1.1. **Construction of Gauss quadrature.** This subsection replaces [Book, §10.1, §10.2 until Thm. 10.2].

For \( a < b \), \( w(x) > 0 \) on \([a, b]\) almost everywhere (a.e.) with \( \int_a^b w(x)dx < \infty \), let for \( f \in C[a, b] \),

\[
I(f) := \int_a^b w(x)f(x)dx.
\]

For \( \{x_0, \ldots, x_n\} \in [a, b] \), \( w_0, \ldots, w_n \in \mathbb{R} \), let

\[
Q_n(f) := \sum_{i=0}^n w_i f(x_i).
\]

**Theorem 1.1.** Let \( Q_n \) be exact on \( P_n \) (so \( w_i = \int_a^b w(x)L_i^{(n)}(x)dx \)). Then it is exact on \( P_m \) for an \( m > n \) if and only if

\[
\pi_{n+1} \perp_{\langle \cdot, \cdot \rangle_w} P_{m-n-1},
\]

where \( \pi_{n+1}(x) := \prod_{i=0}^n (x - x_i) \) and \( \langle g, k \rangle_w := \int_a^b w(x)g(x)k(x)dx \).

**Proof.** Suppose \( Q_n \) is exact on \( P_m \). Then for all \( p \in P_{m-n-1} \), it holds that

\[
\int_a^b w(x)p(x)\pi_{n+1}(x)dx = \sum_{i=0}^n w_i p(x_i)\pi_{n+1}(x_i) = 0.
\]

Conversely, let \( \pi_{n+1} \perp_{\langle \cdot, \cdot \rangle_w} P_{m-n-1} \). Each \( p \in P_m \) can be written as \( p = q\pi_{n+1} + r \) where \( q \in P_{m-n-1} \) and \( r \in P_n \). We have

\[
\int_a^b w(x)p(x)dx = \int_a^b w(x)q(x)\pi_{n+1}(x)dx + \int_a^b w(x)r(x)dx = \int_a^b w(x)r(x)dx,
\]

and

\[
\sum_{i=0}^n w_i p(x_i) = \sum_{i=0}^n w_i q(x_i)\pi_{n+1}(x_i) + \sum_{i=0}^n w_i r(x_i) = \sum_{i=0}^n w_i r(x_i).
\]

From the fact that \( Q_n \) is exact on \( P_n \), we infer that the expressions on both right-hand sides are equal, and so \( I(p) = Q_n(p) \), meaning that \( Q_n \) is exact on \( P_m \). \( \square \)

By selecting \( \pi_{n+1} \) such that \( \pi_{n+1} \perp_{\langle \cdot, \cdot \rangle_w} P_n \), that is, by taking \( \{x_0, \ldots, x_n\} \) the roots of the orthogonal polynomial of degree \( n+1 \), and by determining the weights \( w_0, \ldots, w_n \) such that \( Q_n \) is exact on \( P_n \), the resulting \( (n+1) \)-point formula, known as the Gauss formula, is thus exact on \( P_{2n+1} \).

An example of a Gauss formula for \( n = 1 \), \([a, b] = [0, 1]\) and \( w \equiv 1 \) is given in [Book, Example 10.1]. There the weights are computed as \( w_i = \int_a^b w(x)L_i^{(n)}(x)^2 dx \).

---

*Date: April 30, 2019.*
Apparently, with this choice of quadrature points it holds that \( \int_a^b w(x)L_i^{(n)}(x)^2 \, dx = \int_a^b w(x)L_i^{(n)}(x) \, dx \), cf. [Book, Exer. 10.2].

**Proposition 1.2.** For the \((n+1)\)-point Gauss formula \(Q_n\), and \(f \in C^{(2n+2)}[a,b]\), it holds that

\[
I(f) - Q_n(f) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b w(x)\pi_{n+1}^2(x) \, dx,
\]

for some \(\xi \in [a,b]\).

**Proof.** With \(p \in \mathcal{P}_{2n+1}\) being the Hermite interpolation polynomial of \(f\) on \(\{x_0, \ldots, x_n\}\), it holds that \(Q_n(f) = Q_n(p) = I(p)\), and so

\[
I(f) - Q_n(f) = I(f - p)
\]

\[
= \int_a^b w(x)\pi_{n+1}^2(x) \frac{f^{(2n+2)}(\xi(x))}{(2n+2)!} \, dx = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \int_a^b w(x)\pi_{n+1}^2(x) \, dx,
\]

for some \(\xi \in [a,b]\), where we have used that \(w(x)\pi_{n+1}^2(x) \geq 0\) on \([a,b]\). \(\square\)
2. Additions to Chapter 11 book

2.1. A upper bound for the error of best approximation from a spline space. For \( a = x_0 < x_1 < \ldots < x_m = b \), and \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), let

\[
S_n^{(-1)} := \{ f : [a, b] \to \mathbb{R} : f|_{[x_i, x_{i+1}]} \in \mathcal{P}_n, \ i = 0, \ldots, m - 1 \}.
\]

For \( k \in \{0, \ldots, n\} \), we set

\[
S_n^{(k)} := C^k[a, b] \cap S_n^{(-1)}.
\]

Obviously \( \mathcal{P}_n(a, b) \subseteq S_n^{(n)} \supseteq \ldots \supseteq S_n^{(-1)} \). It holds that

\[
\dim S_n^{(k)} = (n + 1)m - (k + 1)(m - 1) = m(n - k) + k + 1,
\]

where the subtraction of \((k + 1)(m - 1)\) corresponds to the loss of \( k + 1 \) degrees of freedom at each of the points \( x_1, \ldots, x_{m-1} \) as a consequence of the \( C^k \) constraint.

So in particular \( \dim S_n^{(n)} = n + 1 \), meaning that \( S_n^{(n)} \) is simply equal to \( \mathcal{P}_n(a, b) \). From here on we exclude this non-interesting case, and consider \( k \in \{0, \ldots, n - 1\} \). In some books, any of these spaces \( S_n^{(k)} \) are called spline spaces, and in other books the name spline is used exclusively for functions from \( S_n^{(n-1)} \). The piecewise polynomial functions discussed in [Book, §11.2-3], [Book, §11.4], [Book, §11.5], and [Book, §11.6] are in \( S_1^{(0)} \), \( S_3^{(2)} \), \( S_3^{(1)} \), and \( S_n^{(n-1)} \), respectively.

For simplicity in the following let us now consider the equidistant case \( x_i = a + ih \), where \( h = \frac{b-a}{m} \). From [Book, Ch. 6], we know that for \( n \in \mathbb{N}_0 \),

\[
\inf_{s \in S_n^{(-1)}} \| f - s \|_{\infty} \leq \inf_{s \in S_n^{(0)}} \| f - s \|_{\infty} = \mathcal{O}(h^{n+1}\| f^{(n+1)} \|_{\infty}).
\]

Indeed, for example take \( s|_{[x_i, x_{i+1}]} \in S_n^{(0)} \) to be the Lagrange interpolant of degree \( n \in \mathbb{N} \) w.r.t. equidistant interpolation points \( x_i = y_0 < y_1 < \cdots < y_{n+1} = x_{i+1} \), then \( \| (f - s)|_{[x_i, x_{i+1}]} \|_{\infty} = \mathcal{O}(h^{n+1}\| f^{(n+1)} \|_{\infty}) \).

Remarkably, a bound as in (2.2) is also valid for approximation from the much smaller 'true' spline space \( S_n^{(n-1)} \):

\[
\inf_{s \in S_n^{(n-1)}} \| f - s \|_{\infty} = \mathcal{O}(h^{n+1}\| f^{(n+1)} \|_{\infty})
\]

(and thus also for approximation from any 'intermediate space' \( S_n^{(k)} \)). Note that \( \dim S_n^{(n-1)} = m + n \), which thinking of \( m \) 'large' and say \( n = 10 \), is only slightly larger than \( \dim S_n^{(0)} = m + 1 \).

We do not provide a proof for (2.3) in the general case. Instead, in Exer. 6 (first statement of part (f)), the statement is proven for \( n = 3 \). In this exercise, an interpolant \( s \in S_3^{(2)} \) is constructed that realizes this upper bound on the approximation error. Other than the continuous piecewise Lagrange interpolant in \( S_3^{(0)} \), or the \( C^1 \) piecewise cubic Hermite interpolant in \( S_3^{(1)} \), the construction of the interpolant in \( S_3^{(2)} \) is not local, i.e., \( s|_{[x_i, x_{i+1}]} \) does not depend exclusively on \( f|_{[x_i, x_{i+1}]} \).
Exer. 6. Let $I_1 : C[a, b] \to S_1^{(0)}$ the continuous piecewise linear interpolator (i.e., the mapping from a continuous function to its continuous piecewise linear interpolant), and let $I_3 : C^1[a, b] \to S_3^{(2)}$ the “complete cubic spline interpolator” defined by

\begin{equation}
\tag{2.4}
\begin{align*}
  s(x_i) &= f(x_i) \quad (i \in \{0, \ldots, m\}), \\
  s'(x_0) &= f'(x_0), \quad s'(x_m) = f'(x_m).
\end{align*}
\end{equation}

where $s$ is here a shorthand notation for $I_3(f)$. Note that the number of conditions equals dim $S_3^{(2)}$.

The aim of this exercise is to show that \( \|f - I_3(f)\|_\infty = O(h^4\|f^{(4)}\|_\infty) \) assuming \( f \in C^4[a, b] \).

For \( s \in S_3^{(2)} \), it holds that \( s'' \in S_1^{(0)} \), so that for \( i = 1, \ldots, m \),

\[ s''|_{[x_{i-1}, x_i]}(x) = \frac{x_i - x}{h} \sigma_{i-1} + \frac{x - x_{i-1}}{h} \sigma_i, \]

where, for \( i \in \{0, \ldots, m\} \), \( \sigma_i := s''(x_i) \). By integrating this relation twice, we obtain

\begin{equation}
\tag{2.6}
s|_{[x_{i-1}, x_i]}(x) = \frac{(x_i - x)^3}{6h} \sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h} \sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x),
\end{equation}

for some scalars \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \).

By imposing (2.4) we obtain

\[ \alpha_i = \frac{f(x_i)}{h} - \frac{h}{6} \sigma_i, \quad \beta_i = \frac{f(x_{i-1})}{h} - \frac{h}{6} \sigma_{i-1}. \]

(a) By using the continuity of \( s' \) in \( x_1, \ldots, x_{m-1} \) and (2.5), show that

\[ A[\sigma_0 \ldots \sigma_{m-1}]^\top = b, \]

where \( A \in \mathbb{R}^{(m+1) \times (m+1)} \) is defined by

\[ A = \begin{bmatrix}
  4 & 2 \\
  1 & 4 & 1 \\
  & \ddots & \ddots & \ddots \\
  1 & 4 & 1 \\
  2 & 4
\end{bmatrix}, \]

and \( b \in \mathbb{R}^{m+1} \) by

\[ b_i = \begin{cases} 
  12 \left[ \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f'(x_0)}{h} \right] & \text{when } i = 0, \\
  6 \left[ \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \right] & \text{when } i \in \{1, \ldots, m-1\}, \\
  12 \left[ \frac{f'(x_m)}{h} - \frac{f(x_{m}) - f(x_{m-1})}{h} \right] & \text{when } i = m.
\end{cases} \]

Answer: By differentiating (2.6) we infer that

\[ s'|_{[x_{i-1}, x_i]}(x) = -\frac{(x_i - x)^2}{2h} \sigma_{i-1} + \frac{(x - x_{i-1})^2}{2h} \sigma_i + \alpha_i - \beta_i, \]

and so in particular that

\[ s'|_{[x_{i-1}, x_i]}(x_{i-1}) = -\frac{1}{2} h \sigma_{i-1} + \alpha_i - \beta_i = h(\frac{1}{3} \sigma_i - \frac{1}{6} \sigma_{i-1}) + h^{-1}(f(x_i) - f(x_{i-1})), \]

\[ s'|_{[x_{i-1}, x_i]}(x_i) = \frac{1}{2} h \sigma_{i-1} + \alpha_i - \beta_i = h(\frac{1}{3} \sigma_i + \frac{1}{6} \sigma_{i-1}) + h^{-1}(f(x_i) - f(x_{i-1})). \]
Imposing that \( s'\big|_{[x_{i-1}, x_i]}(x_i) = s'\big|_{[x_i, x_{i+1}]}(x_i) \) for \( i = 1, \ldots, m-1 \) yields \( h^3 \sigma_i + \frac{1}{6} \sigma_{i-1} + h^{-1}(f(x_i) - f(x_{i-1}) = h(-\frac{1}{3} \sigma_{i+1} - \frac{1}{6} \sigma_i) + h^{-1}(f(x_{i+1}) - f(x_i)) \) or

\[
\tag{2.7} h \sigma_{i-1} + 4h \sigma_i + h \sigma_{i+1} = 6 \left( \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_i) - f(x_{i-1})}{h} \right),
\]

The conditions \( s'(x_0) = f'(x_0) \) and \( s'(x_m) = f'(x_m) \) yield

\[
\tag{2.8} h \left( \frac{1}{3} \sigma_m - \frac{1}{6} \sigma_{m-1} \right) + h^{-1}(f(x_m) - f(x_{m-1})) = f'(x_m)
\]

Together (2.7) and (2.8) complete the proof.

Elementary linear algebra shows that \( A \) is invertible, and that \( \|A^{-1}\|_{\infty} \leq \frac{1}{2} \), i.e., that \( \max_i \|A^{-1}x_i\| \leq \frac{1}{2} \max_i |x_i| \). (Indeed, writing \( A = 4(I - (I - \frac{1}{4} A)) \) and using that \( \|I - \frac{1}{4} A\|_{\infty} = \frac{1}{2} \), shows that \( \|A^{-1}\|_{\infty} \leq \frac{1}{2} \).

(b) Show that \( \|I_3(f)'\|_{\infty} \leq 3 \|f''\|_{\infty} \). (Hint: Show that for \( s \in S^3_3 \), \( \|s''\|_{\infty} = \max_{0 \leq i \leq m} |s''(x_i)| \), and that \( \max_{0 \leq i \leq m} |b_i| \leq 6 \|f''\|_{\infty} \).

Answer: \( s' \in S^0_1 \), and for any \( g \in S^0_1 \), \( \|g\|_{\infty} = \max_{0 \leq i \leq m} |g(x_i)| \).

Taylor gives

\[
f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(\xi_1)\]
\[
f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2}{2!} f''(\xi_2)
\]

for some \( \xi_1 \in (x_i, x_{i+1}) \), \( \xi_2 \in (x_{i-1}, x_i) \). It shows that for \( 1 \leq i \leq m - 1 \), \( b_i = 6(f''(\xi_1) + f''(\xi_2))/2 = 6f''(\xi) \) for some \( \xi \in [\xi_2, \xi_1] \subset [x_{i-1}, x_{i+1}] \). From \( f(x_1) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(\xi_0) \) for some \( \xi_0 \in (x_0, x_1) \), one infers that \( b_0 = 6f''(\xi_0) \). Since similarly \( b_m = 6f''(\xi_m) \) for some \( \xi_m \in (x_{m-1}, x_m) \), we conclude that \( \max_{0 \leq i \leq m} |b_i| \leq 6 \|f''\|_{\infty} \).

Combining this with \( \max_{0 \leq i \leq m} |\sigma_i| \leq \frac{1}{2} \max_{0 \leq i \leq m} |b_i| \), the proof is completed.

(c) Show that \( I_3 \) is a projector, i.e., that \( I_3(s) = s \) for any \( s \in S^3_3 \).

Answer: From the fact that \( A \) is invertible, we conclude that there exists exactly one -one- \( s \in S^3_3 \) that solves the interpolation problem given by (2.4)-(2.5). In other words given \( y_0, \ldots, y_m, z_0, z_m \in \mathbb{R} \), there is exactly one -one- \( s \in S^3_3 \) with \( s(x_i) = y_i \) (0 \( \leq i \leq m \)) and \( s'(x_i) = z_i \) (i = 0, m).

The latter means that \( I_3(s) = s \) for any \( s \in S^3_3 \).

(d) Show that for any \( p \in S^0_1 \) there exists an \( \bar{s} \in S^3_3 \) with \( \bar{s''} = p \).

Answer: Defining \( \bar{s}(x) := \int_{x_0}^{x} \int_{y}^{y} p(z) \, dz \, dy \), i.e., taking the antiderivative twice, gives a globally \( C^2 \), piecewise cubic function, i.e., a function in \( S^3_3 \), with \( s'' = p \).
(e) Let \( s \in S_3^{(2)} \) be such that \( \bar{s}'' = I_1(f'') \). Show that \( f - I_3(f) = f - \bar{s} - I_3(f - \bar{s}) \), and with that, show that

\[
\|f'' - I_3(f)''\|_{\infty} \leq 4\|f'' - \bar{s}''\|_{\infty} \leq \frac{1}{2}h^2\|f^{(4)}\|_{\infty}.
\]

Answer: The mapping \( I_3 \) is linear (indeed (2.4)-(2.5) show that \( s := I_3(f) \) depends linearly on \( f \)). So \( f - I_3(f) = f - \bar{s} - I_3(f - \bar{s}) \) by (c). So, using (b), we have

\[
\|f'' - I_3(f)''\|_{\infty} = \|f'' - \bar{s}'' - I_3(f - \bar{s})''\|_{\infty}
\]
\[
\leq \|f'' - \bar{s}''\|_{\infty} + \|I_3(f - \bar{s})''\|_{\infty} \leq (1 + 3)\|f'' - \bar{s}''\|_{\infty}.
\]

Now use that \( \bar{s}'' = I_1(f'') \) and that for a \( g \in C^2([a,b]) \),

(2.9)

\[
\|g - I_1(g)\|_{\infty} \leq h^2 \frac{k^2}{8}\|g''\|_{\infty},
\]

which follows from an application of [Book, Thm. 6.2.] on each \([x_i, x_{i+1}]\).

(f) Show that \( I_1(f - I_3(f)) = 0 \), and with that show that

\[
\|f - I_3(f)\|_{\infty} \leq \frac{1}{16}h^4\|f^{(4)}\|_{\infty},
\]

as well as

\[
\|f' - I_3(f)\|_{\infty} \leq \frac{1}{2}h^3\|f^{(4)}\|_{\infty}.
\]

Answer: \( I_1(f - I_3(f)) = 0 \) follows from the fact that \( f \) and \( I_3(f) \) take equal values in \( x_0, \ldots, x_m \). So, using (2.9) and (e), we have

\[
\|f - I_3(f)\|_{\infty} = \|f - I_3(f) - I_1(f - I_3(f))\|_{\infty}
\]
\[
\leq \frac{h^2}{8}\|(f - I_3(f))''\|_{\infty} \leq \frac{h^2}{8}\frac{1}{2}h^2\|f^{(4)}\|_{\infty}.
\]

The second statement follows similarly, by using instead of (2.9),

\[
\|g' - I_1(g)\|_{\infty} \leq h\|g''\|_{\infty},
\]

being a consequence of [Book, Corol. 6.1].
2.2. Construction of a local basis for $S_n^{(n-1)}$. For storing functions from $S_n^{(n-1)}$, or for computing the best approximation w.r.t. the (weighted) $L_2(a,b)$-norm of some function by an element of $S_n^{(n-1)}$, one needs a basis of $S_n^{(n-1)}$. Preferably this basis is local, meaning that the number of basis functions that are non-zero at a point $x \in [a,b]$ is bounded uniformly in $m$ (and $x$).

Remark 2.1. Other than for $n = 1$ (cf. [Book, §11.3]), for $n > 1$, such a local basis that additionally is interpolating does not exist (recall: A basis is (Lagrange) interpolating, when for each basis function there exists a point in $[a,b]$ in which it doesn’t vanish, but in which all other basis functions do vanish.)

A local basis for $S_n^{(n-1)}$ is constructed in Exer. 7. It generalizes the construction of the basis from [Book, §11.3] for $n = 1$ to $n \in \mathbb{N}_0$.

Exer. 7. For convenience, let $a = 0$. With

$$S_{(n)}(x) := \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x - kh)^n_+,$$

see Figure 1, we define $S_{(n,\ell)}(x) := S_{(n)}(x - \ell h)$ for $\ell \in \mathbb{Z}$.

![Figure 1. The functions $S_{(n)}$ for $n = 0, \ldots, 4$.](image)

(a) Show that $S_{(n,\ell)}|_{[0,b]} \in S_n^{(n-1)}$.
(b) Show that supp$S_{(n,\ell)} \subseteq [\ell h, (\ell + n + 1)h]$.
(c) Show that $\dim S_n^{(n-1)} = m + n = \# \{\ell \in \mathbb{Z} : S_{(n,\ell)}|_{[0,b]} \neq 0\}$.
(d) Show that $S_{(n+1,\ell)}(x) = (n+1)(S_{(n,\ell)}(x) - S_{(n,\ell+1)}(x))$ (when $n = 0$ only for $x \notin h\mathbb{Z}$).
(e) From [Book, Exer. 11.6], we know that

$$S_{(n+1,\ell)}(x) = (x - \ell h)S_{(n,\ell)}(x) + ((n+2+\ell)h - x)S_{(n,\ell+1)}(x).$$

Using induction to $n$, from this show that

$$\sum_{\ell \in \mathbb{Z}} S_{(n,\ell)}(x) = h^n n!.$$

Now we are going to show that for all $p \in \mathbb{Z}$,

$$\sum_{\ell \in \mathbb{Z}} c_\ell S_{(n,\ell)}|_{[ph,(p+1)h]} = 0 \implies c_\ell = 0 \text{ for } p - n - 1 < \ell < p + 1. \quad (2.10)$$

(f) Show that (2.10) holds for $n = 0$. 

(g) Now let (2.10) be valid for some \( n \in \mathbb{N}_0 \). Let \( \sum_{\ell \in \mathbb{Z}} c_{\ell} S_{(n+1,\ell)} \) and so \( \sum_{\ell \in \mathbb{Z}} c_{\ell} S'_{(n+1,\ell)} \) vanish on \((ph,(p+1)h)\). Using (d), show that this implies that for some constant \( c \in \mathbb{R} \), \( c_{\ell} = c \) for all \( p - n - 2 < \ell < p + 1 \), and with that, that
\[
\sum_{\ell \in \mathbb{Z}} c_{\ell} S_{(n+1,\ell)}|_{(ph,(p+1)h)} = c \sum_{\ell \in \mathbb{Z}} S_{(n+1,\ell)}|_{(ph,(p+1)h)} = ch^{n+1}(n + 1)!
\]
Conclude that (2.10) is valid for \( n + 1 \), and so for any \( n \in \mathbb{N}_0 \).

(h) Using (a), (c), and (2.10), show that
\[
\{ S_{(n,\ell)}|_{[0,b]} : \ell \in \{-n, \ldots, m - 1\} \}
\]
is a basis for \( S_{n}^{(n-1)} \).
3. Additions to Chapter 12 book

From the book we skip
- the proof of Picard’s Theorem, Thm 12.1.
- §12.3 with the exception of Definition 12.2
- §12.4

The reason to skip §12.4 is that an implicit (1-step) ODE solver cannot be written as $y_{n+1} = y_n + h\Phi(x_n, y_n; h)$ since $\Phi$ doesn’t has $y_{n+1}$ as one of its arguments (in the book they try to solve this by defining $\Phi$ in an implicit way, but this doesn’t lead to an analysis that is correct). Therefore, we replace §12.4 by the analysis of the trapezium rule given below in §3.1.

3.1. Trapeziunrule. Writing

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x)) \, dx,$$

and approximating the integral by the trapezium rule leads to the following implicit one-step method

$$(3.1) \quad y_{n+1} = y_n + \frac{1}{2} h[f(x_n, y_n) + f(x_{n+1}, y_{n+1})].$$

For each $n = 0, \ldots, N - 1$, $y_{n+1}$ is given implicitly as the solution of an equation. The first question is whether this equation has a solution:

3.1.1. Existence.

**Lemma 3.1** (Banach’s fixed point theorem). Let $(X, d)$ be a non-empty complete metric space, and let $F : X \to X$ be a contraction, meaning that for some $K < 1$,

$$d(F(x), F(y)) \leq K d(x, y) \quad (x, y \in X).$$

Then $\exists x \in X$ with $F(x) = x$.

**Proof.** Select $x_0 \in X$ arbitrarily, and define $(x_n)_{n \in \mathbb{N}} \subset X$ by $x_{n+1} = F(x_n)$. Then $d(x_{n+1}, x_n) \leq K d(x_n, x_{n-1}) \leq \cdots \leq K^n d(x_1, x_0)$, and so for $m \geq n$,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leq (K^{m-1} + \cdots + K^n) d(x_1, x_0) \leq \frac{K^n}{1 - K} d(x_1, x_0).$$

So $(x_n)_n$ is a Cauchy sequence, and because $X$ is complete, therefore convergent, say with limit $x$. A consequence of $F$ being a contraction is that $F$ is continuous. So $x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} F(x_n) = F(\lim_{n \to \infty} x_n) = F(x)$.

Now let $z$ be another fixed point of $F$. Then $d(x, z) = d(F(x), F(z)) \leq K d(x, z)$, and so $d(x, z) = 0$, or $x = z$. \qed

Returning to the trapeziunrule, let $f$ be continuous on

$$D := [x_0, x_M] \times [y_0 - C, y_0 + C],$$

and Lipschitz continuous w.r.t. its second variable, i.e., for some constant $L > 0$,

$$|f(x, u) - f(x, v)| \leq L |u - v| \quad ((x, u), (x, v) \in D).$$

Set $Q := \max_{(x, u) \in D} |f(x, u)|$. Fixing $n$, let

$$y_n \in [y_0 - C + hQ, y_0 + C - hQ],$$

and so
which interval is non-empty when \( h \leq \frac{C}{Q} \). Then \( F \) defined by \( F(y) := y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y)] \) maps \([y_0 - C, y_0 + C]\) into \([y_0 - C, y_0 + C]\). Furthermore it holds that for \( y, z \in [y_0 - C, y_0 + C] \),

\[
|F(y) - F(z)| = \frac{1}{2}h|f(x_{n+1}, y) - f(x_{n+1}, z)| \leq \frac{1}{2}hL|y - z|,
\]

Applying the fixed point theorem, with \( M := [y_0 - C, y_0 + C] \), \( d(x, y) := |x - y| \), we conclude that \((3.1)\) has a unique solution whenever \( h \) is sufficiently small such that additionally \( hL < 2 \).

3.1.2. **Approximation of** \( y_{n+1} \). Generally the solution \( y_{n+1} \) of \((3.1)\) cannot be determined exactly. An obvious way to approximate it is to apply a number of iterations \( y^{(i+1)} = F(y^{(i)}) \) starting with say \( y^{(0)} = y_n \), or even better, \( y^{(0)} = y_n + h f(x_n, y_n) \). It holds that

\[
(3.2) \quad |y_{n+1} - y^{(i+1)}| = |F(y_{n+1}) - F(y^{(i)})| \leq \frac{1}{2}hL|y_{n+1} - y^{(i)}|.
\]

Assuming that \( f \) is \( 2 \times \) continuous differentiable as function of its second variable a much faster converging iteration is given by the Newton iteration, described in Book §1.4 starting from Definition 1.6, applied to the equation \( G(y) := y - y_n - \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y)] = 0 \). If \( G''(y_{n+1}) = 1 - \frac{h}{2} \frac{df}{dx}(x_{n+1}, y) \neq 0 \), which is the case when \( h \) is sufficiently small, then for \( y^{(0)} \) being sufficiently close to the solution \( y_{n+1} \), the Newton iteration converges, and

\[
(3.3) \quad \lim_{i \to \infty} \frac{y_{n+1} - y^{(i+1)}}{(y_{n+1} - y^{(i)})^2} = - \frac{G''(y_{n+1})}{2G'(y_{n+1})},
\]

i.e., quadratic convergence, being way more favourable than the ‘\( R \)–linear’ convergence \((3.2)\).

3.1.3. **Truncation error.** Knowing that the recursion \((3.1)\) is well-defined, and that we can approximate its solution at any desired accuracy, we study the accuracy of the approximations for \( y(x_n) \) that are produced by it. First we define a truncation error:

\[
T_n := \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2} (f(x_{n+1}, y(x_{n+1})) + f(x_n, y(x_n))) = \frac{y(x_{n+1}) - y(x_n)}{h} - \frac{1}{2} (y'(x_{n+1}) + y'(x_n)).
\]

To determine the order of the trapezium rule, the usual (and recommended) way is to Taylor all terms in the right-hand side around some suitable point:

\[
\begin{align*}
    y(x_{n+1}) &= y(x_n + \frac{1}{2}h) + \frac{1}{2}h y'(x_n + \frac{1}{2}h) + \frac{1}{2} (\frac{1}{2}h)^2 y''(x_n + \frac{1}{2}h) + \frac{1}{6} (\frac{1}{2}h)^3 y'''(\xi_1) \\
    y(x_n) &= y(x_n + \frac{1}{2}h) - \frac{1}{2}h y'(x_n + \frac{1}{2}h) + \frac{1}{2} (\frac{1}{2}h)^2 y''(x_n + \frac{1}{2}h) - \frac{1}{6} (\frac{1}{2}h)^3 y'''(\xi_2)
\end{align*}
\]

yielding

\[
\frac{y(x_{n+1}) - y(x_n)}{h} = y'(x_n + \frac{1}{2}h) + \frac{1}{24} h^2 y'''(\xi_1) + y'''(\xi_2) + \frac{1}{24} h^2 y'''(\xi).
\]
for some $\xi \in [\xi_1, \xi_2] \subset [x_n, x_{n+1}]$ using the intermediate value theorem. From
\begin{align*}
 y'(x_{n+1}) &= y'(x_n + \frac{1}{2}h) + \frac{1}{2} h y''(x_n + \frac{1}{2}h) + \frac{1}{2} (\frac{1}{2}h)^2 y'''(\eta_1) \\
 y'(x_n) &= y'(x_n + \frac{1}{2}h) - \frac{1}{2} h y''(x_n + \frac{1}{2}h) + \frac{1}{2} (\frac{1}{2}h)^2 y'''(\eta_2)
\end{align*}
we have
\begin{equation}
\frac{1}{2}(y'(x_{n+1}) + y'(x_n)) = y'(x_n + \frac{1}{2}h) + \frac{1}{8} h^2 y'''(\eta)
\end{equation}
for some $\eta \in [\eta_1, \eta_2] \subset [x_n, x_{n+1}]$ again using the intermediate value theorem. This yields
\begin{equation}
T_n = \frac{1}{12} h^2 y'''(\xi) - \frac{1}{8} h^2 y'''(\eta) = -\frac{1}{12} h^2 y'''(\xi) + o(h^2),
\end{equation}
where we have assumed that $y \in C^2$. Consequently, the trapeziumrule has order 2.

An analysis that avoids the $o(h^2)$-term exploits the following trick:

\[
\int_{x_n}^{x_{n+1}} (x - x_{n+1})(x - x_n)y'''(x) \, dx = (x - x_{n+1})(x - x_n)y'''(x)|_{x_n}^{x_{n+1}} + \int_{x_n}^{x_{n+1}} (x_{n+1} + x_n - 2x)y'''(x) \, dx = (x_{n+1} + x_n - 2x)y'(x)|_{x_n}^{x_{n+1}} + \int_{x_n}^{x_{n+1}} 2y'(x) \, dx = 2hT_n.
\]

On the other hand, thanks to $(x - x_{n+1})(x - x_n) \leq 0$ on $[x_n, x_{n+1}]$, it holds that
\[
\int_{x_n}^{x_{n+1}} (x - x_{n+1})(x - x_n)y'''(x) \, dx = y'''(\xi) \int_{x_n}^{x_{n+1}} (x - x_{n+1})(x - x_n) \, dx = -\frac{1}{6} h^3 y'''(\xi)
\]
for some $\xi \in [x_n, x_{n+1}]$, showing that
\[
T_n = -\frac{1}{12} h^2 y'''(\xi).
\]

3.1.4. Global error. Subtracting
\[
y_{n+1} = y_n + \frac{1}{2} h(f(x_{n+1}, y_{n+1}) + f(x_n, y_n))
\]
from
\[
y(x_{n+1}) = y(x_n) + \frac{1}{2} h(f(x_{n+1}, y(x_{n+1})) + f(x_n, y(x_n))) + hT_n
\]
yields the following recursion for the global error $e_n = y(x_n) - y_n$:
\[
e_{n+1} = e_n + \frac{1}{2} h(f(x_{n+1}, y(x_{n+1})) - f(x_{n+1}, y_{n+1}) + f(x_n, y(x_n)) - f(x_n, y_n)) + hT_n
\]
and so using the Lipschitz continuity of $f$ w.r.t. the second variable with constant $L$,
\[
|e_{n+1}| \leq |e_n| + \frac{1}{2} h L (|e_{n+1}| + |e_n|) + h |T_n|
\]
and so for $hL < 2$,\[
(1 - \frac{1}{2} h L)|e_{n+1}| \leq (1 + \frac{1}{2} h L)|e_n| + h |T_n|
\]
giving
\[
|e_{n+1}| \leq \frac{1 + \frac{1}{2} h L}{1 - \frac{1}{2} h L} |e_n| + \frac{h |T_n|}{1 - \frac{1}{2} h L}.
\]
Using that \( e_0 = 0 \), with \( M := \sup_{x \in [x_0, X_M]} |y''(x)| \) one infers that

\[
|e_{n+1}| \leq \sum_{j=0}^{n} \left( 1 + \frac{3}{2}hL \right)^{n-j} \frac{h |T_j|}{1 - \frac{1}{2}hL} \leq \frac{\frac{1}{12} h^3 M}{1 - \frac{1}{2}hL} \sum_{j=0}^{n} \left( 1 + \frac{\frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^j
\]

\[
= \frac{\frac{1}{12} h^3 M}{1 - \frac{1}{2}hL} \frac{1 - \left( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^{n+1}}{1 - \left( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)} = h^2 M \left( \frac{(1 + \frac{1}{2}hL)^{n+1}}{1 - \frac{1}{2}hL} - 1 \right).
\]

Now from \( \frac{1}{1 - \frac{1}{2}hL} = 1 + \frac{1}{2}hL + O(h^2) \), thus \( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} = 1 + hL + O(h^2) \leq e^{hL + O(h^2)} \), and so \( \left( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^n \leq e^{(hL + O(h^2))n} = e^{nhL + O(h)} = e^{nhL}e^{O(h)} = e^{nhL}(1 + O(h)) \) we infer that \( \sup_{nh \leq X_m - x_0} \left( \frac{1 + \frac{1}{2}hL}{1 - \frac{1}{2}hL} \right)^n < \infty \). We conclude the following result:

**Theorem 3.2** (trapeziumrule). Let \( f \) be continuous on \( D := [x_0, X_M] \times [y_0 - C, y_0 + C] \), and Lipschitz continuous w.r.t. its second variable with constant \( L > 0 \), i.e.,

\[
|f(x, u) - f(x, v)| \leq L|u - v| \quad ((x, u), (x, v) \in D).
\]

Let the exact solution \( y \in C^3([x_0, X_M]) \). Then \( e_n = O(h^2) \) uniform in \( n \) and \( h \) with \( nh \leq X_M - x_0 \), and \( hL < 2 \).

### 3.2. Homogeneous recursions.

Together with the next two subsections §3.3 and §3.4, this subsection replaces [Book, §12.7-9]. In particular we provide a proof for Dahlquist’s Equivalence Theorem ([Book, Thm. 12.5]).

As a preparation for the next subsection, we study solutions of homogeneous recursions of the form

\[
(3.5) \quad \alpha_k v_{n+k} + \ldots + \alpha_0 v_n = 0 \quad n = 0, 1, \ldots
\]

where \( \alpha_k \neq 0, \alpha_0 \neq 0 \). In particular, we will be interested in finding conditions on the coefficients \( \alpha_i \) so that \( |v_n| \) remains bounded when \( n \to \infty \).

**Theorem 3.3.** Let \( z_1, \ldots, z_\ell \neq 0 \) be the roots of the characteristic polynomial

\[
\rho(z) := \alpha_k z^k + \ldots + \alpha_0,
\]

where \( z_r \) has multiplicity \( m_r \), so that \( m_1 + \ldots + m_\ell = k \). Then \( (v_n)_{n \geq 0} \) is a linear combination of

\[
(3.6) \quad \left\{ \left( \frac{d^r}{dz^r} z^n \right)|_{z=z_r} \right\}_{n \geq 0} : 1 \leq r \leq \ell, 0 \leq q \leq m_r - 1 \}
\]

**Proof.** Because each solution of (3.5) is uniquely determined by \( v_0, \ldots, v_{k-1} \), these solutions span a linear space of dimension \( \leq k \). It remains to show that the \( k \) sequences given in (3.6) are solutions, and that they are linearly independent.
For each root \( z_r \), and \( 0 \leq q \leq m_r - 1 \), it holds that
\[
\alpha_k \left( \frac{d^n}{dz^n} z^{n+k} \right) \big|_{z=z_r} + \cdots + \alpha_0 \left( \frac{d^n}{dz^n} z^n \right) \big|_{z=z_r} = \left( \frac{d^q}{dz^q} \rho(z) \right) \big|_{z=z_r} = 0,
\]
so that indeed \( \left( \frac{d^q}{dz^q} z^n \right) \big|_{z=z_r} \) satisfies the recursion.

For \( z_1, \ldots, z_k \) being simple roots, using induction one can show that
\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\gamma_1 & \gamma_2 & \cdots & \gamma_k \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_1 & \gamma_2 & \cdots & \gamma_k \\
\end{vmatrix}
= \prod_{1 \leq r < s \leq k} (z_s - z_r) \neq 0.
\]
This means that \( \{ (z^n_1)_{0 \leq n \leq k-1}, \ldots, (z^n_k)_{0 \leq n \leq k-1} \} \) are linearly independent, and so are the infinite sequences.

For the general case, one can show that the determinant of the \( k \times k \) matrix with columns given by the first \( k \) elements of the sequences from (3.6) is equal to
\[
\prod_{1 \leq r < s \leq l} (z_s - z_r)^{m_r} \neq 0.
\]
so that again the infinite sequences are independent. \( \square \)

Theorem 3.3 shows that the solution \( (v_n)_{n \geq 0} \) is a (unique) linear combination of the \( k \) special solutions given in (3.6). The coefficients in this linear combination can be found by solving a \( k \times k \) linear system obtained by equating the linear combination of the \( k \) special solutions, restricted to their first \( k \) entries, to \( (v_0, \ldots, v_{k-1}) \). In other words, denoting the \( k \) special solutions as \( (z_n^{(i)})_{n \geq 0} \) for \( 1 \leq i \leq k \) (where thus \( z_n^{(i)} = z^n \) in case all roots are simple), it holds that \( (v_n)_{n \geq 0} = \sum_{i=1}^{k} \gamma_i (z_n^{(i)})_{n \geq 0} \), where
\[
\begin{bmatrix}
z^{(1)}_0 & \cdots & z^{(k)}_0 \\
\vdots & \ddots & \vdots \\
z^{(1)}_{k-1} & \cdots & z^{(k)}_{k-1}
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\
\vdots \\
\gamma_k
\end{bmatrix}
= \begin{bmatrix} v_0 \\
v_1 \\
\vdots \\
v_{k-1} \end{bmatrix}.
\]

**Corollary 3.4.** If, and only if, all roots \( z \) of the characteristic polynomial \( \rho \) satisfy
\[
|z| \leq 1, \text{ with any of them with modulus } 1 \text{ being simple}
\]
(root condition), then
\[
\sup_{0 \neq (v_0, \ldots, v_{k-1}) \in \mathbb{R}^k} \frac{\sup_{n \geq k} |v_n|}{\max\{|v_0|, \ldots, |v_{k-1}|\}} < \infty.
\]

**Proof.** From \((v_n)_{n \geq 0} = \sum_{i=1}^{k} \gamma_i (z_n^{(i)})_{n \geq 0}\) the sufficiency follows from \(\max_{1 \leq n \leq k} |\gamma_n| \leq ||B^{-1}||_{\infty} \max_{0 \leq n \leq k-1} |v_n|\), and the necessity follows by letting \((v_n)_{0 \leq n \leq k-1}\) run over the special solutions \((z_n^{(i)})_{0 \leq n \leq k-1}\). \( \square \)
3.3. Multi-step methods. We consider
\[
\begin{align*}
  y'(x) &= f(x, y(x)) \quad x \in [x_0, X_M], \\
  y(x_0) &= y_0,
\end{align*}
\]
and assume that the conditions of the Picard theorem are fulfilled on \( D = [x_0, X_M] \times [y_0 - C, y_0 + C] \).

For a given stepsize \( h \), we set \( x_n := x_0 + nh, \ n = 0, \ldots, N := \frac{X_M - x_0}{h} \).

Given starting values \( y_0, \ldots, y_{k-1} \subset [y_0 - C, y_0 + C] \), we consider the multi-step
\[
(3.8) \quad \sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}) \quad n = 0, 1, \ldots, N - k,
\]
where \( \alpha_k \neq 0, \ \alpha_0^2 + \beta_0^2 \neq 0. \)

Example 3.5. A way to arrive at such an approximation scheme is to apply a \((k+1)\)-point Newton-Cotes formula (e.g. the Simpson rule where \( k = 2 \)) to the right-hand side of
\[
y(x_{n+k}) - y(x_n) = \int_{x_n}^{x_{n+k}} y'(s) \, ds = \int_{x_n}^{x_{n+k}} f(s, y(s)) \, ds.
\]

In case \( k > 1 \), a common way to provide suitable \( y_1, \ldots, y_{k-1} \) is to start with \((k-1)\)-steps of a Runge-Kutta method of sufficiently high order (cf. the forthcoming Thm. 3.9).

Possibly after rescaling, for notational convenience in the following we assume that
\[
\alpha_k = 1,
\]
and define the \textit{first} and \textit{second characteristic polynomial} of the multi-step by
\[
\rho(z) := \sum_{j=0}^{k} \alpha_j z^j \quad \text{(cf. Thm. 3.3)}, \quad \sigma(z) := \sum_{j=0}^{k} \beta_j z^j.
\]

For the moment (cf. however Sect. 3.4), we will assume existence and uniqueness of the solution \((y_n)_{0 \leq n \leq N}\), which isn’t obvious for the implicit case \( \beta_k \neq 0 \), and furthermore that \((y_n)_{0 \leq n \leq N} \subset [y_0 - C, y_0 + C] \).

3.3.1. Truncation error. In correspondence with earlier definitions for one-step methods, we define the truncation error \( T_n \) by
\[
(3.9) \quad T_n := h^{-1} \sum_{j=0}^{k} \alpha_j y_{n+j} - \sum_{j=0}^{k} \beta_j f(x_{n+j}, y(x_{n+j})) / \sigma(1).
\]

The division by \( \sigma(1) \) is made to make the definition independent of arbitrary scalings of \( (3.8) \) as the one we applied to make \( \alpha_k = 1 \). The current definition of \( T_n \) coincides with ones given earlier for the Forward Euler and trapezoidal rule. Later, cf. footnote 1, we will see that for a valid multi-step \( \sigma(1) \neq 0 \).

Definition 3.6. The multi-step is said to have \textit{order} \( p \) when, for sufficiently smooth solutions \( x \mapsto y(x) \) on \([x_0, X_M]\), for \( n = 0, \ldots, N - k \), it holds that \( |T_n| = O(h^p) \). (When \( p > 0 \), the method is called \textit{consistent}).
The determination of the order of a multi-step can follow the same lines as in Sect. 3.1.3 for the trapezium rule. That is, using that
\[ T_n = \frac{h^{-1} \sum_{j=0}^{k} \alpha_j y(x_{n+j}) - \sum_{j=0}^{k} \beta_j y'(x_{n+j})}{\sigma(1)}, \]
the order can be determined by replacing each of the terms in the numerator by a Taylor expansion of sufficiently high order around some arbitrary point \( \bar{x} \in [x_n, x_{n+k}] \). When doing so, the leading \( h^{-1} \)– and \( h^0 \)–terms disappear iff
\[ \rho(1) = 0 \text{ and } h^{-1} \sum_{j=0}^{k} \alpha_j (x_{n+j} - \bar{x}) = \sigma(1), \]
respectively. Writing
\[ h^{-1} \sum_{j=0}^{k} \alpha_j (x_{n+j} - \bar{x}) = \sum_{j=0}^{k} \alpha_j \beta_j + h^{-1} (x_n - \bar{x}) \sum_{j=0}^{k} \alpha_j = \rho'(1) - h^{-1} (x_n - \bar{x}) \rho(1), \]
we conclude that the method is consistent if and only if \(^1\)
\[ \rho(1) = 0 \text{ and } \rho'(1) = \sigma(1). \]

Example 3.7. Some examples of multi-step methods, where \( f(x_n, y_n) \) is abbreviated by \( f_n \), are the Euler, implicit Euler, trapezium rule, Adams-Bashforth, Adams-Moulton methods, given by
\[
\begin{align*}
  y_{n+1} &= y_n + hf_n, & \rho(z) &= z - 1, & \sigma(z) &= 1, \\
  y_{n+1} &= y_n + hf_{n+1}, & \rho(z) &= z - 1, & \sigma(z) &= z, \\
  y_{n+1} &= y_n + \frac{1}{2} hf_{n+1} + f_n, & \rho(z) &= z - 1, & \sigma(z) &= \frac{1}{2} (z + 1), \\
  y_{n+4} &= y_{n+3} + \frac{1}{24} h(55f_{n+3} - 59f_{n+2} + 37f_{n+1} - 9f_n), & \rho(z) &= z^4 - z^3, & \sigma(z) &= z, \\
  y_{n+3} &= y_{n+2} + \frac{1}{24} h(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n), & \rho(z) &= z^3 - 2z, & \sigma(z) &= \frac{9z^3 + 19z^2 - 5z + 1}{24},
\end{align*}
\]
respectively.

3.3.2. Global error. Subtracting (3.8) from (3.9), the latter written as \( \sum_{j=0}^{k} \alpha_j y(x_{n+j}) = h \sum_{j=0}^{k} \beta_j f(x_{n+j}, y(x_{n+j})) + h\sigma(1)T_n \), yields for the global error
\[ e_n := y(x_n) - y_n \]
the recursion
\[ \sum_{j=0}^{k} \alpha_j e_{n+j} = h \sum_{j=0}^{k} \beta_j (f(x_{n+j}, y(x_{n+j})) - f(x_{n+j}, y_{n+j})) + h\sigma(1)T_n \quad n = 0, 1, \ldots, N - k. \]
The Lipschitz continuity of \( f \) on \( D \) w.r.t. the second variable means that
\[ |f(x_{n+j}, y(x_{n+j})) - f(x_{n+j}, y_{n+j})| \leq L |e_{n+j}|, \]
or, in other words, that \( f(x_{n+j}, y(x_{n+j})) - f(x_{n+j}, y_{n+j}) = \ell_{n+j} e_{n+j} \) for some \( |\ell_{n+j}| \leq L \). By substituting this we obtain the recursion
\[ \sum_{j=0}^{k} \alpha_j e_{n+j} = h \sum_{j=0}^{k} \beta_j \ell_{n+j} e_{n+j} + h\sigma(1)T_n \quad n = 0, 1, \ldots, N - k. \]

\(^1\) Later we will see that a multi-step can only be convergent when \( \rho \) has no multiple roots with modulus 1. In addition to consistency it means that \( \rho'(1) \neq 0 \), and so indeed \( \sigma(1) \neq 0 \).
We write this scalar $k$-step recursion as an 1-step recursion for $k$-vectors. With the $k$-vectors and the $k \times k$-matrices

$$
\tilde{z}_n := \begin{bmatrix} e_n \\ e_{n+1} \\ \vdots \\ e_{n+k-1} \end{bmatrix}, \quad \tilde{r}_n := \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_n := \text{diag} [0 \cdots 0 \beta_k \ell_{n+k}],
$$

$$
D_n := \begin{bmatrix} 0 \\ \vdots \\ \beta_0 \ell_n & \cdots & \beta_{k-1} \ell_{n+k-1} \\ 0 \\ 1 \end{bmatrix}, \quad A := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
$$

(the latter known as the companion matrix), (3.10) reads as

$$
(I - hE_n) \tilde{z}_{n+1} = (A + hD_n) \tilde{z}_n + h \tilde{r}_n \quad n = 0, 1, \ldots, N - k.
$$

where we have used that $\alpha_k = 1$.

We consider stepsizes with

$$
hL|\beta_k| \leq \frac{1}{2}.
$$

Then $\|hE_n\| \leq \frac{1}{2}$, showing that $I - hE_n$ is invertible, $(I - hE_n)^{-1} = \sum_{i=0}^{\infty} (hE_n)^i$, and $(I - hE_n)^{-1} \leq (1 - \|hE_n\|)^{-1} \leq 2$, so that $\|I - (I - hE_n)^{-1}\| \leq \frac{\|hE_n\|}{1 - \|hE_n\|} = O(h)$. Writing

$$
\tilde{z}_{n+1} = (I - hE_n)^{-1}(A + hD_n) \tilde{z}_n + h (I - hE_n)^{-1} \tilde{r}_n,
$$

we infer

$$
\tilde{z}_{n+1} = A_n \cdots A_0 \tilde{z}_0 + h \sum_{j=0}^n A_n \cdots A_{j+1} \tilde{\sigma}_j.
$$

We have $\|\tilde{\sigma}_j\| \leq 2|\sigma(1)||T_j|$ and from $(n+1)h \leq X_M - x_0$, we conclude that

$$
\|\tilde{z}_{n+1}\| \leq \left[ \|\tilde{z}_0\| + (X_M - x_0)2|\sigma(1)| \right] \max_{0 \leq j \leq n} |T_j| \max_{0 \leq j \leq n} \|A_n \cdots A_j\|.
$$

Writing $A_i$ as $A + F_i$, $F_i = hD_i + ((I - hE_i)^{-1} - I)(A + hD_i)$ so that $\|F_i\| \leq \gamma h$ for some constant $\gamma$. We expand $(A + F_n) \cdots (A + F_j)$ into a sum of $(n-j+1)$-fold products of matrices $A$ and $F_i$. For each $0 \leq \ell \leq n-j + 1$ there are \( \binom{n-j+1}{\ell} \) terms in this sum with $\ell$ factors $F_i$, and so each of these terms contains at most $\ell + 1$ factors of the form $A^p$ for some $1 \leq p \leq n-j + 1$. We conclude that, with

$$
R_N := \max_{1 \leq p \leq N-1} \|A^p\|,
$$

for $0 \leq j \leq n \leq N - k$ it holds that

$$
\|(A + F_n) \cdots (A + F_j)\| \leq \sum_{\ell=0}^{n-j+1} \binom{n-j+1}{\ell} (h\gamma)^\ell R_N^{\ell+1} = R_N (1 + h\gamma R_N)^{n-j+1} \leq R_N (1 + h\gamma R_N)^N \leq R_N e^{h\gamma R_N N} \leq R_N e^{(X_M - x_0)\gamma R_N}.
$$

It remains to find conditions under which $R_N$ is bounded uniformly in $N \to \infty$, i.e., in $h \downarrow 0$. Noting that $\|A^p\| = \sup_{\|\tilde{v}_0\|_2 \leq 1} \|A^p \tilde{v}_0\|_2$, given $\tilde{v}_0 = \begin{bmatrix} v_0 & \cdots & v_{k-1} \end{bmatrix}^T$, with
let \( \vec{v}_p := A^p \vec{v}_0 \). The equivalence of (3.10) and (3.11) in the case \( \beta_0 = \ldots = \beta_k = 0 \), and \( T_n = 0 \) (\( \forall n \)) shows that \( \vec{v}_p = [v_p \cdots v_{p+k-1}]^\top \) where \((v_n)_{n \geq k}\) is defined by
\[
(3.14) \quad \alpha_k v_{n+k} + \ldots + \alpha_0 v_n = 0 \quad n = 0, 1, \ldots
\]
From Corollary 3.4 we conclude that if, and only if all roots \( z \) of \( \rho(z) = \sum_{j=0}^k \alpha_j z^j \) satisfy the root criterion (3.7), then \( R_N \) from (3.13) is bounded uniformly in \( N \), so that the following theorem is valid.

**Remark 3.8.** Despite being excluded in §3.2, this conclusion also holds true when for some \( \ell \geq 1 \), \( \alpha_0 = \ldots = \alpha_{\ell-1} = 0 \), as with the Adams-Bashforth and Adams-Moulton methods. In that case \( \rho \) has a root \( z = 0 \) with multiplicity \( \ell \), and instead of being a \( k \)-step-recursion (3.14) is actually a \((k-\ell)\)-step recursion.

**Theorem 3.9.** Let \((y_n)_{0 \leq n \leq N} \subset [y_0 - C, y_0 + C]\) be a solution of the multi-step (3.8), and let (3.12) be valid. Then if, and generally only if the first characteristic polynomial \( \rho \) satisfies the root condition (3.7), then there exists a constant \( R > 0 \), such that
\[
|e_n| \leq R \{ \max\{|e_0|, \ldots, |e_{k-1}|\} + (X_M - x_0)|\sigma(1)| \max_{0 \leq j \leq n-k} |T_j| \},
\]
\((n = k, \ldots, N := \frac{X_M - x_0}{h})\), with the truncation errors \( T_j \) as defined in (3.9). In particular, if in that case the method is of order \( p \), i.e. \( |T_j| = \mathcal{O}(h^p) \), and \( \max\{|e_0|, \ldots, |e_{k-1}|\} = \mathcal{O}(h^p) \), then \( \max_{0 \leq n \leq N} |e_n| = \mathcal{O}(h^p) \).

The statement of this theorem is often abbreviated by saying that a multi-step is convergent if and only if it is consistent and stable.

To show that the root condition in Theorem 3.9 is generally necessary, consider a consistent multi-step applied to the simple initial value problem
\[
\begin{align*}
y'(x) &= 0, \\
y(x_0) &= y_0,
\end{align*}
\]
which has solution \( y(x) \equiv y_0 \). Given \( y_1, \ldots, y_{k-1} \), the solution of the multi-step is the solution of the recursion \( \sum_{j=0}^k \alpha_j y_{n+j} = 0 \) for \( n = 0, \ldots, N - k \). From \( T_n = 0 \), and \( \sum_{j=0}^k \alpha_j y_{n+j} = y_0 \sum_{j=0}^k \alpha_j = y_0 \rho(1) = 0 \) we have \( \sum_{j=0}^k \alpha_j e_{n+j} = 0 \) for \( n = 0, \ldots, N - k \), and Theorem 3.3 shows that the root condition is generally needed to guarantee boundedness of \((e_n)_{n \geq 0}\).

**3.4. Existence and uniqueness of discrete solutions in the implicit case.**
What is left to discuss is existence and uniqueness of a solution \((y_n)_{0 \leq n \leq N} \subset [y_0 - C, y_0 + C]\) of (3.8). Possibly by decreasing \( X_M \), we may assume that for some \( \varepsilon \in (0, C) \), \( |y(x) - y_0| \leq C - \varepsilon \) for \( x \in [x_0, X_M] \).

We proceed by induction. For some \( n \geq 0 \), let \( y_0, \ldots, y_{n+k} \) exist uniquely in \([y_0 - C, y_0 + C]\). If the root condition is fulfilled, and for some \( p > 0 \), \( \max\{|e_0|, \ldots, |e_{k-1}|\} = \mathcal{O}(h^p) \), and \( |T_j| = \mathcal{O}(h^p) \), then Thm. 3.9 shows that for some constant \( Q \), independent of \( n \), it holds that \( |e_j| \leq Q h^p \), \( 1 \leq j \leq n + k \).

By subtracting (3.8) from (3.9) (the latter multiplied by \( \sigma(1)h \)), and by using that for \( j \leq k - 1 \), \( f(x_{n+1+j}, y(x_{n+1+j})) - f(x_{n+1+j}, y_{n+j+1}) = \ell_{n+1+j} e_{n+1+j} \) all as before, existence and uniqueness of \( y_{n+k+1} \in [y(x_{n+k+1}) - \varepsilon, y(x_{n+k+1}) + \varepsilon] \) is

\[2\text{In the book, a multi-step whose first characteristic polynomial } \rho \text{ satisfies the root condition is called zero-stable.}\]
equivalent to existence and uniqueness of a solution \( e_{n+k+1} \in [-\varepsilon, \varepsilon] \) of

\[
e_{n+k+1} = h\beta_k(f(x_{n+k+1}, y(x_{n+k+1})) - f(x_{n+k+1}, y(x_{n+k+1} - e_{n+k+1})) + \sum_{j=0}^{k-1} h\beta_j e_{n+1+j} + h\sigma(1)T_{n+1} - \sum_{j=0}^{k-1} \alpha_j e_{n+1+j}.
\]

By the induction hypothesis, it holds that \( \Phi(0) = \sum_{j=0}^{k-1} (h\beta_j \ell_{n+1+j} - \alpha_j) e_{n+1+j} + h\sigma(1)T_{n+1} = O(h^p) \), so for \( h \) sufficiently small we have \( |\Phi(0)| \leq \varepsilon/2 \). Furthermore, for \( \xi, \nu \in [-\varepsilon, \varepsilon] \), it holds that \( |\Phi(\xi) - \Phi(\nu)| \leq h|\beta_k|L|\xi - \nu| \leq \frac{1}{2}|\xi - \nu| \). From \( |\Phi(\xi)| \leq |\Phi(0)| + |\Phi(0)| \leq \frac{1}{2}|\xi| + \frac{\xi}{2} \), it follows that \( \Phi : [-\varepsilon, \varepsilon] \to [-\varepsilon, \varepsilon] \). An application of Banach’s fixed point theorem (Lemma 3.1) shows that there exists an unique \( e_{n+k+1} \in [-\varepsilon, \varepsilon] \) with \( e_{n+k+1} = \Phi(e_{n+k+1}) \), and thus that there exists a unique \( y_{n+1+k} \in [y(x_{n+k+1}) - \varepsilon, y(x_{n+k+1}) + \varepsilon] \subset [y_0 - C, y_0 + C] \) that solves (3.8).