Summary \( T \in \mathcal{B}(H) \)

\[ \sigma(p(T)) = p(\sigma(T)) \quad , \quad \sigma(T^{-1}) = \{ \frac{1}{\mu} : \mu \in \sigma(T) \} \]

**Spectral radius** \( \rho(T) := \max \{ |\lambda| : \lambda \in \sigma(T) \} \)

**Numerical range** \( V(T) := \{ \langle Tx, x \rangle : \|x\| = 1 \} \)

- \( T \) unitary \( \Rightarrow \sigma(T) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \)
- \( T \) normal \( \Rightarrow \sigma(T) = V(T)^\perp \)
- \( T \) self-adj \( \Rightarrow V(T) \subseteq \mathbb{R} \) (even equiv.)
  
  \[ \begin{align*} &f(-\|T\|, \|T\|) \cap \sigma(T) \neq \emptyset, \\
  &\rho(T) = \sup \{ |\lambda| : \lambda \in V(T) \} = \|T\|, \\
  &V(T) = [\min \sigma(T), \max \sigma(T)] \end{align*} \]

\( T \) is positive when \( V(T) \subseteq [0, \infty) \). \( T \geq 0 \Leftrightarrow \sigma(T) \subseteq [0, \infty) \) & self-adj

\( T \) self-adj, \( f \in C(\sigma(T), \mathbb{R}) \), then \( f(T) \in \mathcal{B}(H) \) & self-adjoint.

\( T \) invertible, then \( T = UR \) with \( U \) unitary, \( R \geq 0 \).
Ch. 7 Compact operators

Def. $X, Y$ normed lin. spaces. $T \in L(X, Y)$ is compact if for any bounded $\{x_n\} \subset X$, $\{Tx_n\}$ has a converging subseq.

Set of compact ops is denoted as $K(X, Y)$.

**Lemma** $T \in L(X, Y)$ is compact $\iff$ for every bounded $A \subset X$, $\overline{T(A)}$ is compact.

(recall: Subset $A$ of metric space is compact when every seq in $A$ has a converging subseq with limit in $A$)

**Proof** $\iff$ Let $\{x_n\}$ be bounded, then

$\{Tx_n\} \subset \{Tx: n \in \mathbb{N}\}$, so $\{Tx_n\}$ has converging subseq.

$\iff$ Let $A$ be bounded. Let $\{b_n\} \subset \overline{T(A)}$. $\exists \{a_n\} \subset T(A)$ s.t $\|b_n - a_n\| < \frac{1}{n}$. $a_n = T(x_n)$ where $\{x_n\} \subset A$ so bounded.

So $\{a_n\}$ has converging subseq, but then so has $\{b_n\}$ with limit in $\overline{T(A)}$. 
Thm $K(x,y) \subset B(x,y)$ and $K(x,y)$ is lin space.

proof Let $T \in K(x,y)$. Then $\{T x : \|x\| \leq 1\} \subset \{T x : \|x\| = 1\}$ is compact, so bounded, so $T$ is bounded.

Prop $S \subset B(x,y), T \in B(y,z)$ and either $S$ or $T$ is compact, then $TS$ is compact.

proof $\{x_n\} \subset X$ bounded $\Rightarrow$ $\{TS x_n\}$ has a conv subseq.

Prop $T \in B(x,y)$

a) $r(T) = \dim(\text{ran } T) < \infty$, then $T$ compact

b) If either $X$ or $Y$ is finite dim, then $T$ compact

proof a) Let $\{x_n\}$ bounded, then $\{Tx_n\}$ is bounded in finite dimensional space $\text{ran } T$. So with B-W it has converging subseq.

b) $r(T) < \infty$ so apply a).

Prop In infinite dimensional normed lin space $X$ is the identity not compact

proof Unit ball is not compact (Riesz' lemma)
Corollary: In a $\omega$-dim normed linear space $X$, is $T \in \mathcal{L}(X)$ not boundedly invertible.

Proof: Suppose it is, then $I = TT^{-1}$ is compact, which is not.

Then $X, Y$ normed linear spaces. $T \in \mathcal{L}(X, Y)$. Then $\text{ran } T$ (and so $\overline{\text{ran } T}$) is separable.

Proof: $\text{ran } T = \bigcup_{n \in \mathbb{N}} T(B_n(0))$. $T(B_n(0))$ is compact, so separable. (Given $m$, consider open covering with balls of radius $\frac{1}{m}$. Find finite subcovering but then also $T(B_n(0))$ is separable.

Countable union of countable sets is countable.
Thm. \( X \) normed, \( Y \) Banach. Then \( K(X, Y) \) closed in \( B(X, Y) \).

Proof. \( \exists T_n \subset K(X, Y) \) with \( T_n \to T \in B(X, Y) \).

Let \( \{x_n\} \subset X \) be bounded.

\[ \exists \{x_{n(m)}\} \quad \text{with} \quad \{T_{i} x_{n(m)}\} \text{ converging}, \]

\[ \exists \{x_{n(2m)}\} \quad \text{with} \quad \{T_{2} x_{n(2m)}\} \text{ converging}. \]

Subseq. of \( \{x_{n(m)}\} \)

\[ \forall i \text{ 'diagonal'} \quad \{T_{i} x_{n(i)}\} \text{ converging}. \]

Let’s denote \( \{x_{n(m)}\} \) again as \( \{x_{m}\} \).

\[ \| T x_{m} - T x_{2} \| \leq \| T x_{m} - T_{K} x_{m} \| + \| T_{K} (x_{m} - x_{2}) \| + \| T_{2} x_{2} - T x_{2} \| \]

\[ \leq \| T - T_{K} \| \| x_{m} \| + \| T_{K} x_{2} - T x_{2} \| \]

\[ < \frac{\delta}{3} \text{ when } K \text{ is suff. large} \]

For this \( K \), \( \exists N \) s.t. \( \forall m, n \geq N \), 2nd part \( < \frac{\delta}{3} \). So Cauchy, so convergent (\( Y \) is Banach).
Corol X normed, y Banach. \{T_n\}_n \subset B(x,y) with finite rank and \(T_n \to T \in B(x,y)\), then \(T\) is compact.

Let \(T \in B(\ell^2, \ell^2)\) by \(T\{a_n\} = \{a_nb_n\}\) where \(\{b_n\}_n \in C_0\) is compact.

Indeed, \((T_n \{a_n\}) = \{a_nb_n\} \quad i \leq n
= 0 \quad i > n

\[T : L^2(a,b) \to L^2(a,b) \quad (Tf)(x) = \int_a^b k(x,y)f(y)\,dy\]

By appr \(L^2\) by bivariate pol \(\to T\) is compact.
Theorem: $X$ normed, $Y$ Hilbert and $T : X \rightarrow (Y, \langle \cdot, \cdot \rangle)$. Then $\exists T_k \in B(X, Y)$ of operators with finite rank with $\|T_k\| \rightarrow T$.

(not true for $Y$ Banach)

Proof: ran $T$ is separable, and is Hilbert space, so it has an orthonormal basis $\{e_i : i \in \mathbb{N}\}$. Let $P_k$ be orthogonal projector onto $\text{Span} \{e_i : i \in k\}$. $T_k = P_k T$, has finite rank.

Suppose $T_k \not\rightarrow T : \exists \varepsilon > 0 \forall N \exists n \geq N \|T - T_n\| > \varepsilon$, so

$\exists$ subseq of $\{T_k\}$, that we again denote with $\{T_k\}$ with

$\|T - T_k\| > \varepsilon$

So $\exists x_k$ with $\|x_k\| = 1$ and $\|T - T_n - x_k\| > \frac{\varepsilon}{2}$

$T$ is compact, $\exists$ subseq of $\{x_k\}$, that we call $\{x_{k_n}\}$, with $T x_{k_n} \rightarrow y e_y$

$$(T_k - T)(x_{k_n}) = (P_{k_n} - I) T x_{k_n} = (P_{k_n} - I) y + (P_{k_n} - I)(T x_{k_n} - y)$$

$$\frac{\varepsilon}{2} < \|T_k - T\| x_{k_n} \leq \sqrt{\sum_{n=k+1}^{\infty} \langle y, e_n \rangle^2} + 2 \|T x_{k_n} - y\| \rightarrow 0$$

$$\frac{\varepsilon}{2} \langle y, e_n \rangle e_n$$

$$n = k+1$$
Lemma. If $H$ is a Hilbert space, then $\Gamma(T) = r(T^*)$.

Proof. Let $r(T) < \infty$. Ker $T^*$ is closed in subspace of $H$. Write $H = \ker T^* \oplus (\ker T^*)^\perp$. Ker $T^* = (\ran T)^\perp$.

$(\ker T^*)^\perp = \overline{\ran T} = \ran T$.

So $H = \ker T^* \oplus \ran T$, and so $\ran T^* = \ran T^* |_{\ran T}$.

$r(T^*) \leq r(T)$.

Analogously $r(T^*) < \infty \Rightarrow r(T) \leq r(T^*)$.

So either $r(T) < \infty$ or $r(T^*) < \infty \Rightarrow r(T) = r(T^*)$.

or $r(T) = \infty$ and $r(T^*) = \infty \Rightarrow r(T) = r(T^*)$.

Corollary. If $K(H) \ni T \mapsto T^*$.

Proof. When $T \in K(H)$, then $\exists T_n \in B(H)$ with finite rank.

$T_n \to T$.

So $T_n^* \to T^*$ and $r(T_n^*) = r(T_n) < \infty \Rightarrow T^*$ is compact.

When $T^*$ is compact $\Rightarrow (T^*)^* \text{ is compact}$.
Def Let $H$ Hilbert with orthon basis $\{e_n\}$, $T \in \mathcal{B}(H)$ is called Hilbert-Schmidt operator when $\sum_n \|T e_n\|^2 < \infty$.

- Definition is independent of choice of basis.
- $T$ Hilb.-Schm $\iff T^* \text{ is Hilb.-Schm.}$
- $T$ Hilb.-Schm $\Rightarrow T$ compact.

proof: Exerc.
Spectral theory of compact operators

H complex Hilbert, $T \in B(H)$

Point spectrum $\sigma_p(T) = \{ \lambda : \lambda$ eigenvalue of $T \}$

Resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ (i.e. $T - \lambda I$ boundedly invertible)

Thm a) $\dim H = \infty$ & $T \in K(H)$ then $0 \in \sigma(T)$

b) If, additionally, $H$ not separable, then $0 \in \sigma_p(T)$

Proof a) $0 \in \rho(T)$ means $T^{-1} \in B(H)$, but then $I = TT^{-1}$ compact

b) Have to show that $\ker T \neq \{0\}$. 

$T$ compact $\iff$ $T^\perp$ compact, so $\overline{\text{ran } T^\perp}$ separable.

Also $H = \ker T \oplus (\ker T)^\perp$. $\ker T = (\text{ran } T^\perp)^\perp$ so $(\ker T)^\perp = \overline{\text{ran } T^\perp}$. $H$ is not separable, so $\ker T \neq \{0\} \iff 0 \in \sigma_p(T)$. 