Summary H, K Hilbert $T \in B(H,K)$. Adjoint $T^* \in B(K,H)$ by

$$\langle Tx, y \rangle_K = \langle x, T^* y \rangle_H$$

$T^* = T_H^{-1} T' T_K$

$(T^*)^* = T$

T isomorph $\iff$ T^* isomorphism $\quad (T^*)^{-1} = (T^{-1})^*$

$\ker T^* = (\operatorname{ran} T)^\perp$

Now let $T \in B(H)$. T is called normal, self-adjoint, or unitary when $T^*T = TT^*$, $T^* = T$, or $T^* = T^{-1}$.

Spectrum $\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I$ is not invertible $\}$

Contains the eigenvalues of T.

$\sigma(T) \subseteq \{ z \in \mathbb{C} : |z| \leq \|T\| \}$

$\sigma(T)$ is closed

$\sigma(T^*) = \{ \bar{\lambda} : \lambda \in \sigma(T) \}$

Let's call the elements of $\sigma(T)$ the spectral values (of T)

Thm a) $\sigma(p(T)) = p(\sigma(T))$ for any polymial p.

b) When T is invertible, then $\sigma(T^{-1}) = \{ 1/\mu : \mu \in \sigma(T) \}$

Proof

a) When $p = c_{st}$ then $\sigma(p(T)) = \sigma(c_{st} I) = c_{st}\, \sigma(I) = c_{st} = p(\sigma(T))$.

So let $p \neq c_{st}$.

Fixing $\lambda \in \mathbb{C}$, let $q(z) := \lambda - p(z) = c(z-\mu_1)\cdots(z-\mu_n)$ with $c \neq 0$

(otherw $p(z) \equiv \lambda$)

Then $\lambda \notin \sigma(p(T)) \Leftrightarrow \lambda I - p(T)$ invb $\Leftrightarrow q(T)$ invb

$\Leftrightarrow (T-\mu_1 I) \cdots\cdots (T-\mu_n I)$ invb ($\Rightarrow (T-\mu_1 I)$ surj and $(T-\mu_2 I)\cdots(T-\mu_n I)$ inj, but since they commute, both byed.)

$\Leftrightarrow T-\mu_1 I, \ldots\ldots, T-\mu_n I$ invb

$\Leftrightarrow \{ z \in \mathbb{C} : q(z) = 0 \} \cap \sigma(T) = \emptyset$

$\Leftrightarrow \{ z \in \mathbb{C} : \lambda = p(z) \} \cap \sigma(T) = \emptyset \Leftrightarrow \lambda \notin p(\sigma(T))$

b) For $\mu \neq 0$, $\mu^{-1} I - T^{-1} = -T^{-1}\mu^{-1}(\mu I - T)$, so $\mu^{-1} \in \sigma(T^{-1}) \Leftrightarrow \mu \in \sigma(T)$

Use $0 \notin \sigma(T^{-1})$.

<u>prop</u> $U \in B(H)$ unitary, then $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$

<u>proof</u> $\|U\| = 1$, so $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

$$\sigma(U) = \{\bar{\mu} : \mu \in \sigma(U^*)\} = \{\bar{\mu} : \mu \in \sigma(U^{-1})\} = \{1/\bar{\mu} : \mu \in \sigma(U)\}$$
$$\subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \quad \square$$

<u>Def</u> $T \in B(H)$. Spectral radius $r_\sigma(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$

(often denoted as $\rho(T)$)

Numerical range $V(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$

(also known as field of values)

Note $V(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ (as the spectrum)

<u>lemma</u> $T \in B(H)$ normal. Then T invb $\iff \exists \alpha > 0 \ \|Tx\| \geq \alpha \|x\|$

<u>proof</u> $\Rightarrow$ ✓

$\Leftarrow$ cond shows $\operatorname{ran} T$ is closed, $\ker T = \{0\}$

Thanks to T normal, $\|T^* x\| = \|Tx\|$ so also $\ker T^* = \{0\}$

Now see 8 pages earlier. $\square$

closure

<u>lemma</u> Let $T \in B(H)$ normal. Then $\sigma(T) \subseteq \overline{V(T)}$.

<u>proof</u> Let $\lambda \in \sigma(T)$, then $T - \lambda I$ not invertible, so that, since

$T - \lambda I$ is normal, $\nexists\, \alpha > 0$ $\|(T-\lambda I)x\| \geq \alpha \|x\|$ $\forall x \in H$.

So $\exists \{x_n\}$ with $\|x_n\| = 1$ and $\lim (T - \lambda I)x_n = 0$ $\Rightarrow$

$\lim \langle (T-\lambda I)x_n, x_n \rangle = 0 \Leftrightarrow \lim \langle Tx_n, x_n \rangle = \lambda$, i.e. $\lambda \in \overline{V(T)}$ □

<u>Prop</u> Let $S \in B(H)$ self-adjoint. Then

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a) $V(S) \subseteq \mathbb{R}$ (is, when H is complex, even equiv. to self-adjoint)

b) $\sigma(S) \subseteq \mathbb{R}$

c) $\{-\|S\|, \|S\|\} \cap \sigma(S) \neq \emptyset$

d) $r_\sigma(S) = \sup\{|z| : z \in V(S)\} = \|S\|$ (spectral radius = norm)

e) $[\min \sigma(S), \max \sigma(S)] \supseteq V(S)$

<u>proof</u>

a) $\langle Sx, x \rangle = \langle x, S^* x \rangle = \langle x, Sx \rangle = \overline{\langle Sx, x \rangle}$.

b) S is normal, so $\sigma(S) \subseteq \overline{V(S)}$.

c) $\exists \{x_n\}$ with $\|x_n\|=1$ & $\|Sx_n\| \to \|S\|$.

$$\|(\|S\|^2 - S^2)x_n\|^2 = \langle (\|S\|^2-S^2)x_n, (\|S\|^2-S^2)x_n \rangle$$

$$= \|S\|^4 - \|S\|^2\Big(\underbrace{\langle S^2x_n, x_n\rangle}_{\|Sx_n\|^2} + \underbrace{\langle x_n, S^2x_n\rangle}_{\|Sx_n\|^2}\Big) + \|S^2x_n\|^2$$

$$\le 2\|S\|^4 - 2\|S\|^2\|Sx_n\|^2 \to 0$$

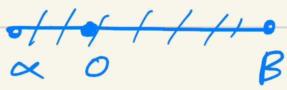
lemma at bottom 2 pages earlier

So $\|S\|^2 \in \sigma(S^2) = \sigma(S)^2$

d) $\|S\| \overset{(c)}{\le} r_\sigma(S) \overset{\sigma(S)\subseteq \overline{V(S)}}{\le} \sup\{|z| : z \in V(S)\} \overset{|\langle Sx,x\rangle| \le \|S\|\|x\|^2}{\le} \|S\|$

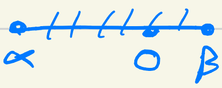
e) $\alpha := \min \sigma(S)$, $\beta := \max\sigma(S)$. Let for some $\|y\|=1$, $\langle Sy,y\rangle > \beta$.

If $r_\sigma(S)=\beta$, then contradiction with d1)

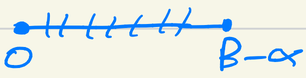


If $r_\sigma(S) = -\alpha$, then $r_\sigma(S-\alpha I) = \beta-\alpha$

$\sigma(S)$



$\sigma(S-\alpha I)$



and $\langle (S-\alpha I)y,y\rangle > \beta-\alpha$ in contradiction with d1) applied to $S-\alpha I$.

Case $\langle Sy,y\rangle < \alpha$ similar.

Positive operators and projectors

<u>Def</u> H complex Hilbert space. $S \in B(H)$ is called **positive** when
$\langle Sx, x \rangle \geq 0 \quad (x \in H)$. Notation $S \geq 0$.

<u>lemma</u> $S \geq 0 \iff \sigma(S) \subseteq [0,\infty)$ & S self-adjoint

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<u>proof</u> $\Rightarrow$ $S \geq 0$ implies $V(S) \subseteq [0,\infty)$, so $S = S^*$, thus normal
and so $\sigma(S) \subseteq \overline{V(S)} \subseteq [0,\infty)$.
$\Leftarrow$ from $S^* = S$ follows $V(S) \subseteq [\min \sigma(S), \max \sigma(S)]$
so positive

<u>rem</u> $T \in B(H) \Rightarrow T^*T \geq 0$.

<u>Def</u> $P \in B(H)$ is called **orthogonal projector** when $P^* = P = P^2$

proj

<u>rem</u> Orth proj is pos: $\langle Px, x \rangle = \langle P^2 x, x \rangle = \langle Px, Px \rangle$

<u>Thm</u> a) H Hilbert, M closed lin subsp. Then $\exists$ orth proj P with $\operatorname{ran} P = M$
$\ker P = M^\perp$. b) For P orth proj, $\operatorname{ran} P$ is closed subsp.

<u>proof</u> a) $H = M \oplus M^\perp$. $x = u + v$. $P : x \mapsto u$

b) $\operatorname{ran} P = \operatorname{ran}(I-P)^\perp$.

Functions of self-adjoint operators

recall when A is normal matrix, then $\exists$ unitary U with $U^*AU = D$
Now let f be well-defined scalar function on $\sigma(A)$.
Then $f(A) := U^* f(D) U$. Note that for $f = p$ polyn., $f(A) = p(A)$

H Hilbert. Set $\mathcal{S}$ of self-adjoint op^s in $B(H)$ is closed.
It is also a real lin space, so $\mathcal{S}$ is a real Banach space.

equipped with sup-norm

<u>lemma</u> Fixing $S \in \mathcal{J}$. $\exists\, \Phi \in B(C(\sigma(S), \mathbb{R}), \mathcal{J})$ isometry s.t

a) $\Phi(p) = p(S)$

b) $\Phi(fg) = \Phi(f)\Phi(g) \quad \forall f, g \in C(\sigma(S), \mathbb{R})$

<u>proof</u> i) Def $\varphi : P \to \mathcal{J}$ by $\varphi(p) = p(S)$. $\varphi(pq) = \varphi(p)\varphi(q)$.

polynomials

$$\|\varphi(p)\|_{B_{H,H}} = \|p(S)\| = r_\sigma(p(S)) = \sup\{|\mu| : \mu \in \sigma(p(S))\}$$

$p(S) = p(S)^*$

$$= \sup\{|\mu| : \mu \in p(\sigma(S))\} = \sup\{|p(\mu)| : \mu \in \sigma(S)\} = \|p\|,$$

so φ is isometry. P dense in $C(\sigma(S), \mathbb{R})$. Since $\mathcal{J}$ is Banach,

(Thm 4.19)

$\exists!$ extension Φ of φ to $C(\sigma(S), \mathbb{R})$ which is an isometry

Let $p_n \to f$, $q_n \to g$ then $p_n q_n \to fg$. From $\varphi(p_n q_n) = \varphi(p_n)\varphi(q_n)$

and boundedness Φ b) follows. $\square$

We will write $\Phi(f)$ as $f(S)$.

Thm Let $S \in B(H)$ with $S \geq 0$. Then

a) $S^{1/2} \geq 0$ & $(S^{1/2})^2 = S$

b) Let $Q \in B(H)$ with $Q \geq 0$ and $Q^2 = S$, then $Q = S^{1/2}$.

proof a) $(S^{1/2})^2 = \Phi(x \mapsto x^{1/2})\,\Phi(x \mapsto x^{\frac{1}{2}}) = \Phi(\underbrace{x \mapsto x}_{\text{polynomial}}) = S^1 = S$

Also $(S^{1/4})^2 = \Phi(x \mapsto x^{1/2}) = S^{1/2}$. $S^{1/2} = S^{1/4} S^{1/4} = (S^{1/4})^* S^{1/4} \geq 0$

b) $QS = QQ^2 = Q^2 Q = SQ$, so $Q\,p(S) = p(S)\,Q$ for any pol p.
$\exists \{p_n\}$ with $p_n(S) \to S^{1/2}$, which implies $QS^{1/2} = S^{1/2}Q$ $(*)$

Part a) show that Q has a square root $Q^{1/2} \geq 0$.
Let $x \in H$ be given. To show that $y := (S^{1/2} - Q)x = 0$

$$\|S^{1/4}y\|^2 + \|Q^{1/2}y\|^2 = \langle S^{1/2}y, y\rangle + \langle Qy, y\rangle = \langle (S^{1/2} + Q)y, y\rangle$$
$$= \langle (S^{1/2} + Q)(S^{1/2} - Q)x, y\rangle \overset{(*)}{=} \langle (S - Q^2)x, y\rangle = 0$$

So $S^{1/4}y = 0 = Q^{1/2}y$, thus $S^{1/2}y = 0 = Qy$.
Now $\|y\|^2 = \langle (S^{\frac{1}{2}} - Q)x, (S^{\frac{1}{2}} - Q)x\rangle = \langle (S^{1/2} - Q)y, x\rangle = 0$

$\square$

rem If $S \geq 0$ invertible, then so is $S^{1/2}$. Indeed $S = S^{1/2}S^{1/2}$ shows $S^{1/2}$ surj & inj.

Thm H complex H space. $T \in B(H)$ invertible

Then $T = UR$ for some unitary U and $R \geq 0$

(compare $z = e^{i\theta}|z|$)

Proof T inv $\Rightarrow T^*$ inv $\Rightarrow T^*T \geq 0$ inv $\Rightarrow R = (T^*T)^{1/2} \geq 0$ inv.

Def $U = TR^{-1}$. Is invertible.

$U^*U = (R^{-1})^* T^* T R^{-1} = I$, and so $U^*UU^{-1} = U^{-1}$.