Summary $T \in B(H)$

$\sigma(p(T)) = p(\sigma(T))$, $\sigma(T^{-1}) = \{ 1/\mu : \mu \in \sigma(T) \}$

Spectral radius $r_\sigma(T) := \max \{ |\lambda| : \lambda \in \sigma(T) \}$

Numerical range $V(T) := \{ \langle Tx, x \rangle : \|x\| = 1 \}$

- T unitary $\Rightarrow \sigma(T) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$
- T normal $\Rightarrow \sigma(T) \subseteq \overline{V(T)}$
- T self-adj $\Rightarrow V(T) \subseteq \mathbb{R}$ (even equiv.),
  $\{ -\|T\|, \|T\| \} \cap \sigma(T) \neq \emptyset$, $r_\sigma(T) = \sup \{ |\lambda| : \lambda \in V(T) \} = \|T\|$,
  $V(T) \subseteq [\min \sigma(T), \max \sigma(T)]$

T is positive when $V(T) \subseteq [0, \infty)$. $T \geq 0 \Leftrightarrow \sigma(T) \subseteq [0, \infty)$ & self-adj

T self-adj, $f \in C(\sigma(T), \mathbb{R})$, then $f(T) \in B(H)$ & self-adjoint.

T invertible, then $T = UR$ with U unitary, $R \geq 0$.

Ch. 7 Compact operators

<u>Def.</u> X, Y normed lin. spaces. $T \in L(X,Y)$ is **compact** if for any bounded $\{x_n\} \subset X$, $\{Tx_n\}$ has a converging subseq.

Set of compact op^s is denoted as $K(X,Y)$.

<u>lemma</u> $T \in L(X,Y)$ is compact $\Leftrightarrow$ for every bounded $A \subset X$ is $\overline{T(A)}$ compact

(recall: Subset A of metric space is compact when every seq in A has a converging subseq with limit in A)

<u>proof</u> $\Leftarrow$ Let $\{x_n\}$ be bounded, then

$\{Tx_n\} \subseteq \overline{\{Tx_n : n \in \mathbb{N}\}}$, so $\{Tx_n\}$ has converging subseq.

$\Rightarrow$ Let A be bounded. Let $\{b_n\} \subseteq \overline{T(A)}$. $\exists \{a_n\} \subseteq T(A)$ s.t. $|b_n - a_n| < \frac{1}{n}$. $a_n = Tx_n$ where $\{x_n\} \subset A$ so bounded

so $\{a_n\}$ has converging subseq, but then so has $\{b_n\}$ with limit in $\overline{T(A)}$.

Thm $K(X,Y) \subset B(X,Y)$ and $K(X,Y)$ is lin space.

proof Let $T \in K(X,Y)$. Then $\{Tx : \|x\| \leq 1\} \subseteq \overline{\{Tx : \|x\| = 1\}}$ is compact, so bounded, so T is bounded.

Prop $S \in B(X,Y)$, $T \in B(Y,Z)$ and either S or T is compact, then TS is compact

proof $\{x_n\} \subset X$ bounded $\Rightarrow$ $\{TSx_n\}$ has a conv subseq.

Prop $T \in B(X,Y)$ a) $r(T) = \dim(\operatorname{ran} T) < \infty$, then T compact
b) If either X or Y is finite dim, then T compact

proof a) Let $\{x_n\}$ bounded, then $\{Tx_n\}$ is bounded in finite dimensional space $\operatorname{ran} T$. So with B-W it has converging subseq.
b) $r(T) < \infty$ so apply a).

Prop In infinite dimensional normed lin space X is the identity not compact

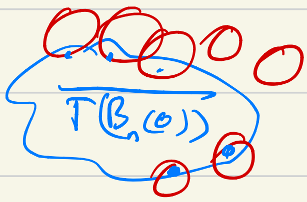
proof Unit ball is not compact (Riesz' lemma)

Corol In ∞-dim normed lin space X, is $T \in K(X)$ not boundedly invertible

proof Suppose it is, then $I = T T^{-1}$ is compact, which is not.

Thm X, Y normed lin spaces. $T \in K(X, Y)$. Then $\operatorname{ran} T$ (and so $\overline{\operatorname{ran} T}$) separable

proof $\operatorname{ran} T = \bigcup_{n \in \mathbb{N}} T(B_n(0))$. $\overline{T(B_n(0))}$ is compact, so separable. (Given m, consider open covering with balls of radius $1/m$. $\exists$ finite subcovering



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but then also $T(B_n(0))$ is separable.

Countable union of countable sets is countable.

Thm X normed, Y Banach. Then $K(X,Y)$ closed in $B(X,Y)$.

proof $\{T_n\} \subset K(X,Y)$ with $T_n \to T \in B(X,Y)$.

Let $\{x_n\} \subset X$ be bounded.

$\exists\ \{x_{n_1(m)}\}$ with $\{T_1 x_{n_1(m)}\}$ converging.

$\exists\ \underbrace{\{x_{n_2(m)}\}}_{\text{subseq of } \{x_{n_1(m)}\}}$ with $\{T_2 x_{n_2(m)}\}$ converging.

$\forall i$ is 'diagonal' $\{T_i x_{n_m(m)}\}$ converging.

Let's denote $\{x_{n_m(m)}\}$ again as $\{x_m\}$

$$\|Tx_m - Tx_\ell\| \le \underbrace{\|Tx_m - T_K x_m\|}_{\le \underbrace{\|T-T_K\|\|x_m\|}_{<\frac{\varepsilon}{3} \text{ when } K \text{ is suff large}}} + \|T_K(x_m - x_\ell)\| + \underbrace{\|T_K x_\ell - Tx_\ell\|}_{<\varepsilon/3}$$

For this K, $\exists N$ s.t $\forall \ell, m \ge N$, 2nd part $< \frac{\varepsilon}{3}$. So Cauchy, so convergent (Y is Banach)

<u>Corol</u> X normed, Y Banach. $\{T_n\} \subset B(X,Y)$ with finite rank and $T_n \to T \in B(X,Y)$, then T compact.

<u>ex</u> $T \in B(\ell^2, \ell^2)$ by $T\{a_k\} = \{a_k b_k\}$ where $\{b_k\} \in c_0$ is compact.

Indeed, $(T_n\{a_k\})_i = \begin{cases} a_i b_i & i \leq n \\ 0 & i > n \end{cases}$

$$T : L^2(a,b) \to L^2(a,b) \quad (Tf)(x) = \int_a^b \underbrace{k(x,y)}_{\text{continuous}} f(y)\,dy$$

By appr k by bivariate pol $\rightsquigarrow$ T is compact.

Thm X normed, Y Hilbert and $T \in K(X,Y)$. Then $\exists \{T_k\} \subset B(X,Y)$ of operators with finite rank with $\{T_k\} \to T$.

(not true for Y Banach)

proof $\overline{\operatorname{ran} T}$ is separable, and is Hilbert space, so it has an orthonormal basis $\{e_n\}$. Let P_k be orthogonal projector onto $\operatorname{span}\{e_1, \dots, e_k\}$. $T_k = P_k T$, has finite rank

Suppose $T_k \not\to T$: $\exists \varepsilon > 0\ \forall N\ \exists n \geq N\ \|T - T_n\| > \varepsilon$, so

$\exists$ subseq of $\{T_k\}$, that we again denote with $\{T_k\}$ with $\|T - T_k\| \geq \varepsilon$

So $\exists \{x_k\}$ with $\|x_k\| = 1$ and $\|(T - T_k)x_k\| > \frac{\varepsilon}{2}$

T is compact, $\exists$ subs of $\{x_k\}$, that we call $\{x_k\}$, with $Tx_k \to y \in Y$

$$(T_k - T)(x_k) = (P_k - I)Tx_k = \underbrace{(P_k - I)y}_{\sum_{n=k+1}^{\infty} <y, e_n> e_n} + (P_k - I)(Tx_k - y)$$

$$\frac{\varepsilon}{2} < \|(T_k - T)x_k\| \leq \sqrt{\sum_{n=k+1}^{\infty} |<y, e_n>|^2} + 2\|Tx_k - y\| \to 0$$

lemma H Hilbert, $T \in B(H)$, then $r(T) = r(T^*)$

proof Let $r(T) < \infty$. $\ker T^*$ closed lin subspace of H

Write $H = \ker T^* \oplus^\perp (\ker T^*)^\perp$. $\ker T^* = (\operatorname{ran} T)^\perp$

$(\ker T^*)^\perp = \overline{\operatorname{ran} T} = \operatorname{ran} T$.

$r(T) < \infty$

So $H = \ker T^* \oplus^\perp \operatorname{ran} T$, and so $\operatorname{ran} T^* = \operatorname{ran} T^*|_{\operatorname{ran} T}$

$r(T^*) \leq r(T)$.

Analogously $r(T^*) < \infty \Rightarrow r(T) \leq r(T^*)$.

So either $r(T) < \infty$ or $r(T^*) < \infty \Rightarrow r(T) = r(T^*)$

or $r(T) = \infty$ and $r(T^*) = \infty \Rightarrow r(T) = r(T^*)$.

Corol H Hilbert. $T \in K(H) \Leftrightarrow T^* \in K(H)$.

proof When $T \in K(H)$, then $\exists \{T_n\} \in B(H)$ with finite rank $T_n \to T$.

So $T_n^* \to T^*$ and $r(T_n^*) = r(T_n) < \infty \Rightarrow T^*$ is compact

When T^* is compact $\Rightarrow \underset{\displaystyle T}{\underset{\shortparallel}{(T^*)^*}}$ is compact.

Def Let H Hilbert with orthon. basis $\{e_n\}$

$T \in B(H)$ is called Hilbert - Schmidt operator when $\sum_1^\infty \|Te_n\|^2 < \infty$.

- Definition is independent of choice basis
- T Hilb.-Schm $\Leftrightarrow$ T^* is Hilb.-Schm.
- T Hilb-Schm $\Rightarrow$ T compact

proof : Exerc.

Spectral theory of compact operators

H complex Hilbert, $T \in B(H)$

$\ker T - \lambda I \neq \{0\}$

Point spectrum $\sigma_p(T) = \{\lambda : \lambda \text{ eigenvalue of } T\}$

Resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ (i.e $T - \lambda I$ boundedly inv)

Thm a) $\dim H = \infty$ & $T \in K(H)$ then $0 \in \sigma(T)$

b) If, additionally, H not separable, then $0 \in \sigma_p(T)$

proof a) $0 \in \rho(T)$ means $T^{-1} \in B(H)$, but then $I = TT^{-1}$ compact $\lightning$

b) Have to show that $\ker T \neq \{0\}$.

T compact $\Leftrightarrow$ T^* compact, so $\overline{\operatorname{ran} T^*}$ separable.

Also $H = \ker T \oplus^{\perp} (\ker T)^{\perp}$. $\ker T = (\operatorname{ran} T^*)^{\perp}$

so $(\ker T)^{\perp} = \overline{\operatorname{ran} T^*}$. H is not separable, so

$\ker T \neq \{0\} \Leftrightarrow 0 \in \sigma_p(T)$.