Summary

Def: $T \in L(X,Y)$ is compact when $\{x_n\}$ bounded $\Rightarrow \{Tx_n\}$ has conv. subseq.

$T \in K(X,Y) \Leftrightarrow$ for every bounded $A \subset X$ is $\overline{T(A)}$ compact.

$K(X,Y)$ is closed lin subspace of $B(X,Y)$ (so $\{T_n\} \subset K(X,Y) \to T \in B(X,Y)$
$\Rightarrow T \in K(X,Y)$)

$T \in B(X,Y)$ & $r(T) < \infty \Rightarrow T \in K(X,Y)$.

If $\dim X = \infty$, then identity not compact

$T \in K(X,Y) \Rightarrow$ $\operatorname{ran} T$ (and $\overline{\operatorname{ran} T}$) is separable.

$T \in B(X,Y)$ & Y Hilbert.
Then $T \in K(X,Y) \Leftrightarrow \exists \{T_k\} \subset B(X,Y)$ with $r(T_k) < \infty$ s.t. $\{T_k\} \to T$.

H Hilbert, $T \in B(H)$. Then a) $r(T) = r(T^*)$
b) $T \in K(H) \Leftrightarrow T^* \in K(H)$

Spectral theory of compact operators

H complex Hilbert, $T \in B(H)$

Point spectrum $\sigma_p(T) = \{\lambda : \lambda \text{ eigenvalue of } T\}$ $\quad \ker T - \lambda I \neq \{0\}$

Resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ (i.e $T - \lambda I$ boundedly inv)

Thm a) $\dim H = \infty$ & $T \in K(H)$ then $0 \in \sigma(T)$

b) If, additionally, H not separable, then $0 \in \sigma_p(T)$

proof a) $0 \in \rho(T)$ means $T^{-1} \in B(H)$, but then $I = TT^{-1}$ compact $\lightning$.

b) Have to show that $\ker T \neq \{0\}$.

T compact $\Leftrightarrow T^*$ compact, so $\overline{\operatorname{ran} T^*}$ separable.

Also $H = \ker T \oplus^{\perp} (\ker T)^{\perp}$. $\ker T = (\operatorname{ran} T^*)^{\perp}$

so $(\ker T)^{\perp} = \overline{\operatorname{ran} T^*}$. H is not separable, so

$\ker T \neq \{0\} \Leftrightarrow 0 \in \sigma_p(T)$.

Thm $T \in K(H)$. Then for $\lambda \neq 0$, $\dim(\underbrace{\ker(T-\lambda I)}_{\text{eigenspace}}) < \infty$

proof Def $M := \ker(T-\lambda I)$ is H-space. Suppose $\dim M = \infty$ and let $\{e_n\}$ ~~orth~~ infinite seq in M. Then $Te_n = \lambda e_n$, and for $n \neq m$ $\|Te_n - Te_m\| = |\lambda| \|e_n - e_m\| = |\lambda|\sqrt{2}$, so no converging subseq. in contradiction with T being compact.

Thm $T \in K(H)$, then for $\lambda \neq 0$, $\operatorname{ran}(T-\lambda I)$ is closed.

proof Let $y_n = (T-\lambda I)x_n$ with $\{y_n\} \to y$.
$x_n = u_n + v_n$ where $u_n = \ker(T-\lambda I)$, $v_n = \ker(T-\lambda I)^\perp$,
so $y_n = (T-\lambda I)v_n$.
We prove that $\{v_n\}$ is bounded. Then, after taking a subseq, $\{Tv_n\}$ is convergent. Since also $\{(T-\lambda I)v_n\}$ is converging, we know that $\{\lambda v_n\}$ is converging, and since $\lambda \neq 0$, $\{v_n\}$ is conv, say to v, and so $y = (T-\lambda I)v$. QED
Suppose $\{v_n\}$ unbounded. Then, after taking subseq, $\|v_n\| \to \infty$.
Define $w_n := \frac{v_n}{\|v_n\|}$, then $(T-\lambda I)w_n = \frac{y_n}{\|v_n\|} \to 0$.
$\{Tw_n\}$ has conv subseq, so by taking this subseq, $\{w_n\}$ are converging, say to w. Then $\|w\| = 1$ and $(T-\lambda I)w = \lim (T-\lambda I)w_n = 0$
so $w \in \ker T-\lambda I$, but also it is $\ker(T-\lambda I)^\perp$, so $w = 0$ ↯ with $\|w\| = 1$

<u>Corol</u> $T \in K(H)$ & $\lambda \neq 0$. Then $(\ker T - \lambda I)^{\perp} = \operatorname{ran} T^* - \bar{\lambda} I$

$(\ker T^* - \bar{\lambda} I)^{\perp} = \operatorname{ran} T - \lambda I$

<u>proof</u> $\ker T - \lambda I = \operatorname{ran} (T^* - \bar{\lambda} I)^{\perp}$

so $(\ker T - \lambda I)^{\perp} = (\operatorname{ran} T^* - \bar{\lambda} I)^{\perp\perp} = \operatorname{ran} T^* - \bar{\lambda} I$

<u>Thm</u> $T \in K(H)$. For any $t>0$, $\{\lambda \in \sigma_p(T) : |\lambda| \geq t\}$ is finite

(later we see $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T)$)

<u>proof</u> Fixing some $t_0 > 0$, let $\{\lambda_n\} \subset \sigma_p(T)$ infinite seq of different eigenvalues, with $|\lambda_n| \geq t_0$, and let $\{e_n\}$ denote corresponding eigenvectors.

Def $M_k = \text{span}\{e_1, \dots, e_k\}$ $(k \geq 1)$, $M_0 = \{0\}$.

$\forall k$ $\{e_1 \dots e_k\}$ is linearly independent.

Select $y_k \in M_k \cap M_{k-1}^{\perp}$ with $\|y_k\| = 1$. Then

$TM_k \subseteq M_k$ & $(T - \lambda_k I) M_k \subseteq M_{k-1}$,

and for $v_{k-1} \in M_{k-1}$ it holds $\|y_k + v_{k-1}\|^2 = \|y_k\|^2 + \|v_{k-1}\|^2 \geq 1$

So for $n > m$,

$$Ty_n - Ty_m = \lambda_n y_n + (T - \lambda_n I) y_n - Ty_m$$
$$= \lambda_n \{ y_n + \underbrace{\lambda_n^{-1} [(T - \lambda_n I) y_n - Ty_m]}_{\in M_{n-1}} \}$$

$\|Ty_n - Ty_m\| \geq |\lambda_n| \geq t_0$ } it cannot have a converging subseq.

<u>Corol</u> $T \in K(H)$. Then $\sigma_p(T)$ is countable

<u>proof</u> $\mathbb{R} = \{0\} \cup \bigcup_{n \geq 1} \mathbb{R} \setminus [-\frac{1}{n}, \frac{1}{n}]$

$\lambda \in \sigma(T) \Leftrightarrow \bar{\lambda} \in \sigma(T^*)$

$\lambda \neq 0$

lemma $T \in B(H)$, $r(T) < \infty$. Then either $\lambda \in \rho(T)$ $(\Leftrightarrow \bar{\lambda} \in \rho(T^*))$,

or $\lambda \in \sigma_p(T)$ and $\bar{\lambda} \in \sigma_p(T^*)$ and $n(T-\lambda I) = n(T^* - \bar{\lambda} I) < \infty$

proof

$M := \operatorname{ran} T$, $N = \ker T^* = M^\perp$. $\dim M < \infty$, so M closed, so

$H = M \oplus M^\perp$. For $x \in H$, write $x = u + v$ $\quad u \in M, v \in M^\perp$.

$$(T-\lambda I)(u+v) = \underbrace{Tu - \lambda u + Tv}_{=:\,\hat{u} \in M} \underbrace{- \lambda v}_{=:\,\hat{v} \in N}, \text{ i.e.}$$

$$\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} (T-\lambda I)|_M & T|_N \\ 0 & -\lambda I|_N \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Set $A = (T-\lambda I)|_M \in L(M,M)$. So $\lambda \in \rho(T) \Leftrightarrow A$ invertible $\Leftrightarrow$ $n(A) = 0$

If $\lambda \notin \rho(T)$, i.e., $n(A) > 0$, then $Au = 0 \Leftrightarrow (T-\lambda I)\begin{pmatrix} u \\ v \end{pmatrix} = 0$ & $v = 0$

i.e $\lambda \in \sigma_p(T)$ and $n(T-\lambda I) = n(A)$

$$I = P_M + P_N$$

(II) $(T^* - \bar\lambda I)(u+v) = (T^* - \bar\lambda I)u - \bar\lambda v =$

$$\underbrace{P_M(T^* - \bar\lambda I)u}_{\substack{\| \\ \hat u \in M}} + \underbrace{P_N T^* u - \bar\lambda v}_{\substack{\| \\ \hat v \in N}}$$

$$\begin{bmatrix} \hat u \\ \hat v \end{bmatrix} = \begin{bmatrix} P_M(T^* - \bar\lambda I)|_M & 0 \\ P_N T^*|_M & -\bar\lambda I|_N \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$P_M(T^* - \lambda I)|_M = A^*$ (exerc 7.20) $\quad n(A) = n(A^*)$

$\bar\lambda \in \rho(T^*) \iff A^*$ is invertible $\iff n(A^*) = 0$

If $\bar\lambda \notin \rho(T^*) \iff n(A^*) > 0 \iff \bar\lambda \in \sigma_p(T^*)$

and $n(T^* - \bar\lambda I) = n(A^*)$

Thm $T \in K(H)$, $\lambda \neq 0$. Then either $\lambda \in \rho(T)$ ($\Leftrightarrow \bar\lambda \in \rho(T^*)$)
or $\lambda \in \sigma_p(T)$ and $\bar\lambda \in \sigma_p(T^*)$ and $n(T-\lambda I) = n(T^* - \bar\lambda I) < \infty$

proof Given $\lambda \neq 0$ $\exists T_F \in B(H)$ with $r(T_F) < \infty$ and
$\|T - T_F\| < |\lambda|$, so that $S := I - \lambda^{-1}(T - T_F)$ is bounded invertible

$$T - \lambda I = T_F - \lambda S = (T_F S^{-1} - \lambda I) S \quad \text{and } r(T_F S^{-1}) < \infty$$

and $T^* - \bar\lambda I = S^* (S^{-*} T_F^* - \bar\lambda I)$

$$\lambda \in \rho(T) \Leftrightarrow \lambda \in \rho(T_F S^{-1}) \Leftrightarrow \bar\lambda \in \rho(S^{-*} T_F^*) \Leftrightarrow \bar\lambda \in \rho(T^*)$$

$$\lambda \notin \rho(T) \Leftrightarrow \lambda \notin \rho(T_F S^{-1}) \Leftrightarrow \lambda \in \sigma_p(T_F S^{-1}) \text{ and}$$
$$\infty > n(T_F S^{-1} - \lambda I) = n(T - \lambda I).$$

Moreover, in this case

$$0 < n(T_F S^{-1} - \lambda I) = n(S^{-*} T_F^* - \bar\lambda I) = n(T^* - \bar\lambda I) \text{ so}$$
in part., $\bar\lambda \in \sigma_p(T^*)$.

Thm (Fredholm alternative). $T \in K(H)$, $\lambda \neq 0$. Consider eq^s

$(T-\lambda I)x = 0 \quad (T^* - \bar{\lambda} I)y = 0 \quad$ (hom eq^s)

$(T-\lambda I)x = p \quad (T^* - \bar{\lambda} I)y = q \quad$ (inhom eq^s)

Then either

a) ($\lambda \in \rho(T)$) Hom. eq^s have only trivial sol, inhom eq^s have unique sol^s that depend continuously on data.

b) ($\lambda \in \sigma_p(T)$) Both hom. eq^s have $m_\lambda < \infty$ independent sol^s $\{x_1, \dots, x_{m_\lambda}\}$ (basis $\ker(T-\lambda I)$) or $\{y_1, \dots, y_{m_\lambda}\}$ (basis $\ker(T^*-\bar{\lambda}I)$).

Inhom eq^s have sol iff $p \in \operatorname{ran}(T-\lambda I) = \ker(T^*-\bar{\lambda}I)^\perp$ (i.e. $p \perp y_i \ \forall i$) or $q \in \operatorname{ran}(T^*-\bar{\lambda}I) = \ker(T-\lambda I)^\perp$ (i.e. $q \perp x_i \ \forall i$)

Moreover

$T-\lambda I : \ker(T-\lambda I)^\perp \to \ker(T^*-\bar{\lambda}I)^\perp$ boundedly inv.

$T^*-\bar{\lambda}I : \ker(T^*-\bar{\lambda}I)^\perp \to \ker(T-\lambda I)^\perp$ boundedly inv.

proof Only things to show are $\operatorname{ran} T-\lambda I = \ker(T^*-\bar{\lambda}I)^\perp$, which follows from $(\operatorname{ran} T-\lambda I)^\perp = \ker(T^*-\bar{\lambda}I)$, and $\operatorname{ran}(T-\lambda I)$ is closed (idem for adjoint side), and, for $\lambda \in \sigma_p(T)$, that

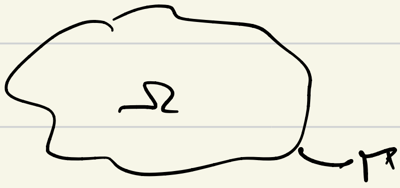
$T-\lambda I : \ker(T-\lambda I)^\perp \to \ker(T^*-\bar{\lambda}I)^\perp$ surj.

It holds that $T-\lambda I : H \to \ker(T^*-\bar{\lambda}I)^\perp$ surj.

Now write $H = \ker(T-\lambda I) \oplus \ker(T-\lambda I)^\perp$

Application

$-\Delta\phi = 0$ on an open $\Omega \subset \mathbb{R}^2$



$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Complete this eq by boundary cond^s. For examp[l]e Dirichlet boundary cond^s $\phi = f$ on $\Gamma = \partial\Omega$

Because eq is homogeneous, it can be reduced to an integral eq on the boundary. Two ways of doing that:

1) $\forall s \in \mathbb{R}^2$ it holds that $-\Delta\left(t \mapsto \frac{-\log|t-s|}{2\pi}\right) = 0$ when $t \neq s$

Search ϕ of the form $\phi(t) = -\frac{1}{\pi}\int_\Gamma \log|t-s|\, z(s)\, ds$

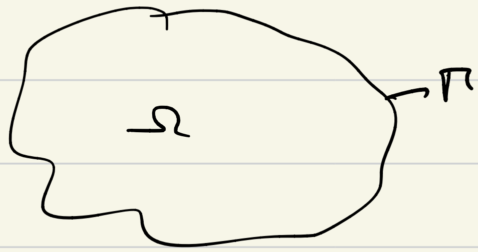
ϕ can be shown to be continuous. Imposing boundary condition means

$$\underbrace{-\frac{1}{\pi}\int_\Gamma \log|t-s|\, z(s)\, ds}_{=:\ (Kz)(t)} = f(t) \qquad t \in \Gamma$$

$(Kz)(t)$ or $Kz = f$ integral eq of 1st kind.

2) $-\Delta\left(\frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial n_s} \log|t-s|\, z(s)\, ds\right) = 0 \qquad s \neq t$

$t \mapsto \frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial n_s} \log|t-s|\, z(s)\, ds$ is discontinuous at Γ. In part.,



$$\lim_{t \to t' \in \Gamma} \frac{1}{\pi}\int \frac{\partial}{\partial n} (\log|t-s|\, z(s)\, ds$$

$$= \underbrace{\frac{1}{\pi}\int_\Gamma \frac{\partial}{\partial n} (\log|t'-s|\, z(s)\, ds}_{(kz)(t')} + z(t').$$

So imposing boundary condition results in

$$(I + k)z = f$$ integral eq of 2nd kind.

$k \in K(L^2(\Gamma))$

<u>Fredholm</u>: Either $-1 \in \rho(k)$ and you have uniq sol, or $-1 \in \sigma_p(k)$, and then there exists a unique solution $z \in \ker(k+I)^\perp$ when $f \in \ker(k^* + I)^\perp$.