Additional exercises with Numerieke Analyse

April 15, 2020

1. (a) Given different points $x_0,\ x_1,\ x_2 \subset [a,b]$ and scalars $y_0,\ y_1,\ y_2,\ z_1$, show that there exists at most one polynomial $p \in P_3$ with $p(x_i) = y_i,\ i = 0,1,2,\ p'(x_1) = z_1$.

   (b) Construct this p in the form $p(x) = p_2(x)+\alpha(x-x_0)(x-x_1)(x-x_2)$ with p_2 being the Langange interpolation polynomial of degree 2 corresponding to the set $\{(x_i, y_i) : i = 0,1,2\}$.

   (c) Let f be four times differentiable. Show that for the polynomial $p \in P_3$ with $p(x_i) = f(x_i),\ i = 0,1,2,\ p'(x_1) = f'(x_1)$ and any $x \in [a,b]$, there exists a $\xi = \xi(x) \in (a,b)$ with
$$f(x) - p(x) = (x-x_0)(x-x_1)^2(x-x_2)\frac{f^{(4)}(\xi)}{4!}.$$

2. (a) Find weights $w_0, \bar{w}_0, w_1, \bar{w}_1$ such that
$$\int_a^b f(x)dx = w_0 f(a) + \bar{w}_0 f'(a) + w_1 f(b) + \bar{w}_1 f'(b)$$
for any $f \in P_3$.

   (b) Show that for $f \in C^4$, the error in this quadrature formula, i.e., true integral minus its approximation, is of the form $C(b-a)^5 f^{(4)}(\xi)$ for some $\xi \in [a,b]$, and give the constant C.

   (c) Splitting the interval into m equal subintervals, give the resulting composite quadrature formula.

   (d) Show that the error in this composite formula is equal to $C\frac{(b-a)^5}{m^4}f^{(4)}(\xi)$ for some $\xi \in [a,b]$.

3. For $a < b$, $\{x_0, \ldots, x_n\} \subset \mathbb{R}$, show that there are unique weights $w_0, \ldots, w_n$ such that $\sum_{i=0}^n w_i f(x_i) = \int_a^b f(x)dx$ for all $f \in P_n$. Show that $w_i = \int_a^b \prod_{k=0,\, k\neq i}^n \frac{x-x_k}{x_i-x_k}\,dx$.

4. (**Runge phenomenon**). See Canvas.

5. (**Adaptive quadrature**). See Canvas.

6. Let $a < b$, $m \in \mathbb{N}$, $h := \frac{b-a}{m}$, $x_i := a + ih$ for $i \in \{0, \ldots, m\}$, and for $n \in \{1, 2, \ldots\}$, let

$$\mathcal{S}_n := \{s \in C^{n-1}(a,b) : s|_{(x_{i-1},x_i)} \in P_n \,(1 \leq i \leq m)\}.$$

Furthermore, let $I_1 : C[a,b]) \to \mathcal{S}_1$ the continuous piecewise linear interpolator, and let $I_3 : C^1[a,b]) \to \mathcal{S}_3$ the "complete cubic spline interpolator" defined by

$$s(x_i) = f(x_i) \quad (i \in \{0, \ldots, m\}), \tag{1}$$

$$s'(x_0) = f'(x_0), \quad s'(x_m) = f'(x_m). \tag{2}$$

where s is here a shorthand notation for $I_3(f)$. The aim of this exercise is to show that $\|f - I_3(f)\|_\infty = \mathcal{O}(h^4)$ when f is sufficiently smooth. From formula (11.5) from the book, we know that each $s \in \mathcal{S}_3$ can be written as

$$s|_{[x_{i-1},x_i]}(x) = \frac{(x_i - x)^3}{6h}\sigma_{i-1} + \frac{(x - x_{i-1})^3}{6h}\sigma_i + \alpha_i(x - x_{i-1}) + \beta_i(x_i - x),$$

for some scalars $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_m$ and, for $i \in \{0, \ldots, m\}$, with $\sigma_i = s''(x_i)$.

By imposing (1) we obtain

$$\alpha_i = \frac{f(x_i)}{h} - \frac{h}{6}\sigma_i, \quad \beta_i = \frac{f(x_{i-1})}{h} - \frac{h}{6}\sigma_{i-1}.$$

(a) By using the continuity of s' in $x_1, \ldots, x_{m-1}$ and (2), show that

$$A[\sigma_0 \ldots \sigma_m]^\top = b,$$

where $A \in \mathbb{R}^{(m+1)\times(m+1)}$ is defined by

$$A = \begin{bmatrix} 4 & 2 & & & \\ 1 & 4 & 1 & & \\ & & \ddots & & \\ & & 1 & 4 & 1 \\ & & & 2 & 4 \end{bmatrix},$$

and $b \in \mathbb{R}^{m+1}$ by

$$b_i = \begin{cases} 12\left[\frac{f(x_1)-f(x_0)}{h^2} - \frac{f'(x_0)}{h}\right] & \text{when } i = 0, \\ 6\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2} & \text{when } i \in \{1, \dots, m-1\}, \\ 12\left[\frac{f'(x_m)}{h} - \frac{f(x_m)-f(x_{m-1})}{h^2}\right] & \text{when } i = m. \end{cases}$$

Elementary linear algebra shows that A is invertible, and that $\|A^{-1}\|_\infty \leq \frac{1}{2}$, i.e., that $\max_i |(A^{-1}x)_i| \leq \frac{1}{2} \max_i |x_i|$ (Indeed, writing $A = 4(I - (I - \frac{1}{4}A))$ and using that $\|I - \frac{1}{4}A\|_\infty = \frac{1}{2}$, shows that $\|A^{-1}\|_\infty \leq \frac{1}{2}$),

(b) Show that $\|I_3(f)''\|_\infty \leq 3\|f''\|_\infty$. (Hint: Show that $\|s''\|_\infty = \max_{0\leq i\leq m} |\sigma_i|$ and that $\max_{0\leq i\leq m} |b_i| \leq 6\|f''\|_\infty$.)

(c) Show that I_3 is a projector, i.e., that $I_3(s) = s$ for any $s \in \mathcal{S}_3$.

(d) Show that for any $p \in \mathcal{S}_1$ there exists a $\bar{s} \in \mathcal{S}_3$ with $\bar{s}'' = p$.

(e) Let $\bar{s} \in \mathcal{S}_3$ be such that $\bar{s}'' = I_1(f'')$. Show that $f - I_3(f) = f - \bar{s} - I_3(f - \bar{s})$, and with that, show that

$$\|f'' - I_3(f)''\|_\infty \leq 4\|f'' - \bar{s}''\|_\infty \leq \tfrac{1}{2}h^2\|f^{(4)}\|_\infty.$$

(f) Show that $I_1(f - I_3(f)) = 0$, and with that show that

$$\|f - I_3(f)\|_\infty \leq \tfrac{1}{16}h^4\|f^{(4)}\|_\infty,$$

as well as

$$\|f' - I_3(f)'\|_\infty \leq Ch^3\|f^{(4)}\|_\infty$$

for some constant $C > 0$.

7. Let $a < b$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $h := \frac{b-a}{m}$ and

$$\mathcal{S}_n := \{s \in C^{n-1}(a,b) : s|_{(a+(i-1)h, a+ih)} \in P_n \ (1 \leq i \leq m)\},$$

being the spline space of degree n w.r.t. the subdivision of $[a,b]$ in m equal subintervals (and with $C^{-1}(a,b)$ being the space of bounded functions on $[a,b]$). For convenience, we take $a = 0$. With

$$S_{(n)}(x) := \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} (x - kh)_+^n,$$

we define $S_{(n,\ell)}(x) := S_{(n)}(x - \ell h)$ for $\ell \in \mathbb{Z}$.

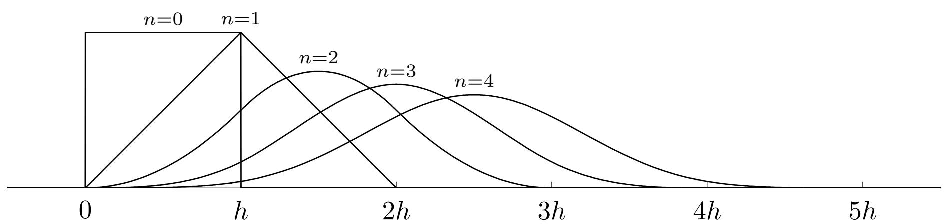

Figure 1: The functions $S_{(n,0)}$ for $n = 0, \ldots, 4$.

(a) Show that $S_{(n,\ell)}|_{[0,b]} \in \mathcal{S}_n$.

(b) Show that $\operatorname{supp} S_{(n,\ell)} \subseteq [\ell h, (\ell + n + 1)h]$.

(c) Show that $\dim \mathcal{S}_n = m + n = \#\{\ell \in \mathbb{Z} : S_{(n,\ell)}|_{[0,b]} \neq 0\}$.

(d) Show that $S'_{(n+1,\ell)}(x) = (n+1)(S_{(n,\ell)}(x) - S_{(n,\ell+1)}(x))$ (when $n = 0$ only for $x \notin h\mathbb{Z}$).

(e) From Exercise 11.6, we know that

$$S_{(n+1,\ell)}(x) = (x - \ell h)S_{(n,\ell)}(x) + ((n + 2 + \ell)h - x)S_{(n,\ell+1)}(x).$$

Using induction to n, from this show that

$$\sum_{\ell \in \mathbb{Z}} S_{(n,\ell)}(x) = h^n n!.$$

Now we are going to show that for all $p \in \mathbb{Z}$,

$$\sum_{\ell \in \mathbb{Z}} c_\ell S_{(n,\ell)}|_{(ph,(p+1)h)} = 0 \Longrightarrow c_\ell = 0 \text{ for } p - n - 1 < \ell < p + 1. \quad (3)$$

(f) Show that (3) holds for $n = 0$.

(g) Now let (3) be valid for some $n \in \mathbb{N}_0$. Let $\sum_{\ell \in \mathbb{Z}} c_\ell S_{(n+1,\ell)}$ and so $\sum_{\ell \in \mathbb{Z}} c_\ell S'_{(n+1,\ell)}$ vanish on $(ph, (p + 1)h)$. Using (7d), show that this implies that for some constant $c \in \mathbb{R}$, $c_\ell = c$ for all

$$p - n - 2 < \ell < p + 1,$$

and with that

$$\sum_{\ell \in \mathbb{Z}} c_\ell S_{(n+1,\ell)}|_{(ph,(p+1)h)} = c \sum_{\ell \in \mathbb{Z}} S_{(n+1,\ell)}|_{(ph,(p+1)h)} = ch^{n+1}(n + 1)!$$

Conclude that (3) is valid for $n + 1$, and so for any $n \in \mathbb{N}_0$.

(h) Using (7a), (7c), and (3), show that

$$\left\{S_{(n,\ell)}|_{[0,b]} : \ell \in \{-n, \dots, m-1\}\right\}$$

is a basis for $\mathcal{S}_n$.

8. (**In case this is a homework assignment: Hand in a Zip file with your code and a PDF document containing answers to the questions**.) Consider the initial value problem(IVP):

$$\begin{cases} y'(x) &= f(x, y(x)) \quad 0 \le x \le 1 \\ y(0) &= 0 \end{cases}$$

where $f(x, y) = (1 + x)(1 + y^2)$.

(a) Verify that the exact solution is given by $y(x) = \tan(x + x^2/2)$.

A possible implementation in Matlab of the Forward Euler method (FE) for solving this IVP is given below:

```
function Euler(N)              % N is the number of the time steps
f=@(x,y)(1+x)*(1+y.^2);        % defines the function f
y=@(x) tan(x+x.^2/2);          % defines exact solution
h=1/N;                         % time step
x=0:h:1;
xfine=0:0.01:1
FE=zeros(1,N+1);               % Forward Euler approximation solution
err=zeros(1,N+1);              % Error values of Forward Euler method
FE(1)=0;
for i=1:N
    FE(i+1)=FE(i)+h*f(x(i),FE(i));
end
for i=1:N+1
   err(i)=y(x(i))-FE(i);
end
plot(xfine,y(xfine));
hold on;
plot(x,FE);
```

The same program in Python reads as follows:

```
import matplotlib.pyplot as plt
import numpy as np

# N is the numer of time steps
def ForwardEuler(N):
    f = lambda x,y : (1+x)*(1+y**2)        # defines function f
    y = lambda x : np.tan(x + x**2 / 2.0)  # defines exact solution
    h = 1.0 / N
    x  = np.linspace(0, 1, N + 1)
    xfine = np.linspace(0, 1, 100)
    FE = np.zeros(N+1)             # Forward Euler approximation solution
    err = np.zeros(N+1)            # Error values of Forward Euler methdo
    FE[0] = 0.0
    for i in range(N):
        FE[i+1] = FE[i] + h * f(x[i], FE[i])

    for i in range(N+1):
        err[i]= y(x[i]) - FE[i];

    plt.plot(xfine, y(xfine), label="Exact solution")
    plt.plot(x,FE, label="Forward Euler approximation");
    plt.legend()
    plt.show()
    return FE, err
```

(b) Run Forward Euler with $h^{-1} = N = 10, 20, 40, 80, 160$ and compare the errors.

(c) To compare the error of FE with other numerical methods, solve the problem with the modified Euler method:

$$\begin{cases} y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2) \text{ where} \\ k_1 = hf(x_i, y_i), \quad k_2 = hf(x_i + h, y_i + k_1) \end{cases}$$

and the following Runge-Kutta scheme:

$$\begin{cases} y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \text{ where} \\ k_1 = f(x_i, y_i),\ k_2 = f(x_i + \frac{h}{2}, y_i + h\frac{k_1}{2}), \\ k_3 = f(x_i + \frac{h}{2}, y_i + h\frac{k_2}{2}),\ k_4 = f(x_i + h, y_i + hk_3) \end{cases}$$

(d) Plot the error in the point $\frac{1}{2}$ vs. N for the three methods, and estimate the order of the methods. To do this, it is most convenient to use a log-log plot.

9. (**In case this is a homework assignment: Hand in a Zip file with your code and a PDF document containing answers to the questions**.) To illustrate the numerical solution of a so-called stiff ODE, consider the IVP

$$\begin{cases} y'(x) &= \lambda(\sin(x) - y) + \cos(x) \quad 0 \leq x \leq 1, \lambda \gg 1 \\ y(0) &= 0 \end{cases}$$

with exact solution $y(x) = \sin(x)$.

(a) Apply the FE method to this problem with $\lambda = 200$ and $h^{-1} = N = 10, 90, 95, 100, 105, 1000$. What do you notice?

(b) Implement the Backward Euler method (BE):

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

and run it with the same values of h and λ. Compare the results.

(c) With $x_i = ih$, define the truncation error of the Backward Euler method by

$$T_i^{(\mathrm{BE})} = \frac{y(x_{i+1}) - y(x_i)}{h} - f(x_{i+1}, y(x_{i+1}))$$

and show that $T_i^{(\mathrm{BE})} = -\frac{h}{2}y''(\xi)$ for some $\xi \in [x_i, x_{i+1}]$.

Similarly, let $T_i^{(\mathrm{FE})} = \frac{y(x_{i+1})-y(x_i)}{h} - f(x_i, y(x_i))$, which is known to be of the form $\frac{h}{2}y''(\eta)$ for some $\eta \in [x_i, x_{i+1}]$.

(d) With $e_i^{(FE)} := y(x_i) - y_i^{(FE)}$ and $e_i^{(BE)} := y(x_i) - y_i^{(BE)}$, show that

$$e_{i+1}^{(FE)} = (1 - h\lambda)e_i^{(FE)} + hT_i^{(FE)},$$

$$e_{i+1}^{(BE)} = \frac{e_i^{(BE)} + hT_i^{(BE)}}{1 + h\lambda}.$$

Show that for $1 \leq i \leq h^{-1}$, $|e_i^{(BE)}| \leq \frac{1}{2}h\|y''\|_\infty$, and, when $h \leq \frac{1}{100}$, $|e_i^{(FE)}| \leq \frac{1}{2}h\|y''\|_\infty$.

Explain the behaviour of the error of FE when $N < 100$. Is there a contradiction with the result of Theorem 12.2 applied to FE?

10. For $x_0 < x_1 < \cdots < x_n$, where $n \geq 2$, and $k \in \mathbb{N}$, consider the spline space

$$S^{(k)} = \{s \in C^{k-1}(x_0, x_n) : s|_{(x_i, x_{i+1})} \in \mathcal{P}_k,\ i = 0, \ldots, n\}.$$

(a) Show that the only $s \in S^{(1)}$ with $s(x_0) = s(x_n) = 0$ and such that for any $i \geq 0$, $j \geq 2$, $i + j \leq n$, $s|_{(x_i, x_{i+j})}$ has $j - 1$ zeros, is the zero function. (Hint: Show that if $s \not\equiv 0$, then there exists i and j as above with $s(x_i) = 0$ and $s(x_{i+j}) = 0$ and $s(x_{i+1}) \neq 0, \ldots, s(x_{i+j-1}) \neq 0$, and derive a contradiction.)

(b) Show that the only $p \in \mathcal{P}_3(a, b)$ with $p(a) = p(b) = 0$ and $p'' \equiv 0$ is the zero polynomial.

(c) Show that there exists at most one natural cubic spline interpolant, i.e., an $t \in S^{(3)}$ with $t''(x_0) = t''(x_n) = 0$ that for some given $y_0, \ldots, y_n$ satisfies $t(x_i) = y_i$ $(0 \leq i \leq n)$. (Hint: suppose two, and consider the difference.)

11. Archimedes (250 v. Chr.) obtained upper and lower bounds for π by measuring the perimeter of regular inscribed or circumscribed polygons for a circle with radius 1. In this exercise we consider inscribed polygons only.

(a) Let $T_0(h)$ be the perimeter of regular inscribed polygon with n sides, where $nh = 1$. Show that $T_0(h) = 2h^{-1} \sin(\pi h)$.

(b) Show that there exist constants (c_i) such that $\forall m \in \mathbb{N}$

$$2\pi - T_0(h) = \sum_{i=1}^{m} c_i h^{2i} + \mathcal{O}(h^{2m+2}) \qquad (h \to 0).$$

(c) Determine α_1, β_1 such that $T_1(h/2) := \alpha_1 T_0(h/2) + \beta_1 T_0(h)$ satisfies

$$2\pi - T_1(h/2) = \mathcal{O}(h^4) \qquad (h \to 0).$$

Huygens used this idea already in 1654. Archimes' measurements went to $n = 96$. Assuming that Huygens used these measurements, which we assume to be exact, what were the errors in the best approximations that they both obtained?

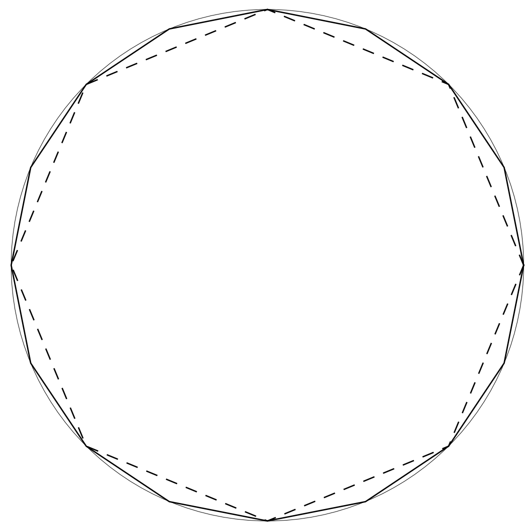

Figure 2: The inscribed regular polygons for $h = 1/8$ and $h = 1/16$.

(d) Improve Huygens, i.e., determine α_2, β_2 such that $T_2(h/4) := \alpha_2 T_1(h/4) + \beta_2 T_1(h/2)$ satisfies

$$2\pi - T_2(h/4) = \mathcal{O}(h^6) \qquad (h \to 0).$$

What is the error in $T_2(1/96)$?

12. To approximate $\sqrt{a}$ ($a > 0$) we apply the Newton scheme to

$$f(x) = x^2 - a = 0.$$

(a) Verify that this yields the following iteration:

$$x_{i+1} = \frac{1}{2}(x_i + \frac{a}{x_i}).$$

(b) Show that for any $x_0 > \sqrt{a}$, the sequence $(x_i)_{i\geq 0}$ is monotone decreasing.

(c) Show that for any $0 < x_0 < \sqrt{a}$, the sequence $(x_i)_{i\geq 1}$ is monotone decreasing.

Now we consider the Newton iteration with $x_0 = \frac{1}{2}(1 + a)$.

(d) Show that x_0 is the result of one step of Newton iteration starting with "x_{-1}"$= 1$, and thus that $x_0 > \sqrt{a}$.

(e) Show that if for some $i \geq 0$, $x_i - x_{i+1} < \epsilon$, that then $x_{i+1} - \sqrt{a} < \epsilon$, which provides a useful stopping criterion.

13. Construct the 3-point Radau formula for the interval $[0, 1]$, being thus the quadrature formula that is exact on $\mathcal{P}_2$ and that has 0 as one of its three quadrature points.

14. (**Interpolation in general**). Let $x_0, \ldots, x_n$ be different points in $[a, b]$, $r_0, \ldots, r_n \in \mathbb{N} = \{1, 2, \ldots\}$, and $N := -1 + \sum_{i=0}^{n} r_i$. Our goal is to show the following:

$$\begin{aligned}&\text{For any } \{y_{i,j} \colon 0 \leq i \leq n,\, 1 \leq j \leq r_i\} \subset \mathbb{R},\\ &\exists!\, p \in P_N \text{ with } p^{(j-1)}(x_i) = y_{i,j} \quad (0 \leq i \leq n,\, 1 \leq j \leq r_i).\end{aligned} \tag{4}$$

(a) Show that (4) can have at most one solution.

(b) Assume that p is a solution of (4). For one i, add one data point y_{i,r_i+1}. Show that the solution q of the new interpolation problem can be found in the form

$$q(x) = p(x) + c \prod_{i=0}^{n} (x - x_i)^{r_i}.$$

Now finish the proof of (4).

(c) For an f that, for $0 \leq i \leq n$, is $r_i - 1$ times continuously differentiable at x_i, and that is $(N + 1)$ times differentiable on (a, b), and with $p \in P_N$ the solution of the interpolation problem with $y_{i,j} := f^{(j-1)}(x_i)$ $(1 \leq j \leq r_i)$, show that for any $x \in (a, b)$, there exists a $\xi = \xi(x) \in [a, b]$ with

$$f(x) - p(x) = \frac{f^{(N+1)}(\xi)}{(N + 1)!} \prod_{i=0}^{n} (x - x_i)^{r_i}.$$

(Generalizes remainder terms for Lagrange, Hermite and Taylor polynomials.)

(d) Show that the interpolation problem where in (4) one or more conditions for $j < r_i$ are omitted, is generally not well-posed: With $N+1$ being the number of imposed conditions, there might be no or multiple solutions in P_N. (Hint: An example can already be found for $n = 0$).

15. With $Q(f)$ denoting the $(n+1)$-point Radau formula from [Book, (10.27)] (where x_k should be read as x_k^*), show that

$$\int_a^b w(x)f(x)\,dx - Q(f) = \frac{f^{(2n+1)}(\xi)}{(2n+1)!}\int_a^b w(x)(x-a)\prod_{k=1}^{n}(x-x_k^*)^2\,dx.$$

for some $\xi \in [a,b]$. (Hint: Use exer. 14).

16. Let $\|\cdot\|$ be a submultiplicative norm on $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$), i.e. $\|AB\| \le \|A\|\|B\|$), for example a matrix norm induced by a vector norm ($\|A\| := \sup_{0\neq\vec{x}\in\mathbb{R}^n}\frac{\|Ax\|}{\|x\|}$), sometimes also called a subordinate norm.

Show that if $\|T\| < 1$, then $I - T$ is invertible, $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$, $\|(I-T)^{-1}\| \le (1-\|T\|)^{-1}$, and $\|I - (I-T)^{-1}\| \le \frac{\|T\|}{1-\|T\|}$.

17. Consider the 3-term recursion $\alpha_2 v_{n+2} + \alpha_1 v_{n+1} + \alpha_0 v_n = 0$, $n = 0, 1, \ldots$. Give an explicit expression of v_n in terms of the starting values v_0, v_1 and the roots of $\alpha_2 z^2 + \alpha_1 z + \alpha_0 = 0$. Distinguish between the cases of having two different roots, or one double root.

18. Let the roots $z_1, \ldots, z_k$ of $\rho(z) = z^k + \alpha_{k-1}z^{k-1} + \ldots \alpha_1 z + \alpha_0$ be single and unequal to zero. Give the eigenvalues of the $k \times k$ companion matrix

$$A := \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots\ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{k-1} \end{bmatrix}$$

and determine corresponding eigenvectors.

Prove that $\sup_{p\in\mathbb{N}} \|A^p\| < \infty$ if and only if $|z_i| \le 1$ for $1 \le i \le k$.

What is the corresponding statement when ρ has one or more multiple roots?

19. We equip $C[a,b]$ with inner product $\langle f, g\rangle = \int_a^b w(x)f(x)g(x)dx$ where w is a valid weight function. With $q_0 := 1$, $q_1 := x - \frac{\langle x, q_0\rangle}{\langle q_0, q_0\rangle}$, we define

$$q_{n+1} := \Big(x - \frac{\langle xq_n, q_n\rangle}{\langle q_n, q_n\rangle}\Big)q_n - \frac{\langle xq_n, q_{n-1}\rangle}{\langle q_{n-1}, q_{n-1}\rangle}q_{n-1} \quad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

(a) Show inductively that $\{q_0, \ldots, q_n\}$ is an orthogonal basis for $\mathcal{P}_n$ with $q_n - x^n \in \mathcal{P}_{n-1}$.
Hints: Show $q_{n+1} \perp \mathcal{P}_{n-2}$, $q_{n+1} \perp q_n$, $q_{n+1} \perp q_{n-1}$.

(b) Prove that q_n has n different roots on (a, b).

(c) For $n \geq 1$, show that $\frac{\langle xq_n, q_{n-1}\rangle}{\langle q_{n-1}, q_{n-1}\rangle} = \frac{\langle q_n, q_n\rangle}{\langle q_{n-1}, q_{n-1}\rangle} > 0$.

(d) For $n \geq 1$, show that the roots of q_n and q_{n-1} *interlace*, meaning that between any pair of consecutive roots of q_n, there is a root of q_{n-1}, and that between any pair of consecutive roots of q_{n-1} there is a root of q_n.

Figure 3: Interlacing of roots of q_3 and q_4.

Hints: Noting that this property holds for $n = 1$ by definition, let it be true for some $n \geq 1$. Use 19c to show that at each root x of q_n, it holds that

$$q_{n+1}(x)q_{n-1}(x) < 0. \tag{5}$$

Conclude from $\lim_{|z|\to\infty} q_{n+1}(z)q_{n-1}(z) = \infty$ (why?), (5), and the induction hypothesis the existence of roots of q_{n+1} left and right of the interval spanned by the roots of q_n. Next, again from (5) and the induction hypothesis, conclude the existence of a root of q_{n+1} between any pair of consecutive roots of q_n.

20. Let $f \in C^{n+1}[a,b]$, and let p be the Lagrange interpolation polynomial of f on $\{x_0, \ldots, x_n\} \subset [a,b]$. In the expression

$$f(x) - p(x) = \pi_{n+1}(x)\frac{f^{(n+1)}(\xi(x))}{(n+1)!} \text{ for } x \in [a,b],$$

for $x \in \{x_0, \ldots, x_n\}$ the choice of $\xi(x) \in [a,b]$ is arbitrary. In this exercise it will be shown that there exists a choice which makes

$$x \mapsto f^{(n+1)}(\xi(x)) \in C[a,b], \tag{6}$$

and an application of this result will be given.

Note that $g(x) := \frac{f(x)-p(x)}{\pi_{n+1}(x)}(n+1)!$ is continuous at $x \in [a,b] \backslash \{x_0, \ldots, x_n\}$.

(a) Show that

$$\lim_{x \to x_i} g(x_i) = \frac{(n+1)!}{\prod_{j=1,\, j \neq i}^{n} (x_i - x_j)} (f'(x_i) - p'(x_i)),$$

meaning that g has a unique extension to a function in $C[a,b]$.

(b) Show that there exists a $\xi(x_i) \in [a,b]$ with $g(x_i) = f^{(n+1)}(\xi(x_i))$ (use the intermediate value theorem), so that (6) is valid.

(c) Using the definition of a derivative, show that $(\pi_{n+1}(x)g(x))'|_{x=x_i} = \pi'_{n+1}(x_i)g(x_i)$, and conclude that

$$f'(x_i) - p'(x_i) = \pi'_{n+1}(x_i) \frac{f^{(n+1)}(\xi(x_i))}{(n+1)!}.$$

(Note that this improves upon Book §6.5).

21. (Euler backward) To approximate the solution of the usual initial value problem, consider the scheme

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}).$$

$(y_0 = y(x_0))$.

(a) Define a truncation error T_n, and show that $T_n = \mathcal{O}(h)$.

(b) With $e_n := y(x_n) - y_n$, prove that when $hL_f < 1$, $|e_n| = \mathcal{O}(h)$, uniformly in $nh \leq X_M - x_0$.