1. We consider the quadrature formula $w_0f(x_0)$ to approximate $\int_{a}^{b} f(x)dx$.

(a) Show that the formula is exact on $P_0$ if and only if $w_0 = b - a$.

(b) Show that $x_0 = m := \frac{b+a}{2}$ is the unique choice such that the formula is even exact on $P_1$.

(c) Show that the formula from b) can be obtained by integrating the Hermite interpolation polynomial of $f$ with degree 1 and interpolation point $m$.

(d) Show that for sufficiently smooth $f$,

$$\int_{a}^{b} f(x)dx - (b - a)f(m) = \frac{(b-a)^3}{24}f''(\eta)$$

for some $\eta \in [a, b]$.

2. For sufficiently smooth $f$, we consider quadrature formulas for integrals of type $\int_{0}^{h} \sqrt{x}f(x) dx$.

(a) By considering the expression for the quadrature error, explain that generally one cannot expect very accurate results from the application of the trapezium rule.

(b) To design an alternative quadrature rule, let $\phi_1(x)$ the linear interpolation polynomial of $f$ with interpolation points 0 and $h$. Integrate $\int_{0}^{h} \sqrt{x}\phi_1(x) dx$.

(c) Show that the quadrature rule defined in this way has an error equal to

$$-\frac{2}{35}h^{7/2}f''(\xi_1),$$

for some $\xi_1 \in [0, h]$, and compare with (a).
3. On $[-1,1]$, let $T_n(x) = \cos(n \arccos x)$. It is known that $T_n \in P_n$; for $n \geq 1$, $T_n(x) - 2^{n-1}x^n \in P_{n-1}$; $T_n$ has roots $\cos\left(\frac{j\pi}{n}\right)$, $j = 1, \ldots, n$; $T_n(x) = \pm 1$ alternating in the points $x = \cos\left(\frac{j\pi}{n}\right)$, $j = 0, \ldots, n$.

(a) Show that $2^{-n}T_{n+1}$ is the polynomial of degree $n+1$ with leading coefficient 1 which has the smallest maximum norm on $[-1,1]$.

Let $f$ be sufficiently smooth with positive $(n+1)$th derivative on $[-1,1]$.

(b) For $p_n$ being the Lagrange interpolation polynomial of degree $n$ having as interpolation points the zeros of $T_{n+1}$, prove that

$$\max_{x \in [-1,1]} |f(x) - p_n(x)| \leq \frac{2^{-n}}{(n+1)!} \max_{y \in [-1,1]} f^{(n+1)}(y).$$

(c) Prove that

$$\min_{q_n \in P_n} \max_{x \in [-1,1]} |f(x) - q_n(x)| \geq \frac{2^{-n}}{(n+1)!} \min_{y \in [-1,1]} f^{(n+1)}(y).$$

4. We consider $C[a,b]$ with inner product $\langle f, g \rangle = \int_a^b w(x)f(x)g(x)dx$ where $w$ is a positive, continuous and integrable weight function on $(a,b)$.

(a) With $q_0 := 1$, $q_1 := x - \frac{\langle x, q_0 \rangle}{\langle q_0, q_0 \rangle}$, and

$$q_{n+1} := \left(x - \frac{\langle x, q_n \rangle}{\langle q_n, q_n \rangle}\right)q_n - \frac{\langle xq_n - q_n-1 \rangle}{\langle q_{n-1}, q_{n-1} \rangle} q_{n-1} \quad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

show using induction that $\{q_0, \ldots, q_n\}$ is an orthogonal basis for $P_n$ with $q_n - x^n \in P_{n-1}$.

(b) Show that $q_n$ has $n$ different zeros on $(a,b)$. 

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