1. (a) Given different points $x_0, x_1, x_2 \subset [a, b]$ and scalars $y_0, y_1, y_2, z_1$, show that there exists at most one polynomial $p \in P_3$ with $p(x_i) = y_i, i = 0, 1, 2, p'(x_1) = z_1$.

(b) Construct this $p$ in the form $p(x) = p_2(x) + \alpha(x-x_0)(x-x_1)(x-x_2)$ with $p_2$ being the Lagrange interpolation polynomial of degree 2 corresponding to the set $\{(x_i, y_i) : i = 0, 1, 2\}$.

(c) Let $f$ be four times differentiable. Show that for the polynomial $p \in P_3$ with $p(x_i) = f(x_i), i = 0, 1, 2, p'(x_1) = f'(x_1)$ and any $x \in [a, b]$, there exists a $\xi = \xi(x) \in (a, b)$ with

$$f(x) - p(x) = (x-x_0)(x-x_1)^2(x-x_2)\frac{f^{(4)}(\xi)}{4!}.$$

2. (a) Find weights $w_0, \bar{w}_0, w_1, \bar{w}_1$ such that

$$\int_a^b f(x)dx = w_0f(a) + \bar{w}_0f'(a) + w_1f(b) + \bar{w}_1f'(b)$$

for any $f \in P_3$.

(b) Show that for $f \in C^4$, the error in this quadrature formula, i.e., true integral minus its approximation, is of the form $C(b-a)^5 f^{(4)}(\xi)$ for some $\xi \in [a, b]$, and give the constant $C$.

(c) Splitting the interval into $m$ equal subintervals, give the resulting composite quadrature formula.

(d) Show that the error in this composite formula is equal to $C\frac{(b-a)^5}{m^4} f^{(4)}(\xi)$ for some $\xi \in [a, b]$.

3. For $a < b$, $\{x_0, \ldots, x_n\} \subset \mathbb{R}$, show that there are unique weights $w_0, \ldots, w_n$ such that $\sum_{i=0}^n w_i f(x_i) = \int_a^b f(x)dx$ for all $f \in P_n$. Show that $w_i = \int_a^b \prod_{k=0, k \neq i}^n \frac{x-x_k}{x_i-x_k} dx$. 

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**Additional exercises with Numerieke Analyse**

March 1, 2019

1. (a) Given different points $x_0, x_1, x_2 \subset [a, b]$ and scalars $y_0, y_1, y_2, z_1$, show that there exists at most one polynomial $p \in P_3$ with $p(x_i) = y_i, i = 0, 1, 2, p'(x_1) = z_1$.

(b) Construct this $p$ in the form $p(x) = p_2(x) + \alpha(x-x_0)(x-x_1)(x-x_2)$ with $p_2$ being the Lagrange interpolation polynomial of degree 2 corresponding to the set $\{(x_i, y_i) : i = 0, 1, 2\}$.

(c) Let $f$ be four times differentiable. Show that for the polynomial $p \in P_3$ with $p(x_i) = f(x_i), i = 0, 1, 2, p'(x_1) = f'(x_1)$ and any $x \in [a, b]$, there exists a $\xi = \xi(x) \in (a, b)$ with

$$f(x) - p(x) = (x-x_0)(x-x_1)^2(x-x_2)\frac{f^{(4)}(\xi)}{4!}.$$

2. (a) Find weights $w_0, \bar{w}_0, w_1, \bar{w}_1$ such that

$$\int_a^b f(x)dx = w_0f(a) + \bar{w}_0f'(a) + w_1f(b) + \bar{w}_1f'(b)$$

for any $f \in P_3$.

(b) Show that for $f \in C^4$, the error in this quadrature formula, i.e., true integral minus its approximation, is of the form $C(b-a)^5 f^{(4)}(\xi)$ for some $\xi \in [a, b]$, and give the constant $C$.

(c) Splitting the interval into $m$ equal subintervals, give the resulting composite quadrature formula.

(d) Show that the error in this composite formula is equal to $C\frac{(b-a)^5}{m^4} f^{(4)}(\xi)$ for some $\xi \in [a, b]$.

3. For $a < b$, $\{x_0, \ldots, x_n\} \subset \mathbb{R}$, show that there are unique weights $w_0, \ldots, w_n$ such that $\sum_{i=0}^n w_i f(x_i) = \int_a^b f(x)dx$ for all $f \in P_n$. Show that $w_i = \int_a^b \prod_{k=0, k \neq i}^n \frac{x-x_k}{x_i-x_k} dx$. 

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6. Let $a < b$, $m \in \mathbb{N}$, $h := \frac{b-a}{m}$, $x_i := a + ih$ for $i \in \{0, \ldots, m\}$, and for $n \in \{1, 2, \ldots\}$, let

$$S_n := \{ s \in C^{n-1}(a,b) : s|_{(x_{i-1},x_i)} \in P_n (1 \leq i \leq m) \}.$$ 

Furthermore, let $I_1 : C[a,b] \rightarrow S_1$ the continuous piecewise linear interpolator, and let $I_3 : C^1[a,b] \rightarrow S_3$ the “complete cubic spline interpolator” defined by

$$s(x_i) = f(x_i) \quad (i \in \{0, \ldots, m\}), \quad (1)$$
$$s'(x_0) = f'(x_0), \quad s'(x_m) = f'(x_m). \quad (2)$$

where $s$ is here a shorthand notation for $I_3(f)$. The aim of this exercise is to show that $\| f - I_3(f) \|_\infty = O(h^4)$ when $f$ is sufficiently smooth. From formula (11.5) from the book, we know that each $s \in S_3$ can be written as

$$s|_{[x_{i-1},x_i]}(x) = \frac{(x_i - x)^3}{6h} \sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h} \sigma_i + \alpha_i (x - x_{i-1}) + \beta_i (x_i - x),$$

for some scalars $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_m$ and, for $i \in \{0, \ldots, m\}$, with $\sigma_i = s''(x_i)$.

By imposing (1) we obtain

$$\alpha_i = \frac{f(x_i)}{h} - \frac{h}{6} \sigma_i, \quad \beta_i = \frac{f(x_{i-1})}{h} - \frac{h}{6} \sigma_{i-1}.$$ 

(a) By using the continuity of $s'$ in $x_1, \ldots, x_{m-1}$ and (2), show that

$$A[\sigma_0 \ldots \sigma_m]^\top = b,$$

where $A \in \mathbb{R}^{(m+1) \times (m+1)}$ is defined by

$$A = \begin{bmatrix}
4 & 2 \\
1 & 4 & 1 \\
& & \ddots \\
1 & 4 & 1 \\
& & & 2 & 4
\end{bmatrix},$$

2
and $b \in \mathbb{R}^{m+1}$ by

$$b_i = \begin{cases} 
12\left[\frac{f(x_i) - f(x_0)}{h} - \frac{f'(x_0)}{h^2}\right] & \text{when } i = 0, \\
6\left[\frac{f(x_i) - 2f(x_i) + f(x_{i-1})}{h^2}\right] & \text{when } i \in \{1, \ldots, m - 1\}, \\
12\left[\frac{f'(x_m)}{h} - \frac{f(x_m) - f(x_{m-1})}{h^2}\right] & \text{when } i = m.
\end{cases}$$

Elementary linear algebra shows that $A$ is invertible, and that $\|A^{-1}\|_\infty \leq \frac{1}{2}$, i.e., that $\max_i |(A^{-1}x)_i| \leq \frac{1}{2} \max_i |x_i|$ (Indeed, writing $A = 4(I - (I - \frac{1}{4}A))$ and using that $\|I - \frac{1}{2}A\|_\infty = \frac{1}{2}$, shows that $\|A^{-1}\|_\infty \leq \frac{1}{2}$).

(b) Show that $\|I_3(f)''\|_\infty \leq 3\|f''\|_\infty$. (Hint: Show that $\|s''\|_\infty = \max_{0 \leq i \leq m} |\sigma_i|$ and that $\max_{0 \leq i \leq m} |b_i| \leq 6\|f''\|_\infty$.)

(c) Show that $I_3$ is a projector, i.e., that $I_3(s) = s$ for any $s \in \mathcal{S}_3$.

(d) Show that for any $p \in \mathcal{S}_1$ there exists a $\bar{s} \in \mathcal{S}_3$ with $\bar{s}'' = p$.

(e) Let $\bar{s} \in \mathcal{S}_3$ be such that $\bar{s}'' = I_1(f'')$. Show that $f - I_3(f) = f - \bar{s} - I_3(f - \bar{s})$, and with that, show that

$$\|f'' - I_3(f)''\|_\infty \leq 4\|f'' - \bar{s}''\|_\infty \leq \frac{1}{2}h^2\|f^{(4)}\|_\infty.$$

(f) Show that $I_1(f - I_3(f)) = 0$, and with that show that

$$\|f - I_3(f)\|_\infty \leq \frac{1}{16}h^4\|f^{(4)}\|_\infty,$$

as well as

$$\|f' - I_3(f)'\|_\infty \leq Ch^3\|f^{(4)}\|_\infty$$

for some constant $C > 0$.

7. Let $a < b$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $h := \frac{b-a}{m}$ and

$$\mathcal{S}_n := \{s \in C^{n-1}(a,b) : s_{|_{(a+(i-1)h, a+ih)}} \in P_n \ (1 \leq i \leq m)\},$$

being the spline space of degree $n$ w.r.t. the subdivision of $[a,b]$ in $m$ equal subintervals (and with $C^{n-1}(a,b)$ being the space of bounded functions on $[a,b]$). For convenience, we take $a = 0$. With

$$S_n(x) := \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x - kh)_+^n,$$

we define $S_{n,\ell}(x) := S_n(x - \ell h)$ for $\ell \in \mathbb{Z}$. 

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Figure 1: The functions $S_{(n,0)}$ for $n = 0, \ldots, 4$.

(a) Show that $S_{(n,\ell)}|_{[0,b]} \in \mathcal{S}_n$.
(b) Show that $\text{supp} S_{(n,\ell)} \subseteq [\ell h, (\ell + n + 1)h]$.
(c) Show that $\dim \mathcal{S}_n = m + n = \# \{ \ell \in \mathbb{Z} : S_{(n,\ell)}|_{[0,b]} \neq 0 \}$.
(d) Show that $S'_{(n+1,\ell)}(x) = (n+1)(S_{(n,\ell)}(x) - S_{(n,\ell+1)}(x))$ (when $n = 0$ only for $x \notin h\mathbb{Z}$).
(e) From Exercise 11.6, we know that $S_{(n+1,\ell)}(x) = (x - \ell h)S_{(n,\ell)}(x) + ((n + 2 + \ell)h - x)S_{(n,\ell+1)}(x)$.

Using induction to $n$, from this show that
$$\sum_{\ell \in \mathbb{Z}} S_{(n,\ell)}(x) = h^n n!.$$ 

Now we are going to show that for all $p \in \mathbb{Z}$,
$$\sum_{\ell \in \mathbb{Z}} c_{\ell} S_{(n,\ell)}|_{(ph,(p+1)h)} = 0 \implies c_{\ell} = 0 \text{ for } p - n - 1 < \ell < p + 1. \quad (3)$$

(f) Show that (3) holds for $n = 0$.
(g) Now let (3) be valid for some $n \in \mathbb{N}_0$. Let $\sum_{\ell \in \mathbb{Z}} c_{\ell} S_{(n+1,\ell)}$ and so $\sum_{\ell \in \mathbb{Z}} c_{\ell} S'_{(n+1,\ell)}$ vanish on $(ph, (p+1)h)$. Using (7d), show that this implies that for some constant $c \in \mathbb{R}$, $c_{\ell} = c$ for all
$$p - n - 2 < \ell < p + 1,$$
and with that
$$\sum_{\ell \in \mathbb{Z}} c_{\ell} S_{(n+1,\ell)}|_{(ph,(p+1)h)} = c \sum_{\ell \in \mathbb{Z}} S_{(n+1,\ell)}|_{(ph,(p+1)h)} = ch^{n+1}(n + 1)!$$

Conclude that (3) is valid for $n + 1$, and so for any $n \in \mathbb{N}_0$. 

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(h) Using (7a), (7c), and (3), show that

\[ S_{(n, \ell)}|_{[0,b]} : \ell \in \{-n, \ldots, m - 1\} \]

is a basis for \( S_n \).

8. (In case this is a homework assignment: Hand in a Zip file with your code and a PDF document containing answers to the questions.) Consider the initial value problem (IVP):

\[
\begin{align*}
   y'(x) &= f(x, y(x)) & 0 \leq x \leq 1 \\
   y(0) &= 0
\end{align*}
\]

where \( f(x, y) = (1 + x)(1 + y^2) \).

(a) Verify that the exact solution is given by \( y(x) = \tan(x + x^2/2) \).

A possible implementation in Matlab of the Forward Euler method (FE) for solving this IVP is given below:

```matlab
function Euler(N) % N is the number of the time steps
f=@(x,y)(1+x)*(1+y.^2); % defines the function f
y=@(x) tan(x+x.^2/2); % defines exact solution
h=1/N; % time step
x=0:h:1; % time step
xfine=0:0.01:1
FE=zeros(1,N+1); % Forward Euler approximation solution
err=zeros(1,N+1); % Error values of Forward Euler method
FE(1)=0;
for i=1:N
    FE(i+1)=FE(i)+h*f(x(i),FE(i));
end
for i=1:N+1
    err(i)=y(x(i))-FE(i);
end
plot(xfine,y(xfine));
hold on;
plot(x,FE);
```

The same program in Python reads as follows:
import matplotlib.pyplot as plt
import numpy as np

# N is the number of time steps
def ForwardEuler(N):
    f = lambda x,y : (1+x)*(1+y**2) # defines function f
    y = lambda x : np.tan(x + x**2 / 2.0) # defines exact solution
    h = 1.0 / N
    x = np.linspace(0, 1, N + 1)
    xfine = np.linspace(0, 1, 100)
    FE = np.zeros(N+1) # Forward Euler approximation solution
    err = np.zeros(N+1) # Error values of Forward Euler method
    FE[0] = 0.0
    for i in range(N):
        FE[i+1] = FE[i] + h * f(x[i], FE[i])

    for i in range(N+1):
        err[i]= y(x[i]) - FE[i];

    plt.plot(xfine, y(xfine), label="Exact solution")
    plt.plot(x,FE, label="Forward Euler approximation");
    plt.legend()
    plt.show()
    return FE, err

(b) Run Forward Euler with $h^{-1} = N = 10, 20, 40, 80, 160$ and compare the errors.

(c) To compare the error of FE with other numerical methods, solve the problem with the modified Euler method:

$$
\begin{align*}
  y_{i+1} &= y_i + \frac{1}{2}(k_1 + k_2) \\
  k_1 &= hf(x_i, y_i), \quad k_2 = hf(x_i + h, y_i + k_1)
\end{align*}
$$

and the following Runge-Kutta scheme:

$$
\begin{align*}
  y_{i+1} &= y_i + \frac{h^6}{6}(k_1 + 2k_2 + 2k_3 + 2k_4 + k_6) \\
  k_1 &= f(x_i, y_i), \quad k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1), \\
  k_3 &= f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2), \quad k_4 = f(x_i + h, y_i + hk_3)
\end{align*}
$$
(d) Plot the error in the point \( \frac{1}{2} \) vs. \( N \) for the three methods, and estimate the order of the methods. To do this, it is most convenient to use a log-log plot.

9. (In case this is a homework assignment: Hand in a Zip file with your code and a PDF document containing answers to the questions.) To illustrate the numerical solution of a so-called stiff ODE, consider the IVP

\[
\begin{aligned}
    y'(x) &= \lambda (\sin(x) - y) + \cos(x) & 0 \leq x \leq 1, \lambda \gg 1 \\
    y(0) &= 0
\end{aligned}
\]

with exact solution \( y(x) = \sin(x) \).

(a) Apply the FE method to this problem with \( \lambda = 200 \) and \( h^{-1} = N = 10, 90, 95, 100, 105, 1000 \). What do you notice?

(b) Implement the Backward Euler method (BE):

\[
y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})
\]

and run it with the same values of \( h \) and \( \lambda \). Compare the results.

(c) With \( x_i = ih \), define the truncation error of the Backward Euler method by

\[
T_{i}^{(BE)} = \frac{y(x_{i+1}) - y(x_i)}{h} - f(x_{i+1}, y(x_{i+1}))
\]

and show that \( T_{i}^{(BE)} = -\frac{h}{2} y''(\xi) \) for some \( \xi \in [x_i, x_{i+1}] \).

Similarly, let \( T_{i}^{(FE)} = \frac{y(x_{i+1}) - y(x_i)}{h} - f(x_i, y(x_i)) \), which is known to be of the form \( \frac{h}{2} y''(\eta) \) for some \( \eta \in [x_i, x_{i+1}] \).

(d) With \( e_{i}^{(FE)} := y(x_i) - y_i^{(FE)} \) and \( e_{i}^{(BE)} := y(x_i) - y_i^{(BE)} \), show that

\[
\begin{aligned}
    e_{i+1}^{(FE)} &= (1 - h\lambda)e_{i}^{(FE)} + hT_{i}^{(FE)}, \\
    e_{i+1}^{(BE)} &= e_{i}^{(BE)} + hT_{i}^{(BE)} \frac{1}{1 + h\lambda}.
\end{aligned}
\]

Show that for \( 1 \leq i \leq h^{-1} \), \( |e_{i}^{(BE)}| \leq \frac{1}{2} h\|y''\|_{\infty} \), and, when \( h \leq \frac{1}{100} \), \( |e_{i}^{(FE)}| \leq \frac{1}{2} h\|y''\|_{\infty} \).

Explain the behaviour of the error of FE when \( N < 100 \). Is there a contradiction with the result of Theorem 12.2 applied to FE?
10. For \( x_0 < x_1 < \cdots < x_n \), where \( n \geq 2 \), and \( k \in \mathbb{N} \), consider the spline space

\[ S^{(k)} = \{ s \in C^{k-1}(x_0, x_n) : s|_{(x_i, x_{i+1})} \in \mathcal{P}_k, \; i = 0, \ldots, n \}. \]

(a) Show that the only \( s \in S^{(1)} \) with \( s(x_0) = s(x_n) = 0 \) and such that for any \( i \geq 0, \; j \geq 2, \; i + j \leq n \), \( s|_{(x_i, x_{i+j})} \) has \( j - 1 \) zeros, is the zero function. (Hint: Show that if \( s \neq 0 \), then there exists \( i \) and \( j \) as above with \( s(x_i) = 0 \) and \( s(x_{i+j}) = 0 \) and \( s(x_{i+1}) \neq 0, \ldots, s(x_{i+j-1}) \neq 0 \), and derive a contradiction.)

(b) Show that the only \( p \in \mathcal{P}_3(a, b) \) with \( p(a) = p(b) = 0 \) and \( p'' \equiv 0 \) is the zero polynomial.

(c) Show that there exists at most one natural cubic spline interpolant, i.e., an \( t \in S^{(3)} \) with \( t''(x_0) = t''(x_n) = 0 \) that for some given \( y_0, \ldots, y_n \) satisfies \( t(x_i) = y_i \) (\( 0 \leq i \leq n \)). (Hint: suppose two, and consider the difference.)

11. Archimedes (250 v. Chr.) obtained upper and lower bounds for \( \pi \) by measuring the perimeter of regular inscribed or circumscribed polygons for a circle with radius 1. In this exercise we consider inscribed polygons only.

(a) Let \( T_0(h) \) be the perimeter of regular inscribed polygon with \( n \) sides, where \( nh = 1 \). Show that \( T_0(h) = 2h^{-1}\sin(\pi h) \).

(b) Show that there exist constants \( (c_i) \) such that \( \forall m \in \mathbb{N} \)

\[ 2\pi - T_1(h/2) = \sum_{i=1}^{m} c_i h^{2i} + \mathcal{O}(h^{2m+2}) \quad (h \to 0). \]

(c) Determine \( \alpha_1, \beta_1 \) such that \( T_1(h/2) := \alpha_1 T_0(h/2) + \beta_1 T_0(h) \) satisfies

\[ 2\pi - T_1(h/2) = \mathcal{O}(h^4) \quad (h \to 0). \]

Huygens used this idea already in 1654. Archimedes’ measurements went to \( n = 96 \). Assuming that Huygens used these measurements, which we assume to be exact, what were the errors in the best approximations that they both obtained?
Figure 2: The inscribed regular polygons for $h = 1/8$ and $h = 1/16$.

(d) Improve Huygens, i.e., determine $\alpha_2$, $\beta_2$ such that $T_2(h/4) := \alpha_2 T_1(h/4) + \beta_2 T_1(h/2)$ satisfies

$$2\pi - T_2(h/4) = O(h^6) \quad (h \to 0).$$

What is the error in $T_2(1/96)$?

12. To approximate $\sqrt{a}$ ($a > 0$) we apply the Newton scheme to

$$f(x) = x^2 - a = 0.$$

(a) Verify that this yields the following iteration:

$$x_{i+1} = \frac{1}{2}(x_i + \frac{a}{x_i}).$$

(b) Show that for any $x_0 > \sqrt{a}$, the sequence $(x_i)_{i \geq 0}$ is monotone decreasing.

(c) Show that for any $0 < x_0 < \sqrt{a}$, the sequence $(x_i)_{i \geq 1}$ is monotone decreasing.

Now we consider the Newton iteration with $x_0 = \frac{1}{2}(1 + a)$. 

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(d) Show that \( x_0 \) is the result of one step of Newton iteration starting with \( "x_{-1}\)" = 1, and thus that \( x_0 > \sqrt{a} \).

(e) Show that if for some \( i \geq 0 \), \( x_i - x_{i+1} < \epsilon \), that then \( x_{i+1} - \sqrt{a} < \epsilon \), which provides a useful stopping criterion.

13. Construct the 3-point Radau formula for the interval \([0, 1]\), being thus the quadrature formula that is exact on \( P_2 \) and that has 0 as one of its three quadrature points.

14. (Interpolation in general). Let \((x_i)_{i \in \mathbb{N}_0} \subset \mathbb{R}\) be a sequence of pairwise disjoint points. Let \((r_i)_{i \in \mathbb{N}_0} \subset \mathbb{N}_0\) such that \( A := \{i : r_i \neq 0\} \) is non-empty and finite, and set \( N := (\sum_{i=0}^{\infty} r_i) - 1 \). Our goal is to show the following:

   For any \( \cup_{i \in A} \{y_{i,j} : 1 \leq j \leq r_i\} \subset \mathbb{R} \),
   \[ \exists! p \in P_N \text{ with } p^{(j-1)}(x_i) = y_{i,j} \quad (i \in A, 1 \leq j \leq r_i). \] (4)

   (a) Show that (4) can have at most one solution.
   (b) Show that for any \( j \in \mathbb{N}_0 \),
   \[ \left( \frac{d^r_j}{dx^r_j} \prod_{i \in A} (x - x_i)^{r_i} \right)(x_j) \neq 0 \quad \left( \frac{d^0}{dx^0} := \text{Id} \right) \]
   (Hint: Show that \( \left( \frac{d^r_j}{dx^r_j} ((x - x_j)^{r_j}g(x)) \right)(x_j) = g(x_j) \)).
   (c) Assume that \( p \) is a solution of (4). For one \( i \), say \( i^* \), increase \( r_{i^*} \) by one, and add some \( y_{i^*,r_{i^*}+1} \). Show that the solution \( q \) of the new interpolation problem can be found in the form
   \[ q(x) = p(x) + c \prod_{i \in A} (x - x_i)^{r_i}. \]

   Now finish the proof of (4).
   (d) For an \( f \) that, for \( i \in A \), is \( r_i - 1 \) times continuously differentiable at \( x_i \), and that is \( (N + 1) \) times differentiable on \((a, b) \supset \) \( (\min_{i \in A} x_i, \max_{i \in A} x_i) \), and with \( p \in P_N \) the solution of the interpolation problems with \( y_{i,j} := f^{(j-1)}(x_i) \) \( (1 \leq j \leq r_i) \), show that for any \( x \in (a, b) \), there exists a \( \xi = \xi(x) \) with
   \[ f(x) - p(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i \in A} (x - x_i)^{r_i}. \]
(Generalizes remainder terms for Lagrange, Hermite and Taylor polynomials.)

(e) Show that the interpolation problem where in (4) one or more conditions for \( j < r_i \) are omitted, is generally not well-posed: With \( N + 1 \) being the number of imposed conditions, there might be no or multiple solutions in \( P_N \).

15. With \( Q(f) \) denoting the \((n+1)\)-point Radau formula from [Book, (10.27)] (where \( x_k \) should read as \( x^*_k \)), show that
\[
\int_a^b w(x)f(x)\,dx - Q(f) = \frac{f^{(2n+1)}(\xi)}{(2n+1)!} \int_a^b w(x)(x-a) \prod_{k=1}^n (x-x_k^*)^2 \,dx.
\]
for some \( \xi \in [a,b] \). (Hint: Use exer. 14).

16. Let \( \| \cdot \| \) be a submultiplicative norm on \( \mathbb{R}^{n \times n} \) (or \( \mathbb{C}^{n \times n} \)), i.e. \( \| AB \| \leq \| A \| \| B \| \), for example a matrix norm induced by a vector norm (\( \| A \| := \sup_{0 \neq x \in \mathbb{R}^n} \frac{\| Ax \|}{\| x \|} \)), sometimes also called a subordinate norm.

Show that if \( \| T \| < 1 \), then \( I - T \) is invertible, \( (I - T)^{-1} = \sum_{n=0}^\infty T^n \), \( \|(I - T)^{-1}\| \leq (1 - \| T \|)^{-1} \), and \( \| I - (I - T)^{-1} \| \leq \frac{\| T \|}{1 - \| T \|} \).

17. Consider the 3-term recursion \( \alpha_2 v_{n+2} + \alpha_1 v_{n+1} + \alpha_0 v_n = 0 \), \( n = 0, 1, \ldots \). Give an explicit expression of \( v_n \) in terms of the starting values \( v_0, v_1 \) and the roots of \( \alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0 \). Distinguish between the cases of having two different roots, or one double root.

18. Let the roots \( z_1, \ldots, z_k \) of \( \rho(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \ldots + \alpha_1 z + \alpha_0 \) be single and unequal to zero. Give the eigenvalues of the \( k \times k \) companion matrix
\[
A := \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\alpha_0 & -\alpha_1 & \cdots & -\alpha_{k-1}
\end{bmatrix}
\]
and determine corresponding eigenvectors.

Prove that \( \sup_{p \in \mathbb{N}} \| A^p \| < \infty \) if and only if \( |z_i| \leq 1 \) for \( 1 \leq i \leq k \).

What is the corresponding statement when \( \rho \) has one or more multiple roots?