Rightarrow$ Y refl (use $J_Y T = T'' J_X$). evaluation maps

- complementary subspace U & V of lin space X: $X = U \oplus V$.
  1-1 corr with projectors.
- U & V are topological compl when $P: x \mapsto u_x$ is bounded.

- Banach $X = U \oplus V$ & U & V closed $\Rightarrow$ U & V top compl.

$\{x_n\} \rightharpoonup x$ when $f(x_n) \to f(x)$ $\forall f \in X'$.

$\{f_n\} \xrightarrow{*} f$ when $f_n(x) \to f(x)$ $\forall x \in X$.

(weak or weak-* convergence)

Prop. Let H Hilbert.

a) $x_n \rightharpoonup x \iff \langle x_n, y\rangle \to \langle x, y\rangle \quad \forall y$

b) If $\dim H = \infty$ and $\{e_n\}$ orthon. seq, then $e_n \rightharpoonup 0$

(so generally weak conv is weaker than convergence)

proof: Exerc.

Lemma Let $S \subset X$ with $\overline{\operatorname{Sp} S} = X$. Then if $\{f_n\} \subset X'$ is bounded, and $\{f_n(s)\}$ converges $\forall s \in S$. Then $\exists f \in X'$ s.t. $\{f_n\} \xrightarrow{*} f$.

proof Let $C := \sup\{\|f_n\|_{X'} : n \in \mathbb{N}\}$. Let $x \in X$. We show $\{f_n(x)\}$ conv.
Let $\varepsilon > 0$ be given. $\exists t \in \operatorname{Sp} S$ with $\|x - t\| < \frac{\varepsilon}{3C}$.

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(t)|}_{1} + \underbrace{|f_n(t) - f_m(t)|}_{2} + \underbrace{|f_m(t) - f_m(x)|}_{3}$$

1 & 3 $< \frac{\varepsilon}{3}$. t is finite lin combi of $s \in S$. So for n, m large enough, 2 $< \frac{\varepsilon}{3}$. So $\{f_n(x)\}$ Cauchy in $\mathbb{F}$, so conv.

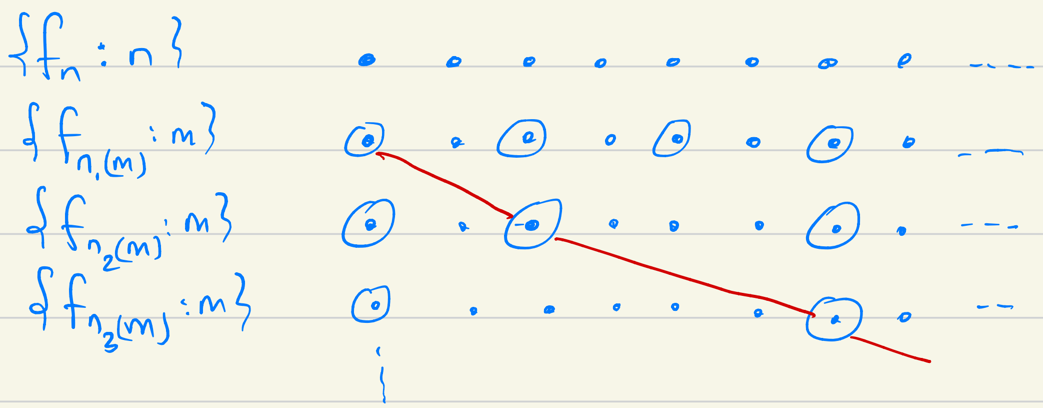
Def $f(x) := \lim f_n(x)$. Obv $f \in X^*$, $|f(x)| \leq \sup\{|f_n(x)| : n\} \leq C\|x\|$

so $f \in X'$. $\square$

<u>Thm</u> Let X <u>separable</u> normed lin. space. $\{f_n\} \subset X'$ bounded.
Then $\{f_n\}$ has a weak-$*$ convergent subseq.

(generalizes Bolzano-W. for finite dimensional X')

<u>proof</u> Let $\overline{\{s_k\}} = X$. $\{f_n(s_1)\}$ is bounded seq in $\mathbb{F}$, so (BW) has a conv subseq $\{f_{n_1(m)}(s_1) : m \in \mathbb{N}\}$. $\{f_{n_1(m)}(s_2) : m \in \mathbb{N}\}$ has conv subs $\{f_{n_2(m)}(s_2) : m \in \mathbb{N}\}$ etc.



Consider 'diagonal' $\{f_{n_m(m)} : m \in \mathbb{N}\}$.
$\forall k$ is $\{f_{n_m(m)}(s_k) : m\}$ conv, since for $m \geq k$ it is a subseq of $\{f_{n_k(m)}(s_k) : m\}$.

$X = \overline{\{s_k\}} \subseteq \overline{\mathrm{Sp}\{s_k\}}$, so prev. lemma shows $\{f_{n_m(m)}\} \xrightarrow{*} f$ for some $f \in X'$.

<u>Thm</u> Let X <u>refl</u> and $\{x_n\} \subset X$ bounded. Then $\{x_n\}$ has a weakly conv subseq.

(generalizes B-W when $\dim X < \infty$)

<u>proof</u>

$Y := \overline{\mathrm{Sp}\{x_n\}}$ is separable $\left(\bigcup_{k\geq 1} \left\{ \sum_{1}^{k} \alpha_i x_i : \alpha_i \in \mathbb{Q} + i\mathbb{Q} \right\} \text{ is dense} \right)$,

and refl (because closed in refl X). So Y'' is separable,

and so Y' separable (was a cons. of HB).

$\{J_Y(x_n)\}$ bounded seq in Y'', so, previous thm, $\exists$ subseq $\{J_Y(x_{n(m)})\}$

that is weak-$*$ converg. to some $\varphi \in Y''$, thus to some $J_Y(z)$ for some $z \in Y$

(because Y is refl). For each $f \in X'$, it holds $f|_Y \in Y'$, and so

$\lim f(x_{n(m)}) = \lim f|_Y(x_{n(m)}) = \lim \left(J_Y(x_{n(m)})\right)(f|_Y) =$

$J_Y(z)(f|_Y) = f|_Y(z) = f(z)$, i.e. $\{x_{n(m)}\} \rightharpoonup z$.

$\square$

Ch6 Lin. operators on Hilbert spaces

H, K Hilb. $T \in B(H,K)$. We show $\exists!\, T^*$, the adjoint of T,

$$\langle Tx, y\rangle_K = \langle x, T^* y\rangle_H \qquad (x \in H,\ y \in K)$$

Uniqueness ✓. Existence: Recall Riesz $T_H : H \to H'$ by $(T_H y)(x) = \langle x, y\rangle_H$ is bijective, isometric, conj linear.

Recall T isomorphism $\Rightarrow$ dual T' is isomorphism.

$H \leftarrow H' \leftarrow K' \leftarrow K$

prop a) $T^* = T_H^{-1} T' T_K \in B(K,H)$ & $\|T^*\| = \|T\|$

b) $(T^*)^* = T$

c) T isomorphism $\Leftrightarrow$ T^* isomorphism

proof $\langle x, T_H^{-1} T' T_K y\rangle_K = (T' T_K y)(x) = (T_K y)(Tx) = \langle Tx, y\rangle_K$

$T_H^{-1} T' T_K \in L(K,H)$ & $\|T_H^{-1} T' T_K\| = \|T'\| = \|T\|$.

b) $\langle x, (T^*)^* y\rangle_H = \langle T^* x, y\rangle_K = \overline{\langle y, T^* x\rangle_K} = \overline{\langle Ty, x\rangle_H} = \langle x, Ty\rangle_H$

c) T iso $\Rightarrow T'$ iso $\Leftrightarrow T^*$ iso. T^* iso $\Rightarrow (T^*)^* = T$ iso.

<u>lemma</u> H, K Hilbert, $T_1, T_2 \in B(H,K)$

a) $(\mu T_1 + \lambda T_2)^* = \bar{\mu} T_1^* + \bar{\lambda} T_2^*$ (so $B(H,K) \to B(K,H) : T \mapsto T^*$ is conj linear isometry)

b) $T \in B(H,K)$ $S \in B(K,Z)$ (Z Hilb) then $(ST)^* = T^* S^*$.

cf. $\ker T = {}^{\circ}(\operatorname{ran} T')$

c) $\|T^* T\| = \|T\|^2$ d) $\ker T = (\operatorname{ran} T^*)^\perp$

e) $\ker T^* = (\operatorname{ran} T)^\perp$ f) $\ker T^* = \{0\} \Leftrightarrow \overline{\operatorname{ran} T} = K$

g) When T, or equiv T^*, isomorph., then $(T^*)^{-1} = (T^{-1})^*$.

cf. $(T')^{-1} = (T^{-1})'$

<u>proof</u> c) $\|T^* T\| \le \|T^*\| \|T\| = \|T\|^2$

$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^* T x \rangle \le \|x\|^2 \|T^* T\|$

d) From $\langle Tx, y \rangle = \langle x, T^* y \rangle$ follows $x \in \ker T \Leftrightarrow x \in (\operatorname{ran} T^*)^\perp$

f) $\ker T^* = \{0\} \overset{(e)}{\Leftrightarrow} (\operatorname{ran} T)^\perp = \{0\} \Leftrightarrow K = \underbrace{(\operatorname{ran} T)^{\perp\perp} = \overline{\operatorname{ran} T}}_{\text{true for any subsp}}$

g) Let $TT^{-1} = I$, then $I = I^* = (TT^{-1})^* = (T^{-1})^* T^*$

Anal. $T^* (T^{-1})^* = I$.

Prop H, K Hilb. Equiv are a) T invertible

b) $\ker T^* = \{0\}$ & $\exists \alpha > 0$ $\|Tx\| \geq \alpha\|x\|$

proof a) $\Rightarrow$ b) $\operatorname{ran} T = K$, so $\ker T^* = (\operatorname{ran} T)^\perp = \{0\}$

Invertible means boundedly invertible (open mapping)

so $\|x\| \leq \|T^{-1}\| \, \|Tx\|$.

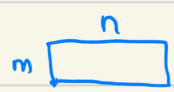
b) $\Rightarrow$ a) $\ker T^* = \{0\} \Leftrightarrow \overline{\operatorname{ran} T} = K$

Second prop shows T is injective, and $\operatorname{ran} T$ closed:

Let $\{Tx_n\} \to y$. Then $\{Tx_n\}$ Cauchy, so $\{x_n\}$ Cauchy

so convergent to some x, but then $y = Tx$. $\square$

An $m \times n$ matrix A is an element of $B(\mathbb{F}^n, \mathbb{F}^m)$.



From

$$\langle A\vec{x}, \vec{y} \rangle_{\mathbb{F}^m} = \langle \vec{x}, \overline{A}^T \vec{y} \rangle_{\mathbb{F}^n} \qquad (\forall x, y)$$

it follows that $A^* = \overline{A}^T$.

Def $T \in B(H) = B(H,H)$ is called **normal** when $TT^* = T^*T$.

Prop If T is normal, then $\|Tx\| = \|T^*x\| \quad \forall x$

proof $\|Tx\|^2 - \|T^*x\|^2 = \langle Tx, Tx \rangle - \langle T^*x, T^*x \rangle$

$= \langle x, (T^*T - TT^*)x \rangle = 0 \quad \square$

Def $T \in B(H)$ is called **self-adjoint** when $T = T^*$

Prop Set S of self-adjoint operators in $B(H)$ is closed.

proof Let $S \supset \{T_n\} \to T$. Then $\{T_n^*\} \to T^*$

$(\|T_n - T\| = \|T_n^* - T^*\|)$, so $\{T_n\} \to T^*$.

Prop Let $T \in B(H)$. Then TT^*, T^*T, $\frac{1}{2}(T+T^*)$, $\frac{1}{2i}(T-T^*)$ are self-adjoint

$\left(T = \frac{1}{2}(T+T^*) + i \frac{1}{2i}(T-T^*) \right)$

Def $T \in B(H)$ is called unitary when $T^* = T^{-1}$.

Lemma Let H a complex Hilbert space, $T \in B(H)$.
Then $\langle Tx, x \rangle = 0 \ \forall x \Rightarrow T = 0$

proof exerc. 6.8

(not true for real Hilb: rotation over $90°$ in $\mathbb{R}^2$)

Prop H complex Hilb. space. $T \in B(H)$.
a) $T^*T = I \iff T$ is isometry
b) T unitary $\iff T$ is isometric isomorphism

ex right-shift (not unitary)

proof a) $\Rightarrow$ $\|Tx\|^2 = \langle T^*Tx, x \rangle = \|x\|^2$

$\Leftarrow$ $\langle T^*Tx, x \rangle = \|Tx\|^2 = \langle x, x \rangle$. Apply previous prop onto $I - T^*T$.

b) $\Rightarrow$ Unitary implies isometry by a). Unitary map is invertible.
$\Leftarrow$ $T^*T = I$ by a).

write $y = Tx$

$$TT^*y \stackrel{\downarrow}{=} TT^*Tx = Tx = y \qquad \square$$

Prop Set of unitary operators in $B(H)$ is closed. **Proof** exer.

Spectrum of an operator

H complex Hilbert space, $T \in B(H)$

- Spectrum of T, $\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ not invertible}\}$

- (λ, x) is eigenpair of T when $Tx = \lambda x$ & $x \neq 0$
  (eigenvalue, eigenvector)

Then $\ker(T - \lambda I) \neq \{0\}$ so $\lambda \in \sigma(T)$.

- Other possibility for $\lambda \in \sigma(T)$ is $\operatorname{ran}(T - \lambda I) \neq H$ (not when $\dim H < \infty$)

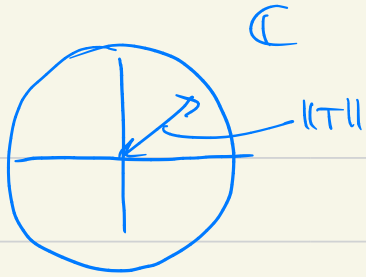
For X normed lin space, $T \in B(X)$, $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not boundedly invertible}\}$

So another possibility for $\lambda \in \sigma(T)$

Prop a) $\sigma(T) \subseteq \{ z \in \mathbb{C} : |z| \leq \|T\| \}$

b) $\sigma(T)$ is closed

c) $\sigma(T^*) = \{ \bar{\lambda} : \lambda \in \sigma(T) \}$



proof a) For $|\lambda| > \|T\|$ is $\|\lambda^{-1} T\| < 1$ so

$I - \lambda^{-1} T$ inv, but then $T - \lambda I$ inv.

b) Let $T - \lambda I$ inv (i.e. $\lambda \notin \sigma(T)$). Then

$$T - \mu I = T - \lambda I + (\lambda - \mu) I = (T - \lambda I) \left(I + (\lambda - \mu)(T - \lambda I)^{-1} \right)$$

is invb when $|\lambda - \mu| < \|(T - \lambda I)^{-1}\|^{-1}$

c) $T - \lambda I$ inv $\iff (T - \lambda I)^* = T^* - \bar{\lambda} I$ inv.

ex $S : \ell^2 \to \ell^2 : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$

$(Sx)_1 = (\lambda x)_1 \Rightarrow x_1 = 0$; $(Sx)_2 = (\lambda x)_2 \Rightarrow x_2 = 0$ etc.

$\rightsquigarrow$ S has no eigenvalues

$S^* : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$

$S^* x = \lambda x \quad x_2 = \lambda x_1, \ x_3 = \lambda x_2, \dots \Rightarrow x = (1, \lambda, \lambda^2, \dots) \in \ell^2$

iff $|\lambda| < 1$

$\|S^*\|=1$. So $\{\lambda\in\mathbb{C}: |\lambda|<1\}\subseteq\sigma(S^*)\subseteq\{\lambda\in\mathbb{C}: |\lambda|\leq 1\}$

and $\sigma(S^*)$ is closed $\Rightarrow$ $\sigma(S^*)=\{\lambda\in\mathbb{C}: |\lambda|\leq 1\}$,

and so $\sigma(S)=\{\lambda\in\mathbb{C}: |\lambda|\leq 1\}$.

Thm a) $\sigma(p(T))=p(\sigma(T))$ for any polymial p.

b) When T is invertible, then $\sigma(T^{-1})=\{1/\mu : \mu\in\sigma(T)\}$

proof

a) When $p=c_{st}$ then $\sigma(p(T))=\sigma(c_{st}I)=c_{st}\,\sigma(I)=c_{st}=p(\sigma(T))$.

So let $p\neq c_{st}$.

Fixing $\lambda\in\mathbb{C}$, let $q(z):=\lambda-p(z)=c(z-\mu_1)\cdots(z-\mu_n)$ with $c\neq 0$

(otherw $p(z)\equiv\lambda$)

Then $\lambda\notin\sigma(p(T))\Leftrightarrow\lambda I-p(T)$ invb $\Leftrightarrow q(T)$ invb

$\Leftrightarrow (T-\mu_1 I)\cdots\cdots(T-\mu_n I)$ invb ($\Rightarrow (T-\mu_1 I)$ surj and $(T-\mu_2 I)\cdots(T-\mu_n I)$ inj, but since they commute, both byect.)

$\Leftrightarrow T-\mu_1 I, \cdots\cdots, T-\mu_n I$ invb

$\Leftrightarrow \{z\in\mathbb{C}: q(z)=0\}\cap\sigma(T)=\emptyset$

$\Leftrightarrow \{z\in\mathbb{C}: \lambda=p(z)\}\cap\sigma(T)=\emptyset \Leftrightarrow \lambda\notin p(\sigma(T))$

b) For $\mu\neq 0$, $\mu^{-1}I-T^{-1}=-T^{-1}\mu^{-1}(\mu I-T)$, so $\mu^{-1}\in\sigma(T^{-1})\Leftrightarrow \mu\in\sigma(T)$

Use $0\notin\sigma(T^{-1})$.