

OPTIMAL PRECONDITIONING FOR PROBLEMS OF NEGATIVE ORDER

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ABSTRACT. Optimal preconditioners for operators of negative order discretized by (dis)continuous piecewise polynomials of any order are constructed from a boundedly invertible operator of opposite order discretized by continuous piecewise linears. Besides the cost of the application of the latter discretized operator, the other cost of the preconditioner scales linearly with the number of mesh cells. Compared to earlier proposals, the preconditioner has the following advantages: It does not require the inverse of a non-diagonal matrix; it applies without any mildly grading assumption on the mesh; and it does not require a barycentric refinement of the mesh underlying the trial space.

1. INTRODUCTION

1.1. Operator preconditioning. This paper is about the construction of preconditioners for discretized boundedly invertible linear operators of negative order using the concept of ‘operator preconditioning’ ([Hip06]). The idea is to precondition the discretized operator by a discretized operator of opposite order. This is an appealing idea, but it turns out that in order to get a uniformly well-conditioned system, as well as a preconditioner that can be implemented efficiently, the second discretization has to be carefully chosen dependent on the first one.

For a Hilbert space $\mathcal{H}$, and a densely embedded reflexive Banach space $\mathcal{W} \hookrightarrow \mathcal{H}$, consider the Gelfand triple

$$\mathcal{W} \hookrightarrow \mathcal{H} \simeq \mathcal{H}' \hookrightarrow \mathcal{W}'.$$

For A being a boundedly invertible coercive linear operator $\mathcal{W}' \rightarrow \mathcal{W}$, and $\mathcal{V}_\mathcal{T} \subset \mathcal{H}$ being a finite dimensional subspace of $\mathcal{W}'$, let $(A_\mathcal{T}v)(\tilde{v}) := (Av)(\tilde{v})$ ($v, \tilde{v} \in \mathcal{V}_\mathcal{T}$). For B being a boundedly invertible coercive linear operator $\mathcal{W} \rightarrow \mathcal{W}'$, and $\mathcal{W}_\mathcal{T}$ being a finite dimensional subspace of $\mathcal{W}$, let $(B_\mathcal{T}w)(\tilde{w}) := (Bw)(\tilde{w})$ ($w, \tilde{w} \in \mathcal{W}_\mathcal{T}$).

A typical example is given by the case that for the boundary Γ of some domain, $\mathcal{H} = L_2(\Gamma)$, $\mathcal{W} = H^{\frac{1}{2}}(\Gamma)$, A is the single layer integral operator, B the hypersingular integral operator, $\mathcal{T}$ is a partition from an infinite collection of partitions $\mathbb{T}$, $\mathcal{V}_\mathcal{T}$ is a trial space of discontinuous piecewise polynomials w.r.t. $\mathcal{T}$, and $\mathcal{W}_\mathcal{T}$ is a suitable subspace of $\mathcal{W}$, which thus cannot be equal to $\mathcal{V}_\mathcal{T}$. Besides as boundary integral equations, coercive linear operators of order -1 also appear in various domain decomposition type methods in the equations for normal fluxes on interfaces.

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In order to precondition $A_{\mathcal{T}} : \mathcal{V}_{\mathcal{T}} \rightarrow \mathcal{V}'_{\mathcal{T}}$ with $B_{\mathcal{T}} : \mathcal{W}_{\mathcal{T}} \rightarrow \mathcal{W}'_{\mathcal{T}}$ we need to be able to ‘identify’ $\mathcal{V}'_{\mathcal{T}}$ with $\mathcal{W}_{\mathcal{T}}$, similar to the identification of $\mathcal{W}''$ with $\mathcal{W}$. Let $\dim \mathcal{W}_{\mathcal{T}} = \dim \mathcal{V}_{\mathcal{T}}$ and

$$(1.1) \quad \inf_{\mathcal{T} \in \mathbb{T}} \inf_{0 \neq v \in \mathcal{V}_{\mathcal{T}}} \sup_{0 \neq w \in \mathcal{W}_{\mathcal{T}}} \frac{\langle v, w \rangle_{\mathcal{H}}}{\|v\|_{\mathcal{V}_{\mathcal{T}}} \|w\|_{\mathcal{W}_{\mathcal{T}}}} > 0.$$

Then $D_{\mathcal{T}}$ defined by $(D_{\mathcal{T}}v)(w) := \langle v, w \rangle_{\mathcal{H}}$ ($v \in \mathcal{V}_{\mathcal{T}}, w \in \mathcal{W}_{\mathcal{T}}$) is a uniformly boundedly invertible linear map $\mathcal{V}_{\mathcal{T}} \rightarrow \mathcal{W}'_{\mathcal{T}}$, and so its adjoint $D'_{\mathcal{T}}$ is such a map $\mathcal{W}'_{\mathcal{T}} \rightarrow \mathcal{V}_{\mathcal{T}}$. We conclude that the preconditioned system $D_{\mathcal{T}}^{-1} B_{\mathcal{T}} (D'_{\mathcal{T}})^{-1} A_{\mathcal{T}}$ is uniformly boundedly invertible $\mathcal{V}_{\mathcal{T}} \rightarrow \mathcal{V}_{\mathcal{T}}$.

Equipping $\mathcal{V}_{\mathcal{T}}$ and $\mathcal{W}_{\mathcal{T}}$ with bases $\Xi_{\mathcal{T}}$ and $\Psi_{\mathcal{T}}$, respectively, the matrix representation of the preconditioned system reads as $D_{\mathcal{T}}^{-1} B_{\mathcal{T}} D_{\mathcal{T}}^{-T} A_{\mathcal{T}}$, with ‘stiffness matrices’ $A_{\mathcal{T}} := (A_{\mathcal{T}} \Xi_{\mathcal{T}})(\Xi_{\mathcal{T}})$ and $B_{\mathcal{T}} := (B_{\mathcal{T}} \Psi_{\mathcal{T}})(\Psi_{\mathcal{T}})$, and ‘generalized mass matrix’ $D_{\mathcal{T}} := \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{\mathcal{H}}$. Regardless of the choice of the bases, the spectral condition number of this matrix is equal to that of $D_{\mathcal{T}}^{-1} B_{\mathcal{T}} (D'_{\mathcal{T}})^{-1} A_{\mathcal{T}}$, and thus uniformly bounded.

After an earlier proposal from [Ste02], the currently commonly followed construction of a suitable pair $(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}})$ is the one from [BC07]. Here $\mathcal{V}_{\mathcal{T}}$ is the space of piecewise constants w.r.t. a partition $\mathcal{T}$ of a two-dimensional domain or manifold equipped with the usual basis $\Xi_{\mathcal{T}}$, and $\mathcal{W}_{\mathcal{T}}$, defined as the span of a collection $\Psi_{\mathcal{T}}$, is a subspace of the space of continuous piecewise linears w.r.t. a barycentric refinement of $\mathcal{T}$ constructed by subdividing each triangle into 6 subtriangles by connecting its vertices and midpoints with its barycenter. In [HUT16] the inf-inf-sup condition (1.1) was demonstrated for families of partitions including locally refined ones that satisfy a certain mildly-grading condition from [Ste03].

A problem with the constructions from both [Ste02, BC07] is that the matrix $D_{\mathcal{T}}$ is *not* diagonal, so that its inverse has to be approximated. Knowing that $D_{\mathcal{T}}^{-1} B_{\mathcal{T}} D_{\mathcal{T}}^{-T}$ is *not* well-conditioned, because $A_{\mathcal{T}}$ is not whereas their product is uniformly well-conditioned, the accuracy with which $D_{\mathcal{T}}^{-1}$ has to be approximated such that it gives rise to a uniform preconditioner increases with an increasing (minimal) mesh-size.

1.2. Contributions from this paper. For the aforementioned $\mathcal{V}_{\mathcal{T}}$ and $\Xi_{\mathcal{T}}$, in this work a space $\mathcal{W}_{\mathcal{T}}$, given as the span of a collection $\Psi_{\mathcal{T}}$, will be constructed such that (1.1) is valid, and $D_{\mathcal{T}} = \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{\mathcal{H}}$ is *diagonal*. Thanks to the latter, the corresponding biorthogonal projector is *local*, which allows to demonstrate the inf-inf-sup stability *without any mildly grading assumption* on the partitions.

Each function in $\Psi_{\mathcal{T}}$ equals a function from the space $\mathcal{S}_{\mathcal{T}}^{0,1}$ of continuous piecewise linears w.r.t. $\mathcal{T}$, plus a linear combination of ‘bubble functions’ from a space denoted as $\mathcal{B}_{\mathcal{T}}$. Since the decomposition of $\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}}$ into $\mathcal{S}_{\mathcal{T}}^{0,1}$ and $\mathcal{B}_{\mathcal{T}}$ is stable w.r.t. the $\mathcal{W}$ -norm, instead of simply defining $(B_{\mathcal{T}}w)(\tilde{w}) := (Bw)(\tilde{w})$, a suitable boundedly invertible linear operator $B_{\mathcal{T}} : \mathcal{W}_{\mathcal{T}} \rightarrow \mathcal{W}'_{\mathcal{T}}$ will be constructed from a diagonal scaling on the bubble space and a boundedly invertible linear operator $B_{\mathcal{T}}^{\mathcal{S}^{0,1}} : \mathcal{S}_{\mathcal{T}}^{0,1} \rightarrow (\mathcal{S}_{\mathcal{T}}^{0,1})'$, e.g. $(B_{\mathcal{T}}^{\mathcal{S}^{0,1}}w)(\tilde{w}) := (Bw)(\tilde{w})$. The total cost of the resulting preconditioner is the sum of the cost of the application of $B_{\mathcal{T}}^{\mathcal{S}^{0,1}}$ plus a cost that scales linearly in $\#\mathcal{T}$. In any case for $\mathcal{T}$ being a uniform refinement of some initial coarse partition, a $B_{\mathcal{T}}^{\mathcal{S}^{0,1}}$ of multi-level type can be found ([BPV00]) whose cost scales linearly in $\#\mathcal{T}$. By this use of the stable decomposition, other

than in [Ste02, BC07], there is no need to discretize the hypersingular operator on a refinement of $\mathcal{T}$. The whole approach relies on *existence* of bubble functions with certain properties (which e.g. are verified by continuous piecewise linears w.r.t. the barycentric refinement), whereas these functions themselves do not enter the implementation.

The construction of the biorthogonal collection $\Psi_{\mathcal{T}}$, and with that of the preconditioner, is based on a general principle. It applies in any space dimension, and, as we will see, it applies equally well when $\mathcal{V}_{\mathcal{T}}$ is the space of *continuous piecewise linears*. Higher order discretizations will be covered as well.

The construction applies equally well on manifolds. The coefficients of the functions from $\Psi_{\mathcal{T}}$ in terms of functions from $\mathcal{S}_{\mathcal{T}}^{0,1}$ and the bubble functions are given as inner products between functions of $\mathcal{V}_{\mathcal{T}}$ and $\mathcal{S}_{\mathcal{T}}^{0,1}$. Since in the manifold case, however, generally these inner products cannot be evaluated exactly, we present an alternative slightly modified construction in which the true L_2 -inner product is replaced by a mesh-dependent one by an element-wise freezing of the Jacobian. It still yields a uniform preconditioner on general, possibly locally refined partitions, whereas now the formula for the expansion coefficients of the functions of $\Psi_{\mathcal{T}}$ that was derived in the domain case, applies verbatim in the manifold case.

1.3. Notations. In this work, by $\lambda \lesssim \mu$ we will mean that λ can be bounded by a multiple of μ , independently of parameters which λ and μ may depend on, with the sole exception of the space dimension d , or in the manifold case, on the parametrization of the manifold that is used to define the finite element spaces on it. Obviously, $\lambda \gtrsim \mu$ is defined as $\mu \lesssim \lambda$, and $\lambda \approx \mu$ as $\lambda \lesssim \mu$ and $\lambda \gtrsim \mu$.

For normed linear spaces $\mathcal{Y}$ and $\mathcal{Z}$, in this paper for convenience over $\mathbb{R}$, $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$ will denote the space of bounded linear mappings $\mathcal{Y} \rightarrow \mathcal{Z}$ endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Z})}$. The subset of invertible operators in $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$ with inverses in $\mathcal{L}(\mathcal{Z}, \mathcal{Y})$ will be denoted as $\mathcal{L}\text{is}(\mathcal{Y}, \mathcal{Z})$. The *condition number* of a $C \in \mathcal{L}\text{is}(\mathcal{Y}, \mathcal{Z})$ is defined as $\kappa_{\mathcal{Y}, \mathcal{Z}}(C) := \|C\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Z})} \|C^{-1}\|_{\mathcal{L}(\mathcal{Z}, \mathcal{Y})}$.

For $\mathcal{Y}$ a reflexive Banach space and $C \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$ being *coercive*, i.e.,

$$\inf_{0 \neq y \in \mathcal{Y}} \frac{(Cy)(y)}{\|y\|_{\mathcal{Y}}^2} > 0,$$

both C and $\mathfrak{R}(C) := \frac{1}{2}(C + C')$ are in $\mathcal{L}\text{is}(\mathcal{Y}, \mathcal{Y}')$ with

$$\|\mathfrak{R}(C)\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Y}')} \leq \|C\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Y}')},$$

$$\|C^{-1}\|_{\mathcal{L}(\mathcal{Y}', \mathcal{Y})} \leq \|\mathfrak{R}(C)^{-1}\|_{\mathcal{L}(\mathcal{Y}', \mathcal{Y})} = \left(\inf_{0 \neq y \in \mathcal{Y}} \frac{(Cy)(y)}{\|y\|_{\mathcal{Y}}^2} \right)^{-1}.$$

The set of coercive $C \in \mathcal{L}\text{is}(\mathcal{Y}, \mathcal{Y}')$ is denoted as $\mathcal{L}\text{is}_c(\mathcal{Y}, \mathcal{Y}')$. If $C \in \mathcal{L}\text{is}_c(\mathcal{Y}, \mathcal{Y}')$, then $C^{-1} \in \mathcal{L}\text{is}_c(\mathcal{Y}', \mathcal{Y})$ and $\|\mathfrak{R}(C^{-1})^{-1}\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Y}')} \leq \|C\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Y}')}^2 \|\mathfrak{R}(C)^{-1}\|_{\mathcal{L}(\mathcal{Y}', \mathcal{Y})}$.

Two countable collections $\Upsilon = (v_i)_i$ and $\tilde{\Upsilon} = (\tilde{v}_i)_i$ in a Hilbert space will be called *biorthogonal* when $\langle \Upsilon, \tilde{\Upsilon} \rangle = [\langle v_j, \tilde{v}_i \rangle]_{ij}$ is an *invertible diagonal* matrix, and *biorthonormal* when it is the *identity* matrix.

1.4. Organization. In Sect. 2 the general principles of operator preconditioning are recalled. In Sect. 3-6, it is applied to operators of negative order discretized with discontinuous piecewise polynomials, first in the domain- and then in the manifold-case, first for piecewise constants, and then for higher order polynomials

using an additive subspace correction method. In Sect. 7, the same program is followed for trial spaces of continuous piecewise polynomials.

In Sect. 8 we briefly discuss the case when the partitions underlying the trial space and that of the preconditioner are different. The application of this setting is given by the case that for a construction of a preconditioner of multi-level type one would like to have a partition that is built by recurrent refinements from some initial coarse partition, whereas the given partition underlying the trial space is not of this type.

Finally, in Sect. 9 we report on some numerical results obtained with the new preconditioner, and compare them with those obtained with the preconditioner from [BC07, HUT16].

2. OPERATOR PRECONDITIONING

The exposition in this section largely follows [Hip06, Sect. 2] closely. Let $\mathcal{V}, \mathcal{W}$ be reflexive Banach spaces. We will search a ‘preconditioner’ G for an $A \in \mathcal{L}is(\mathcal{V}', \mathcal{V})$, i.e. an operator $G \in \mathcal{L}is(\mathcal{V}', \mathcal{V})$ (whose application is ‘easy’ compared to that of A^{-1}). For future applications in Sect. 6 of constructing additive subspace correction methods, it is useful when additionally the preconditioner is coercive, i.e., being an operator in $\mathcal{L}is_c(\mathcal{V}', \mathcal{V})$. The following result is easily verified.

Proposition 2.1. *If $B \in \mathcal{L}is(\mathcal{W}, \mathcal{W}')$ and $D \in \mathcal{L}is(\mathcal{V}, \mathcal{W}')$, then*

$$G := D^{-1}B(D')^{-1} \in \mathcal{L}is(\mathcal{V}', \mathcal{V}),$$

and

$$\begin{aligned} \|G\|_{\mathcal{L}(\mathcal{V}', \mathcal{V})} &\leq \|D^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{V})}^2 \|B\|_{\mathcal{L}(\mathcal{W}, \mathcal{W}')} , \\ \|G^{-1}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} &\leq \|D\|_{\mathcal{L}(\mathcal{V}, \mathcal{W}')}^2 \|B^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{W})} . \end{aligned}$$

If even $B \in \mathcal{L}is_c(\mathcal{W}, \mathcal{W}')$, then $G \in \mathcal{L}is_c(\mathcal{V}', \mathcal{V})$, and

$$\|\Re(G)^{-1}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq \|D\|_{\mathcal{L}(\mathcal{V}, \mathcal{W}')}^2 \|\Re(B)^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{W})} .$$

Remark 2.2. We recall that by an application of the *closed range theorem*, $D \in \mathcal{L}(\mathcal{V}, \mathcal{W}')$ is in $\mathcal{L}is(\mathcal{V}, \mathcal{W}')$ if and only if for all $w \in \mathcal{W}$ there exists a $v \in \mathcal{V}$ with $(Dv)(w) \neq 0$, and

$$0 < \inf_{0 \neq v \in \mathcal{V}} \sup_{0 \neq w \in \mathcal{W}} \frac{(Dv)(w)}{\|v\|_{\mathcal{V}} \|w\|_{\mathcal{W}}} \quad \left(= \|D^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{V})}^{-1} \right).$$

In particular we are interested in finding a preconditioner $G_{\mathcal{T}}$, of the form $G_{\mathcal{T}} = D_{\mathcal{T}}^{-1}B_{\mathcal{T}}(D'_{\mathcal{T}})^{-1}$, for an operator $A_{\mathcal{T}} \in \mathcal{L}is(\mathcal{V}_{\mathcal{T}}, \mathcal{V}'_{\mathcal{T}})$ where $\mathcal{V}_{\mathcal{T}}$ is some finite dimensional space. For that goal, in view of Proposition 2.1 we search some finite dimensional space $\mathcal{W}_{\mathcal{T}}$ with

$$(2.1) \quad \dim \mathcal{W}_{\mathcal{T}} = \dim \mathcal{V}_{\mathcal{T}},$$

and operators $B_{\mathcal{T}} \in \mathcal{L}is(\mathcal{W}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$ and $D_{\mathcal{T}} \in \mathcal{L}is(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$.

A typical setting is that, for some reflexive Banach spaces $\mathcal{V}$ and $\mathcal{W}$, and operators $A \in \mathcal{L}is_c(\mathcal{V}, \mathcal{V}')$ and $B \in \mathcal{L}is_c(\mathcal{W}, \mathcal{W}')$, we have $\mathcal{V}_{\mathcal{T}} \subset \mathcal{V}$ (thus equipped with $\|\cdot\|_{\mathcal{V}}$), $(A_{\mathcal{T}}u)(v) := (Au)(v)$ and, for a suitable $\mathcal{W}_{\mathcal{T}} \subset \mathcal{W}$ (thus equipped with $\|\cdot\|_{\mathcal{W}}$),

take $(B_{\mathcal{T}}w)(z) := (Bw)(z)$. In this case $A_{\mathcal{T}} \in \mathcal{L}\text{is}_c(\mathcal{V}_{\mathcal{T}}, \mathcal{V}'_{\mathcal{T}})$ and $B_{\mathcal{T}} \in \mathcal{L}\text{is}_c(\mathcal{W}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$ with

$$\begin{aligned} \|A_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{V}'_{\mathcal{T}})} &\leq \|A\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}, \quad \|\Re(A_{\mathcal{T}})^{-1}\|_{\mathcal{L}(\mathcal{V}'_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})} \leq \|\Re(A)^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{V})}, \\ \|B_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})} &\leq \|B\|_{\mathcal{L}(\mathcal{W}, \mathcal{W}')}, \quad \|\Re(B_{\mathcal{T}})^{-1}\|_{\mathcal{L}(\mathcal{W}'_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}})} \leq \|\Re(B)^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{W})}. \end{aligned}$$

A possible construction of a suitable $D_{\mathcal{T}}$ is discussed in the next proposition.

Proposition 2.3 (Fortin projector ([For77])). *For some $D \in \mathcal{L}\text{is}(\mathcal{V}, \mathcal{W}')$, let $(D_{\mathcal{T}}v)(w) := (Dv)(w)$. Then $D_{\mathcal{T}} \in \mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$ and*

$$\|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})} \leq \|D\|_{\mathcal{L}(\mathcal{V}, \mathcal{W}')}. \quad (2.2)$$

Assuming (2.1), $D_{\mathcal{T}} \in \mathcal{L}\text{is}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$ if, and in case $\mathcal{W}$ is a Hilbert space, only if there exists a projector $P_{\mathcal{T}} \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ onto $\mathcal{W}_{\mathcal{T}}$ with $(D_{\mathcal{T}})(\text{Id} - P_{\mathcal{T}})\mathcal{W} = 0$, and

$$(2.2) \quad \|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}'_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})} \leq \|P_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}, \mathcal{W})} \|D^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{V})}.$$

Proof. The first statement is obvious. Now let us assume existence of a (Fortin) projector $P_{\mathcal{T}}$. Then for $v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$,

$$\begin{aligned} \|D^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{V})}^{-1} \|v_{\mathcal{T}}\|_{\mathcal{V}} &\leq \sup_{0 \neq w \in \mathcal{W}} \frac{(Dv_{\mathcal{T}})(w)}{\|w\|_{\mathcal{W}}} = \sup_{0 \neq w \in \mathcal{W}} \frac{(Dv_{\mathcal{T}})(P_{\mathcal{T}}w)}{\|w\|_{\mathcal{W}}} \\ &\leq \|P_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}, \mathcal{W})} \sup_{0 \neq w_{\mathcal{T}} \in \mathcal{W}_{\mathcal{T}}} \frac{(Dv_{\mathcal{T}})(w_{\mathcal{T}})}{\|w_{\mathcal{T}}\|_{\mathcal{W}}}, \end{aligned}$$

which together with Remark 2.2 and (2.1) shows that $D_{\mathcal{T}} \in \mathcal{L}\text{is}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$, in particular (2.2).

Conversely (cf. [Bra01, Remark 4.9]), assume $D_{\mathcal{T}} \in \mathcal{L}\text{is}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}'_{\mathcal{T}})$. Then given $w \in \mathcal{W}$, let $w_{\mathcal{T}}$ be the first component of the solution $(w_{\mathcal{T}}, v_{\mathcal{T}}) \in \mathcal{W}_{\mathcal{T}} \times \mathcal{V}_{\mathcal{T}}$ of the well-posed saddle point problem

$$\begin{aligned} \langle w_{\mathcal{T}}, z_{\mathcal{T}} \rangle_{\mathcal{W}} + (D_{\mathcal{T}}v_{\mathcal{T}})(z_{\mathcal{T}}) &= \langle w, z_{\mathcal{T}} \rangle_{\mathcal{W}} \quad (z_{\mathcal{T}} \in \mathcal{W}_{\mathcal{T}}), \\ (D_{\mathcal{T}}u_{\mathcal{T}})(w_{\mathcal{T}}) &= (D_{\mathcal{T}}u_{\mathcal{T}})(w) \quad (u_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}). \end{aligned}$$

Then $P_{\mathcal{T}} := w \mapsto w_{\mathcal{T}}$ is a valid Fortin projector. $\square$

In applications, one usually has a *family* of spaces $\mathcal{V}_{\mathcal{T}}$ and aims at a *uniform* preconditioner $G_{\mathcal{T}}$. In the setting of Proposition 2.3 it means that one searches a Fortin projector $P_{\mathcal{T}}$ such that $\|P_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}, \mathcal{W})}$ is *uniformly* bounded.

2.1. Implementation. Given a finite collection $\Upsilon = \{v\}_{v \in \Upsilon}$ in a linear space, we set the *synthesis operator*

$$\mathcal{F}_{\Upsilon} : \mathbb{R}^{\#\Upsilon} \rightarrow \text{span } \Upsilon : \mathbf{c} \mapsto \mathbf{c}^{\top} \Upsilon := \sum_{v \in \Upsilon} c_v v.$$

Equipping $\mathbb{R}^{\#\Upsilon}$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$, and identifying $(\mathbb{R}^{\#\Upsilon})'$ with $\mathbb{R}^{\#\Upsilon}$ using the corresponding Riesz map, we infer that the adjoint of $\mathcal{F}_{\Upsilon}$, known as the *analysis operator*, satisfies

$$\mathcal{F}'_{\Upsilon} : (\text{span } \Upsilon)' \rightarrow \mathbb{R}^{\#\Upsilon} : f \mapsto f(\Upsilon) := [f(v)]_{v \in \Upsilon}.$$

A collection Υ is a *basis* for its span when $\mathcal{F}_{\Upsilon} \in \mathcal{L}\text{is}(\mathbb{R}^{\#\Upsilon}, \text{span } \Upsilon)$ (and so $\mathcal{F}'_{\Upsilon} \in \mathcal{L}\text{is}((\text{span } \Upsilon)', \mathbb{R}^{\#\Upsilon})$.)

Now let $\Xi_{\mathcal{T}} = \{\xi\}_{\xi \in \Xi_{\mathcal{T}}}$ and $\Psi_{\mathcal{T}} = \{\psi\}_{\psi \in \Psi_{\mathcal{T}}}$ be bases for $\mathcal{V}_{\mathcal{T}}$ and $\mathcal{W}_{\mathcal{T}}$, respectively. Then in coordinates the preconditioned system reads as

$$\mathcal{F}_{\Xi_{\mathcal{T}}}^{-1} G_{\mathcal{T}} A_{\mathcal{T}} \mathcal{F}_{\Xi_{\mathcal{T}}} = \mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}} := \mathbf{D}_{\mathcal{T}}^{-1} \mathbf{B}_{\mathcal{T}} \mathbf{D}_{\mathcal{T}}^{-\top} \mathbf{A}_{\mathcal{T}},$$

where

$$\mathbf{A}_{\mathcal{T}} := \mathcal{F}'_{\Xi_{\mathcal{T}}} A_{\mathcal{T}} \mathcal{F}_{\Xi_{\mathcal{T}}}, \quad \mathbf{B}_{\mathcal{T}} := \mathcal{F}'_{\Psi_{\mathcal{T}}} B_{\mathcal{T}} \mathcal{F}_{\Psi_{\mathcal{T}}}, \quad \mathbf{D}_{\mathcal{T}} := \mathcal{F}'_{\Psi_{\mathcal{T}}} D_{\mathcal{T}} \mathcal{F}_{\Xi_{\mathcal{T}}}.$$

By identifying a map in $\mathcal{L}(\mathbb{R}^{\#\Xi_{\mathcal{T}}}, \mathbb{R}^{\#\Xi_{\mathcal{T}}})$ with a $\#\Xi_{\mathcal{T}} \times \#\Xi_{\mathcal{T}}$ matrix by equipping $\mathbb{R}^{\#\Xi_{\mathcal{T}}}$ with the canonical basis $\{e_{\xi}\}_{\xi \in \Xi_{\mathcal{T}}}$, one has

$$(\mathbf{A}_{\mathcal{T}})_{\xi\tilde{\xi}} = \langle \mathcal{F}'_{\Xi_{\mathcal{T}}} A_{\mathcal{T}} \mathcal{F}_{\Xi_{\mathcal{T}}} e_{\tilde{\xi}}, e_{\xi} \rangle = (A_{\mathcal{T}} \mathcal{F}_{\Xi_{\mathcal{T}}} e_{\tilde{\xi}})(\mathcal{F}_{\Xi_{\mathcal{T}}} e_{\xi}) = (A_{\mathcal{T}} \tilde{\xi})(\xi),$$

and similarly,

$$(\mathbf{B}_{\mathcal{T}})_{\psi\tilde{\psi}} = (B_{\mathcal{T}} \tilde{\psi})(\psi), \quad (\mathbf{D}_{\mathcal{T}})_{\psi\xi} = (D_{\mathcal{T}} \xi)(\psi).$$

Concerning $\mathbf{D}_{\mathcal{T}}$, preferably it is such that its inverse can be applied in linear complexity, as is the case when $\mathbf{D}_{\mathcal{T}}$ is *diagonal*.

Remark 2.4. Clearly $\sigma(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}) = \sigma(G_{\mathcal{T}} A_{\mathcal{T}})$ and so for the spectral condition number we have

$$\kappa_S(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}) := \rho(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}) \rho((\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}})^{-1}) \leq \kappa_{\mathcal{V}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}}(G_{\mathcal{T}} A_{\mathcal{T}}),$$

which thus holds true *independently* of the choice of the basis $\Xi_{\mathcal{T}}$ for $\mathcal{V}_{\mathcal{T}}$ (this in contrast to having an efficient implementation). Furthermore, in view of an application of Conjugate Gradients, in case $A_{\mathcal{T}}$ and $B_{\mathcal{T}}$ are coercive and *self-adjoint*, then $\mathbf{A}_{\mathcal{T}}$ and $\mathbf{G}_{\mathcal{T}}$ are symmetric and positive definite. Equipping $\mathbb{R}^{\dim \mathcal{V}_{\mathcal{T}}}$ with $\|\cdot\| := \|(\mathbf{G}_{\mathcal{T}})^{-\frac{1}{2}} \cdot\|$ or $\|\cdot\| := \|(\mathbf{A}_{\mathcal{T}})^{\frac{1}{2}} \cdot\|$, in that case we have

$$\kappa_{(\mathbb{R}^{\dim \mathcal{V}_{\mathcal{T}}}, \|\cdot\|), (\mathbb{R}^{\dim \mathcal{V}_{\mathcal{T}}}, \|\cdot\|)}(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}) = \kappa_S(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}}).$$

3. PRECONDITIONING AN OPERATOR OF NEGATIVE ORDER DISCRETIZED BY PIECEWISE CONSTANTS: CONSTRUCTION OF $\mathcal{W}_{\mathcal{T}}$ AND $\mathbf{D}_{\mathcal{T}}$.

For a bounded polytopal domain $\Omega \subset \mathbb{R}^d$, a measurable, closed, possibly empty $\gamma \subset \partial\Omega$, and an $s \in [0, 1]$, we take

$$\mathcal{W} := [L_2(\Omega), H_{0,\gamma}^1(\Omega)]_{s,2}, \quad \mathcal{V} := \mathcal{W}',$$

where $H_{0,\gamma}^1(\Omega)$ is the closure in $H^1(\Omega)$ of the $C^\infty(\Omega) \cap H^1(\Omega)$ functions that vanish at γ .¹ The role of $D \in \text{Lis}(\mathcal{V}, \mathcal{W}')$ in Proposition 2.3 is going to be played by the duality pairing

$$(Dv)(w) := \langle v, w \rangle_{L_2(\Omega)},$$

which satisfies $\|D\|_{\mathcal{L}(\mathcal{V}, \mathcal{W}')} = \|D^{-1}\|_{\mathcal{L}(\mathcal{W}', \mathcal{V})} = 1$.

Let $(\mathcal{T})_{\mathcal{T} \in \mathbb{T}}$ be a family of *conforming* partitions of Ω into (open) *uniformly shape regular* d -simplices, where we assume that γ is the (possibly empty) union of $(d-1)$ -faces of $T \in \mathcal{T}$. Thanks to the conformity and the uniform shape regularity, for $d > 1$ we know that neighbouring $T, T' \in \mathcal{T}$, i.e. $\bar{T} \cap \bar{T}' \neq \emptyset$, have uniformly comparable sizes. For $d = 1$, we impose this uniform '*K-mesh property*' explicitly.²

¹In the domain case, it is easy to generalize the results to $s \in [0, \frac{3}{2})$, or even to $s \in (-\frac{1}{2}, \frac{3}{2})$.

²For our convenience, throughout this paper we consider trial spaces w.r.t. conforming partitions into uniformly shape regular d -simplices. It will however become clear that families of non-conforming partitions into uniformly shape regular d -simplices or hyperrectangles that satisfy a uniform K -mesh property can be dealt with as well.

For $\mathcal{T} \in \mathbb{T}$, we define $N_{\mathcal{T}}$ as the set of vertices of $\mathcal{T}$ that are not on γ , and for $\nu \in N_{\mathcal{T}}$ we set its valence

$$d_{\mathcal{T},\nu} := \#\{T \in \mathcal{T} : \nu \in \overline{T}\}.$$

For $T \in \mathcal{T}$, with N_T denoting the set of its vertices we set $N_{\mathcal{T},T} := N_{\mathcal{T}} \cap N_T$, and define $h_T := |T|^{1/d}$.

We take

$$\boxed{\mathcal{V}_{\mathcal{T}}} = \mathcal{S}_{\mathcal{T}}^{-1,0} := \{u \in L_2(\Omega) : u|_T \in \mathcal{P}_0(T \in \mathcal{T})\} \subset \mathcal{V},$$

and, as a first ingredient in the construction of a suitable $\mathcal{W}_{\mathcal{T}}$,

$$\mathcal{S}_{\mathcal{T}}^{0,1} := \{u \in H_{0,\gamma}^1(\Omega) : u|_T \in \mathcal{P}_1(T \in \mathcal{T})\},$$

equipped with the usual bases

$$(3.1) \quad \Xi_{\mathcal{T}} = \{\xi_T : T \in \mathcal{T}\}, \quad \Phi_{\mathcal{T}} = \{\phi_{\mathcal{T},\nu} : \nu \in N_{\mathcal{T}}\},$$

respectively, defined by

$$(3.2) \quad \xi_T := \begin{cases} 1 & \text{on } T, \\ 0 & \text{on } \Omega \setminus T, \end{cases} \quad \phi_{\mathcal{T},\nu}(\nu') = \delta_{\nu,\nu'} \quad (\nu, \nu' \in N_{\mathcal{T}}).$$

We are going to construct a collection $\Psi_{\mathcal{T}} \subset H_{0,\gamma}^1(\Omega)$ that is biorthogonal to $\Xi_{\mathcal{T}}$, and such that

$$\boxed{\mathcal{W}_{\mathcal{T}}} := \text{span } \Psi_{\mathcal{T}} \subset \mathcal{W}$$

has ‘approximation properties’. These two properties of $\Psi_{\mathcal{T}}$ will allow us to construct a suitable Fortin projector, and they will give rise to a diagonal matrix $\mathbf{D}_{\mathcal{T}}$.

The construction of $\Psi_{\mathcal{T}}$ builds on two collections $\Theta_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}}$ in $H_{0,\gamma}^1(\Omega)$ whose cardinalities are equal to that of $\Xi_{\mathcal{T}}$, the first being biorthogonal to $\Xi_{\mathcal{T}}$, and the second having approximation properties and being inside $\mathcal{S}_{\mathcal{T}}^{0,1}$.³

Let $\Theta_{\mathcal{T}} = \{\theta_T : T \in \mathcal{T}\} \subset H_{0,\gamma}^1(\Omega)$ be such that

$$(3.3) \quad \langle \theta_T, \xi_{T'} \rangle_{L_2(\Omega)} \approx \delta_{TT'} \|\theta_T\|_{L_2(\Omega)} \|\xi_{T'}\|_{L_2(\Omega)}, \quad (T, T' \in \mathcal{T}).$$

An obvious construction of such $\Theta_{\mathcal{T}}$ will be presented shortly. Defining $\Sigma_{\mathcal{T}} = \{\sigma_{\mathcal{T},T} : T \in \mathcal{T}\} \subset \mathcal{S}_{\mathcal{T}}^{0,1}$ by

$$\sigma_{\mathcal{T},T} := \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1} \phi_{\mathcal{T},\nu},$$

we have

$$\sum_{T \in \mathcal{T}} \sigma_{\mathcal{T},T} = \sum_{\nu \in N_{\mathcal{T}}} \phi_{\mathcal{T},\nu},$$

being equal to the constant function 1 on $\Omega \setminus \bigcup_{\{T \in \mathcal{T} : \overline{T} \cap \gamma \neq \emptyset\}} \overline{T}$, which is an instance of an ‘approximation property’.

We now define

$$\Psi_{\mathcal{T}} := \{\psi_{\mathcal{T},T} : T \in \mathcal{T}\} \subset \mathcal{S}_{\mathcal{T}}^{0,1} \oplus \text{span } \Theta_{\mathcal{T}},$$

by

$$(3.4) \quad \psi_{\mathcal{T},T} = \sigma_{\mathcal{T},T} + \frac{\langle \mathbb{1} - \sigma_{\mathcal{T},T}, \xi_T \rangle_{L_2(\Omega)}}{\langle \theta_T, \xi_T \rangle_{L_2(\Omega)}} \theta_T - \sum_{T' \in \mathcal{T} \setminus \{T\}} \frac{\langle \sigma_{\mathcal{T},T}, \xi_{T'} \rangle_{L_2(\Omega)}}{\langle \theta_{T'}, \xi_{T'} \rangle_{L_2(\Omega)}} \theta_{T'},$$

³Or inside another subspace $\hat{\mathcal{S}}_{\mathcal{T}} \subset \mathcal{W}$ for which one is able to construct a $B_{\mathcal{T}}^{\hat{\mathcal{S}}} \in \mathcal{L}is_c(\hat{\mathcal{S}}_{\mathcal{T}}, \hat{\mathcal{S}}_{\mathcal{T}}')$ with uniformly bounded $\|B_{\mathcal{T}}^{\hat{\mathcal{S}}}\|_{\mathcal{L}(\hat{\mathcal{S}}_{\mathcal{T}}, \hat{\mathcal{S}}_{\mathcal{T}}')}$ and $\|\Re(B_{\mathcal{T}}^{\hat{\mathcal{S}}})^{-1}\|_{\mathcal{L}(\hat{\mathcal{S}}_{\mathcal{T}}', \hat{\mathcal{S}}_{\mathcal{T}})}$.

The third term at the right-hand side corrects $\sigma_{\mathcal{T},T}$ such that it becomes orthogonal to $\xi_{T'}$ for $T' \neq T$, whereas the second term restores the ‘approximation property’.

Lemma 3.1. *It holds that*

$$(3.5) \quad \sum_{T \in \mathcal{T}} \psi_{\mathcal{T},T} = \sum_{T \in \mathcal{T}} \sigma_{\mathcal{T},T} + \sum_{T \in \mathcal{T}} \frac{\langle \mathbb{1} - \sum_{T' \in \mathcal{T}} \sigma_{\mathcal{T},T'}, \xi_T \rangle_{L_2(\Omega)}}{\langle \theta_T, \xi_T \rangle_{L_2(\Omega)}} \theta_T,$$

and

$$(3.6) \quad \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{L_2(\Omega)} = \text{diag}\{\langle \mathbb{1}, \xi_T \rangle_{L_2(\Omega)} : T \in \mathcal{T}\}.$$

Proof. Writing $\mathbb{1} - \sigma_{\mathcal{T},T} = \sum_{T' \in \mathcal{T} \setminus \{T\}} \sigma_{\mathcal{T},T'} + (\mathbb{1} - \sum_{T' \in \mathcal{T}} \sigma_{\mathcal{T},T'})$, (3.5) follows from (3.4). The biorthonormality of $\Xi_{\mathcal{T}}$ and $\{\theta_T / \langle \theta_T, \xi_T \rangle_{L_2(\Omega)} : T \in \mathcal{T}\}$ shows (3.6). $\square$

An easy construction of $\Theta_{\mathcal{T}}$ that we consider is to take, on a reference d -simplex $\hat{T}$, some ‘bubble’ function $\theta_{\hat{T}} \in H_0^1(\hat{T}) \cap C(\hat{T})$ with $\theta_{\hat{T}} \geq 0$, that is symmetric in the barycentric coordinates with $\int_{\hat{T}} \theta_{\hat{T}} dx = |\hat{T}|$, and then for each d -simplex T , to set $\theta_T := \theta_{\hat{T}} \circ F_{\hat{T},T}$ with $F_{\hat{T},T} : T \rightarrow \hat{T}$ being an affine bijection. From $\int_T \theta_T dx = |T|$ and $\|\theta_T\|_{L_2(\Omega)} \approx \sqrt{|T|}$, (3.3) follows.⁴

Expanding $\sigma_{\mathcal{T},T}$ in terms of the nodal basis, and using that $\int_T \phi_{\mathcal{T},\nu} dx = \frac{|T|}{d+1}$, with this $\Theta_{\mathcal{T}}$ we arrive at the expression

$$(3.7) \quad \psi_{\mathcal{T},T} := \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1} \phi_{\mathcal{T},\nu} + \left(1 - \frac{1}{d+1} \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1}\right) \theta_T - \sum_{T' \in \mathcal{T} \setminus \{T\}} \left(\frac{1}{d+1} \sum_{\nu \in N_{\mathcal{T},T} \cap N_{\mathcal{T},T'}} d_{\mathcal{T},\nu}^{-1}\right) \theta_{T'}.$$

As a consequence of (3.6), the biorthogonal ‘Fortin’ projector $P_{\mathcal{T}} : L_2(\Omega) \rightarrow H_{0,\gamma}^1(\Omega)$ with $\text{ran } P_{\mathcal{T}} = \mathcal{W}_{\mathcal{T}}$ and $\text{ran}(\text{Id} - P_{\mathcal{T}}) = \mathcal{V}_{\mathcal{T}}^{\perp L_2(\Omega)}$ exists, and is given by

$$P_{\mathcal{T}} u = \sum_{T \in \mathcal{T}} \frac{\langle u, \xi_T \rangle_{L_2(\Omega)}}{\langle \mathbb{1}, \xi_T \rangle_{L_2(\Omega)}} \psi_{\mathcal{T},T}.$$

3.1. Boundedness of $P_{\mathcal{T}}$. To proceed, we list a few properties of the collections $\Xi_{\mathcal{T}}$, $\Theta_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}}$. For $T \in \mathcal{T}$, we set $\omega_{\mathcal{T}}^{(0)}(T) := \overline{T}$, and for $i = 0, 1, \dots$, define

$$R_{\mathcal{T}}^{(i+1)}(T) := \{T' \in \mathcal{T} : \overline{T'} \cap \omega_{\mathcal{T}}^{(i)}(T) \neq \emptyset\}, \quad \omega_{\mathcal{T}}^{(i+1)}(T) := \cup_{T' \in R_{\mathcal{T}}^{(i+1)}(T)} \overline{T'}.$$

It holds that

$$(3.8) \quad \text{supp } \xi_T \subset \omega_{\mathcal{T}}^{(0)}(T), \quad \|\xi_T\|_{L_2(\Omega)} \approx h_T^{d/2}, \quad \langle \mathbb{1}, \xi_T \rangle_{L_2(\Omega)} \approx h_T^d,$$

$$(3.9) \quad \text{supp } \sigma_{\mathcal{T},T} \subset \omega_{\mathcal{T}}^{(1)}(T), \quad \|\sigma_T\|_{L_2(\Omega)} \lesssim h_T^{d/2-k} \quad (k \in \{0, 1\}),$$

$$(3.10) \quad \text{supp } \theta_T \subset \omega_{\mathcal{T}}^{(0)}(T), \quad \|\theta_T\|_{H^1(\Omega)} \lesssim h_T^{-1} \|\theta_T\|_{L_2(\Omega)}.$$

From these properties we infer that

$$\text{supp } \psi_{\mathcal{T},T} \subset \omega_{\mathcal{T}}^{(1)}(T),$$

and, by additionally using (3.3), that for $k \in \{0, 1\}$

$$\left\| \frac{\langle \sigma_{\mathcal{T},T} - \delta_{TT'} \mathbb{1}, \xi_{T'} \rangle_{L_2(\Omega)}}{\langle \theta_{T'}, \xi_{T'} \rangle_{L_2(\Omega)}} \theta_{T'} \right\|_{H^k(\Omega)} \lesssim h_T^{-k} \|\sigma_{\mathcal{T},T} - \delta_{TT'} \mathbb{1}\|_{L_2(\text{supp } \xi_{T'})} \lesssim h^{d/2-k},$$

⁴Although the definition (3.4) of $\psi_{\mathcal{T},T}$ is independent of the scaling of $\theta_{T'}$ ’s, for convenience here we fixed some (arbitrary) scaling of these functions.

showing that

$$\|\psi_{\mathcal{T},T}\|_{H^k(\Omega)} \lesssim h_T^{d/2-k} \quad (k \in \{0,1\}).$$

Remark 3.2. Although we specified (to some extent) a collection $\Theta_{\mathcal{T}}$, we emphasize that in the end the definition of the preconditioner will *not* depend on the choice of $\Theta_{\mathcal{T}}$. Only the derivation of qualitative properties of this preconditioner builds on *existence* of a collection $\Theta_{\mathcal{T}}$ with properties (3.3), (3.10) (where $\omega_{\mathcal{T}}^{(0)}(T)$ can be read as $\omega_{\mathcal{T}}^{(\ell)}(T)$ for some constant ℓ), and the forthcoming (5.1)-(5.2).

Theorem 3.3. *It holds that $\sup_{T \in \mathbb{T}} \|P_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}, \mathcal{W})} < \infty$.*

Proof. We have

$$\begin{aligned} \|P_{\mathcal{T}}u\|_{H^k(T)} &\leq \sum_{T' \in R_{\mathcal{T}}^{(1)}(T)} \|\psi_{\mathcal{T},T'}\|_{H^k(T)} \frac{\|u\|_{L_2(T')} \|\xi_{T'}\|_{L_2(T')}}{|\langle \mathbb{1}, \xi_{T'} \rangle_{L_2(\Omega)}|} \\ (3.11) \quad &\lesssim h_T^{-k} \|u\|_{L_2(\omega_{\mathcal{T}}^{(1)}(T))} \quad (k \in \{0,1\}), \end{aligned}$$

which in particular shows that

$$(3.12) \quad \sup_{T \in \mathbb{T}} \|P_{\mathcal{T}}\|_{\mathcal{L}(L_2(\Omega), L_2(\Omega))} < \infty.$$

To continue, we revisit the construction of $\mathcal{W}_{\mathcal{T}}$ and its basis $\Psi_{\mathcal{T}}$ by temporarily including in $N_{\mathcal{T}}$, and thus in $N_{\mathcal{T},T}$ for $T \in \mathcal{T}$, also vertices of $\mathcal{T}$ that are on the Dirichlet boundary γ . Consequently, for the ‘new’ $\psi_{\mathcal{T},T}$, (3.5) shows that

$$(3.13) \quad \sum_{T \in \mathcal{T}} \psi_{\mathcal{T},T} = \sum_{\nu \in N_{\mathcal{T}}} \phi_{\mathcal{T},\nu} = \mathbb{1} \text{ on } \Omega.$$

For any $\nu \in N_{\mathcal{T}}$, we select an $(d-1)$ -face e of a $T \in \mathcal{T}$ with $\nu \in e$ and $e \subset \gamma$ if $\nu \in \gamma$, and define the functional

$$g_{\mathcal{T},\nu}(u) := \int_e u \, ds.$$

By the trace theorem and homogeneity arguments (see e.g [SZ90, (3.6)]), one infers that

$$(3.14) \quad |g_{\mathcal{T},\nu}(u)| \leq |e|^{-1} \|u\|_{L_1(e)} \lesssim h_T^{-\frac{d}{2}} \|u\|_{L_2(T)} + h_T^{-\frac{d}{2}+1} |u|_{H^1(T)}.$$

For $T \in \mathcal{T}$, we select a $\nu \in N_T$ with $\nu \in \gamma$ if $\overline{T} \cap \gamma \neq \emptyset$, and define

$$g_{\mathcal{T},T} := g_{\mathcal{T},\nu},$$

and a Scott-Zhang ([SZ90]) type interpolator $\Pi_{\mathcal{T}} : H^1(\Omega) \rightarrow \mathcal{W}_{\mathcal{T}}$ by

$$\Pi_{\mathcal{T}}u = \sum_{T \in \mathcal{T}} g_{\mathcal{T},T}(u) \psi_{\mathcal{T},T}.$$

It satisfies

$$\|\Pi_{\mathcal{T}}u\|_{H^k(T)} \lesssim h_T^{-k} \|u\|_{L_2(\omega_{\mathcal{T}}^{(2)}(T))} + h_T^{1-k} |u|_{H^1(\omega_{\mathcal{T}}^{(2)}(T))} \quad (k \in \{0,1\}).$$

Invoking (3.13) and using that $g_{\mathcal{T},T}(\mathbb{1}) = 1$, we infer that for $k \in \{0, 1\}$

$$\begin{aligned}
 \|(\text{Id} - \Pi_{\mathcal{T}})u\|_{H^k(T)} &= \inf_{p \in \mathcal{P}_0} \|(\text{Id} - \Pi_{\mathcal{T}})(u - p)\|_{H^k(T)} \\
 &\leq \inf_{p \in \mathcal{P}_0} \|u - p\|_{H^k(T)} + h_T^{-k} \|u - p\|_{L_2(\omega_{\mathcal{T}}^{(2)}(T))} + h_T^{1-k} |u|_{H^1(\omega_{\mathcal{T}}^{(2)}(T))} \\
 (3.15) \quad &\approx \inf_{p \in \mathcal{P}_0} h_T^{-k} \|u - p\|_{L_2(\omega_{\mathcal{T}}^{(2)}(T))} + h_T^{1-k} |u|_{H^1(\omega_{\mathcal{T}}^{(2)}(T))} \\
 &\approx h_T^{1-k} |u|_{H^1(\omega_{\mathcal{T}}^{(2)}(T))}
 \end{aligned}$$

by an application of the Bramble-Hilbert lemma (cf. [SZ90, (4.2)]).

Noting that the ‘new’ $\psi_{\mathcal{T},T}$ differs only from the ‘old’, original one when $\bar{T} \cap \gamma \neq \emptyset$, and that for those T and $u \in H_{0,\gamma}^1(\Omega)$ it holds that $g_{\mathcal{T},T}(u) = 0$, we conclude that $\text{ran } \Pi_{\mathcal{T}}|_{H_{0,\gamma}^1(\Omega)}$ is included in the original space $\mathcal{W}_{\mathcal{T}}$, which we consider again from here on. Using that $P_{\mathcal{T}}$ is a projector onto this $\mathcal{W}_{\mathcal{T}}$, for $u \in H_{0,\gamma}^1(\Omega)$ writing $P_{\mathcal{T}}u = \Pi_{\mathcal{T}}u + P_{\mathcal{T}}(\text{Id} - \Pi_{\mathcal{T}})u$, using (3.11) and (3.15) for $k \in \{0, 1\}$ we arrive at

$$\begin{aligned}
 \|P_{\mathcal{T}}u\|_{H^1(T)} &\lesssim \|\Pi_{\mathcal{T}}u\|_{H^1(T)} + h_T^{-1} \|(\text{Id} - \Pi_{\mathcal{T}})u\|_{L_2(\omega_{\mathcal{T}}^{(1)}(T))} \\
 &\lesssim \|u\|_{H^1(\omega_{\mathcal{T}}^{(2)}(T))} + h_T^{-1} \|(\text{Id} - \Pi_{\mathcal{T}})u\|_{L_2(\omega_{\mathcal{T}}^{(1)}(T))} \\
 &\lesssim \|u\|_{H^1(\omega_{\mathcal{T}}^{(3)}(T))},
 \end{aligned}$$

and consequently,

$$\sup_{\mathcal{T} \in \mathbb{T}} \|P_{\mathcal{T}}\|_{\mathcal{L}(H_{0,\gamma}^1(\Omega), H_{0,\gamma}^1(\Omega))} < \infty.$$

In combination with (3.12), the proof is completed by an application of the Riesz-Thorin interpolation theorem. $\square$

Defining $D_{\mathcal{T}}$ by $(D_{\mathcal{T}}v)(w) := (Dv)(w)$, from Proposition 2.3 we conclude that $D_{\mathcal{T}} \in \mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')$ with $\|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')} \leq 1$ and $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})} < \infty$, which result is thus valid *without any additional assumptions on the mesh grading*. The latter is a consequence of the fact that we were able to equip $\mathcal{V}_{\mathcal{T}}$ and $\mathcal{W}_{\mathcal{T}}$ with local biorthogonal bases. (Compare [Ste03, eq. (2.30)] for conditions on the mesh grading without having local biorthogonal bases). Additionally, the biorthogonality has the important advantage of the matrix

$$\boxed{D_{\mathcal{T}}} = \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{L_2(\Omega)} = \text{diag}\{|T| : T \in \mathcal{T}\}$$

being *diagonal*.

Remark 3.4. Other than the spaces $\mathcal{V}_{\mathcal{T}} = \mathcal{S}_{\mathcal{T}}^{-1,0}$, the spaces $\mathcal{W}_{\mathcal{T}}$ cannot be expected to be nested under refinements of $\mathcal{T}$. This hampers the use of these spaces for the construction of a biorthogonal (wavelet) decomposition of $\mathcal{V}$, or more generally, that of a ‘stable’ *bi*-orthogonal multilevel decomposition ([Dah96]) substituting for the common orthogonal multilevel decomposition that, for $s \geq \frac{1}{2}$, is known not to be stable in $\mathcal{V}$ ([Osw98]).

To see this non-nestedness, let us consider the $d = 1$ shift-invariant case. In case of nestedness, there would be a $\psi \in H_0^1(-1, 2)$ with $\sum_{j \in \mathbb{Z}} \psi(\cdot - j) = \mathbb{1}$ and $\int_{-1}^0 \psi dx = \int_1^2 \psi dx = 0$, such that for some constants c_i , $\psi = \sum_{i=-1}^2 c_i \psi(2 \cdot - i)$. Integrating this refinement equation over $(-1, 0)$ and $(1, 2)$ yields $c_{-1} = c_2 = 0$, which, by a repeated application of the refinement equation shows that $\text{supp } \psi \subseteq [0, 1]$. This contradicts with $\psi \in H^1(\mathbb{R})$ and $\sum_{j \in \mathbb{Z}} \psi(\cdot - j) = \mathbb{1}$.

4. CONSTRUCTION OF $\mathcal{W}_{\mathcal{T}}$ AND $\mathbf{D}_{\mathcal{T}}$ IN THE MANIFOLD CASE

Let Γ be a compact d -dimensional Lipschitz, piecewise smooth manifold in $\mathbb{R}^d$ with or without boundary $\partial\Gamma$. For some closed measurable $\gamma \subset \partial\Gamma$ and $s \in [0, 1]$, let

$$\mathcal{W} := [L_2(\Gamma), H_{0,\gamma}^1(\Gamma)]_{s,2}, \quad \mathcal{V} := \mathcal{W}'.$$

We assume that Γ is given as the essentially disjoint union of $\cup_{i=1}^p \overline{\kappa_i(\Omega_i)}$, with, for $1 \leq i \leq p$, $\kappa_i: \mathbb{R}^d \rightarrow \mathbb{R}^d$ being some smooth regular parametrization, and $\Omega_i \subset \mathbb{R}^d$ an open polytope. W.l.o.g. assuming that for $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$, we define

$$\kappa: \Omega := \cup_{i=1}^p \Omega_i \rightarrow \cup_{i=1}^p \kappa_i(\Omega_i) \text{ by } \kappa|_{\Omega_i} = \kappa_i.$$

Let $\mathbb{T}$ be a family of conforming partitions $\mathcal{T}$ of Γ such that, for $1 \leq i \leq p$, $\kappa^{-1}(\mathcal{T}) \cap \Omega_i$ is a uniformly shape regular conforming partition of Ω_i into d -simplices (that for $d = 1$ satisfies a uniform K -mesh property). We assume that γ is a (possibly empty) union of ‘faces’ of $T \in \mathcal{T}$ (i.e., sets of type $\kappa_i(e)$, where e is a $(d-1)$ -dimensional face of $\kappa_i^{-1}(T)$).

As in Sect. 3, for $\mathcal{T} \in \mathbb{T}$, we define $N_{\mathcal{T}}$ as the set of vertices of $\mathcal{T}$ that are not on γ , set $d_{\mathcal{T},\nu} := \#\{T \in \mathcal{T}: \nu \in T\}$, and for $T \in \mathcal{T}$, define $h_T := |T|^{1/d}$ and $N_{\mathcal{T},T} := N_{\mathcal{T}} \cap N_T$, with N_T being the set of the vertices of T .

We set

$$\begin{aligned} \boxed{\mathcal{V}_{\mathcal{T}}} &= \mathcal{S}_{\mathcal{T}}^{-1,0} := \{u \in L_2(\Gamma): u \circ \kappa|_{\kappa^{-1}(T)} \in \mathcal{P}_0(T \in \mathcal{T})\} \subset \mathcal{V}, \\ \mathcal{S}_{\mathcal{T}}^{0,1} &:= \{u \in H_{0,\gamma}^1(\Gamma): u \circ \kappa|_{\kappa^{-1}(T)} \in \mathcal{P}_1(T \in \mathcal{T})\}, \end{aligned}$$

equipped with $\Xi_{\mathcal{T}} = \{\xi_T: T \in \mathcal{T}\}$ and $\Phi_{\mathcal{T}} = \{\phi_{\mathcal{T},\nu}: \nu \in N_{\mathcal{T}}\}$, respectively, defined by $\xi_T := 1$ on T , $\xi_T := 0$ elsewhere, and $\phi_{\mathcal{T},\nu}(\nu') = \delta_{\nu,\nu'}$ ($\nu, \nu' \in N_{\mathcal{T}}$). Furthermore, we define $\Sigma_{\mathcal{T}} = \{\sigma_{\mathcal{T},T}: T \in \mathcal{T}\} \subset \mathcal{S}_{\mathcal{T}}^{0,1}$ and $\Theta_{\mathcal{T}} = \{\theta_T: T \in \mathcal{T}\} \subset H_{0,\gamma}^1(\Gamma)$ by $\sigma_{\mathcal{T},T} := \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1} \phi_{\mathcal{T},\nu}$, $\theta_T := \theta_{\kappa^{-1}(T)} \circ \kappa^{-1}$ on T and $\theta_T := 0$ elsewhere. Thanks to our assumption of $\theta_{\kappa^{-1}(T)} \geq 0$, it holds that $\langle \theta_T, \xi_T \rangle_{L_2(\Gamma)} \approx \langle \theta_{\kappa^{-1}(T)}, \xi_{\kappa^{-1}(T)} \rangle_{L_2(\kappa^{-1}(T))} \approx \|\theta_T\|_{L_2(\Gamma)} \|\xi_T\|_{L_2(\Gamma)}$ (cf. (3.3)).

Now defining $\Psi_{\mathcal{T}} := \{\psi_{\mathcal{T},T}: T \in \mathcal{T}\}$, and $\mathcal{W}_{\mathcal{T}} := \text{span } \Psi_{\mathcal{T}} \subset \mathcal{W}$, by

$$\psi_{\mathcal{T},T} := \sigma_{\mathcal{T},T} + \frac{\langle \mathbb{1} - \sigma_{\mathcal{T},T}, \xi_T \rangle_{L_2(\Gamma)}}{\langle \theta_T, \xi_T \rangle_{L_2(\Gamma)}} \theta_T - \sum_{T' \in \mathcal{T} \setminus \{T\}} \frac{\langle \sigma_{\mathcal{T},T}, \xi_{T'} \rangle_{L_2(\Gamma)}}{\langle \theta_{T'}, \xi_{T'} \rangle_{L_2(\Gamma)}} \theta_{T'},$$

$(Dv)(w) := \langle v, w \rangle_{L_2(\Gamma)}$, and $(D_{\mathcal{T}}v)(w) := (Dv)(w)$, the analysis from Sect. 3 applies verbatim by only changing $\langle \cdot, \cdot \rangle_{L_2(\Omega)}$ into $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$. It yields that $\|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')} \leq 1$, $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}')} < \infty$, and $\mathbf{D}_{\mathcal{T}} = \text{diag}\{\langle \mathbb{1}, \xi_T \rangle_{L_2(\Gamma)}: T \in \mathcal{T}\}$.

A hidden problem, however, is that the computation of $\mathbf{D}_{\mathcal{T}}$, and that of the scalar products $\langle \delta_{TT'} \mathbb{1} - \sigma_{\mathcal{T},T}, \xi_{T'} \rangle_{L_2(\Gamma)}$ involve integrals over Γ that generally have to be approximated using numerical quadrature. Recalling that, for $s > 0$, the preconditioner $\mathbf{G}_{\mathcal{T}} = \mathbf{D}_{\mathcal{T}}^{-1} \mathbf{B}_{\mathcal{T}} \mathbf{D}_{\mathcal{T}}^{-\top}$ is not a uniformly well-conditioned matrix, it is a priori not clear which quadrature errors are allowable, in particular when $\mathcal{T}$ is far from being quasi-uniform. For this reason, in the next subsection we propose a slightly modified construction of $\mathcal{W}_{\mathcal{T}}$ and $\mathbf{D}_{\mathcal{T}}$ that does not require the evaluation of integrals over Γ . (Also the scalar products $\langle \theta_T, \xi_T \rangle_{L_2(\Gamma)}$ involve integrals over Γ , but their accurate evaluation is not critical, cf. Remark 3.2.)

As a preparation, in the next lemma we present a non-standard inverse inequality on the family $(\mathcal{V}_{\mathcal{T}})_{\mathcal{T} \in \mathbb{T}}$. Proofs of this inequality for $d \leq 3$ can be found in

[DFG⁺04, GHS05]. It turns out that our construction of ‘local’ collections $\Psi_{\mathcal{T}}$ that are biorthogonal to $\Xi_{\mathcal{T}}$ and whose spans have approximation properties allows for a very simple proof.

Lemma 4.1 (inverse inequality). *With $h_{\mathcal{T}}|_T := h_T$, it holds that*

$$\|h_{\mathcal{T}}v_{\mathcal{T}}\|_{L_2(\Gamma)} \lesssim \|v_{\mathcal{T}}\|_{H_{0,\gamma}^1(\Gamma)'} \quad (v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}).$$

Proof. For $P_{\mathcal{T}} : L_2(\Omega) \rightarrow H_{0,\gamma}^1(\Gamma)$ defined by

$$P_{\mathcal{T}}u = \sum_{T \in \mathcal{T}} \frac{\langle u, \xi_T \rangle_{L_2(\Gamma)}}{\langle \mathbb{1}, \xi_T \rangle_{L_2(\Gamma)}} \psi_{\mathcal{T},T}.$$

we have $\text{ran } P_{\mathcal{T}} = \mathcal{W}_{\mathcal{T}}$ and $\text{ran}(\text{Id} - P_{\mathcal{T}}) = \mathcal{V}_{\mathcal{T}}^{\perp L_2(\Gamma)}$, and as follows from (3.11),

$$\|P_{\mathcal{T}}u\|_{H^1(\Gamma)} \lesssim \|h_{\mathcal{T}}^{-1}u\|_{L_2(\Gamma)} \quad (u \in L_2(\Gamma)).$$

The proof is completed by

$$\|v_{\mathcal{T}}\|_{H_{0,\gamma}^1(\Gamma)'} = \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_{\mathcal{T}}, w \rangle_{L_2(\Gamma)}}{\|w\|_{H^1(\Gamma)}} \geq \frac{\langle v_{\mathcal{T}}, P_{\mathcal{T}}h_{\mathcal{T}}^2v_{\mathcal{T}} \rangle_{L_2(\Gamma)}}{\|P_{\mathcal{T}}h_{\mathcal{T}}^2v_{\mathcal{T}}\|_{H^1(\Gamma)}} \gtrsim \frac{\langle h_{\mathcal{T}}v_{\mathcal{T}}, h_{\mathcal{T}}v_{\mathcal{T}} \rangle_{L_2(\Gamma)}}{\|h_{\mathcal{T}}v_{\mathcal{T}}\|_{L_2(\Gamma)}}. \square$$

4.1. Modified construction. To avoid the need for the evaluation of integrals over Γ , given $\mathcal{T} \in \mathbb{T}$, on $L_2(\Gamma)$ we define an additional, ‘mesh-dependent’ scalar product

$$\langle u, v \rangle_{\mathcal{T}} := \sum_{T \in \mathcal{T}} \frac{|T|}{|\kappa^{-1}(T)|} \int_{\kappa^{-1}(T)} u(\kappa(x))v(\kappa(x))dx.$$

It is constructed from

$$\langle u, v \rangle_{L_2(\Gamma)} = \int_{\Omega} u(\kappa(x))v(\kappa(x))|\partial\kappa(x)|dx$$

by replacing on each $\kappa^{-1}(T)$, the Jacobian $|\partial\kappa|$ by its average $\frac{|T|}{|\kappa^{-1}(T)|}$ over $\kappa^{-1}(T)$.⁵

We now redefine $\Psi_{\mathcal{T}} := \{\psi_{\mathcal{T},T} : T \in \mathcal{T}\}$ and $\mathcal{W}_{\mathcal{T}} := \text{span } \Psi_{\mathcal{T}} \subset \mathcal{W}$ by

$$\psi_{\mathcal{T},T} := \sigma_{\mathcal{T},T} + \frac{\langle \mathbb{1} - \sigma_{\mathcal{T},T}, \xi_T \rangle_{\mathcal{T}}}{\langle \theta_T, \xi_T \rangle_{\mathcal{T}}} \theta_T - \sum_{T' \in \mathcal{T} \setminus \{T\}} \frac{\langle \sigma_{\mathcal{T},T}, \xi_{T'} \rangle_{\mathcal{T}}}{\langle \theta_{T'}, \xi_{T'} \rangle_{\mathcal{T}}} \theta_{T'},$$

and $(D_{\mathcal{T}}v_{\mathcal{T}})(w_{\mathcal{T}}) := \langle v_{\mathcal{T}}, w_{\mathcal{T}} \rangle_{\mathcal{T}}$ ($(v_{\mathcal{T}}, w_{\mathcal{T}}) \in \mathcal{V}_{\mathcal{T}} \times \mathcal{W}_{\mathcal{T}}$). Then, as in the domain case,

$$D_{\mathcal{T}} = \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{\mathcal{T}} = \text{diag}\{\langle \mathbb{1}, \xi_T \rangle_{\mathcal{T}} : T \in \mathcal{T}\} = \text{diag}\{|T| : T \in \mathcal{T}\},$$

and

$$(4.1) \quad \psi_{\mathcal{T},T} = \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1} \phi_{\mathcal{T},\nu} + \left(1 - \frac{1}{d+1} \sum_{\nu \in N_{\mathcal{T},T}} d_{\mathcal{T},\nu}^{-1}\right) \theta_T - \sum_{T' \in \mathcal{T} \setminus \{T\}} \left(\frac{1}{d+1} \sum_{\nu \in N_{\mathcal{T},T} \cap N_{\mathcal{T},T'}} d_{\mathcal{T},\nu}^{-1}\right) \theta_{T'}.$$

thus with coefficients that are *independent* of κ .

What remains is to prove the uniform boundedness of $\|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')}$, and that of $\|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})}$. Because of the definition of $D_{\mathcal{T}}$ in terms of the mesh-dependent scalar product, for doing so we cannot simply rely on Proposition 2.3.

⁵It will be clear from the following that $\frac{|T|}{|\kappa^{-1}(T)|}$ can be read as *any* constant approximation to $|\partial\kappa|$ on $L_{\infty}(\kappa^{-1}(T))$ -distance $\lesssim h_{\kappa^{-1}(T)}$, for example $|\partial\kappa(z)|$ in some $z \in \kappa^{-1}(T)$. Then in the following, the volumes $|T|$ in the expression for $D_{\mathcal{T}}$ should be read as $|\kappa^{-1}(T)||\partial\kappa(z)|$, with which also the computation of $|T|$ is avoided.

Lemma 4.2. *It holds that $\sup_{T \in \mathbb{T}} \|D_T\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}'_T)} < \infty$.*

Proof. In case $s = 0$, i.e., when $\mathcal{W} = L_2(\Gamma) \simeq L_2(\Gamma)' = \mathcal{V}$, the uniform boundedness of $\|D_T\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}'_T)}$ follows directly from $\langle \cdot, \cdot \rangle_T \simeq \|\cdot\|_{L_2(\Gamma)}^2$.

By an interpolation argument in the following it suffices to consider the case $s = 1$, i.e., $\mathcal{W} = H_{0,\gamma}^1(\Gamma)$ and $\mathcal{V} = H_{0,\gamma}^1(\Gamma)'$. By definition of $\langle \cdot, \cdot \rangle_T$, it holds that

$$(4.2) \quad |\langle v, u \rangle_T - \langle v, u \rangle_{L_2(\Gamma)}| \lesssim \|h_T v\|_{L_2(\Gamma)} \|u\|_{L_2(\Gamma)} \quad (v, u \in L_2(\Gamma)).$$

By writing $(D_T v_T)(w_T) = \langle v_T, w_T \rangle_{L_2(\Gamma)} + \langle v_T, w_T \rangle_T - \langle v_T, w_T \rangle_{L_2(\Gamma)}$, the uniform boundedness of $\|D_T\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}'_T)}$ (for $s = 1$) now follows by combining (4.2) and Lemma 4.1. $\square$

The $\langle \cdot, \cdot \rangle_T$ -biorthogonal projector $\check{P}_T : L_2(\Omega) \rightarrow H_{0,\gamma}^1(\Omega)$ with $\text{ran } \check{P}_T = \mathcal{W}_T$ and $\text{ran}(\text{Id} - \check{P}_T) = \mathcal{V}_T^{\perp(\cdot, \cdot)_T}$ exists and is given by $\check{P}_T u = \sum_{T \in \mathbb{T}} |T|^{-1} \langle u, \xi_T \rangle_T \psi_{T,T}$. Since $\langle \cdot, \cdot \rangle_T$ gives rise to a norm that is uniformly equivalent to $\|\cdot\|_{L_2(\Gamma)}$, the proof of Theorem 3.3 again applies, and shows that

$$\sup_{T \in \mathbb{T}} \|\check{P}_T\|_{\mathcal{L}(L_2(\Gamma), L_2(\Gamma))} < \infty, \quad \sup_{T \in \mathbb{T}} \|\check{P}_T\|_{\mathcal{L}(H_{0,\gamma}^1(\Gamma), H_{0,\gamma}^1(\Gamma))} < \infty,$$

as well as

$$(4.3) \quad \|\check{P}_T u\|_{H^1(\Gamma)} \lesssim \|h_T^{-1} u\|_{L_2(\Gamma)} \quad (u \in L_2(\Gamma)).$$

Uniform boundedness of $\|D_T^{-1}\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}'_T)}$, in case $s = 0$, follows from

$$(D_T v_T)(\check{P}_T v_T) = \langle v_T, v_T \rangle_T \simeq \|v_T\|_{L_2(\Gamma)}^2 \gtrsim \|v_T\|_{L_2(\Gamma)} \|\check{P}_T v_T\|_{L_2(\Gamma)}.$$

To conclude, by an interpolation argument, uniform boundedness of $\|D_T^{-1}\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}'_T)}$ for any $s \in [0, 1]$, it is sufficient to verify the case $s = 1$, which can be done with the following modified inverse inequality.

Lemma 4.3. *It holds that*

$$\|h_T v_T\|_{L_2(\Gamma)} \lesssim \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_T, w \rangle_T}{\|w\|_{H^1(\Gamma)}} \quad (v_T \in \mathcal{V}_T).$$

Proof. Similar to proof of Lemma 4.1, using (4.3) for $v_T \in \mathcal{V}_T$ we estimate

$$\sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_T, w \rangle_T}{\|w\|_{H^1(\Gamma)}} \geq \frac{\langle v_T, \check{P}_T h_T^2 v_T \rangle_T}{\|\check{P}_T h_T^2 v_T\|_{H^1(\Gamma)}} \gtrsim \frac{\langle h_T v_T, h_T v_T \rangle_T}{\|h_T v_T\|_{L_2(\Gamma)}} \simeq \|h_T v_T\|_{L_2(\Gamma)}. \quad \square$$

Corollary 4.4. *It holds that*

$$\|v_T\|_{H_{0,\gamma}^1(\Gamma)'} \simeq \sup_{0 \neq w_T \in \mathcal{W}_T} \frac{\langle v_T, w_T \rangle_T}{\|w_T\|_{H^1(\Gamma)}} \quad (v_T \in \mathcal{V}_T),$$

(with ' $\lesssim$ ' being the statement $\sup_{T \in \mathbb{T}} \|D_T^{-1}\|_{\mathcal{L}(\mathcal{W}'_T, \mathcal{V}_T)} < \infty$ for $s = 1$).

Proof. The inequality ' $\gtrsim$ ' is the statement of Lemma 4.2 for $s = 1$.

To prove the other direction, for $v \in L_2(\Gamma)$, (4.2) shows that

$$\|v\|_{H_{0,\gamma}^1(\Gamma)'} = \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v, w \rangle_T}{\|w\|_{H^1(\Gamma)}} \lesssim \|h_T v\|_{L_2(\Gamma)},$$

Taking $v = v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$, from Lemma 4.3 we conclude that

$$\begin{aligned} \|v_{\mathcal{T}}\|_{H_{0,\gamma}^1(\Gamma)'} &\lesssim \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_{\mathcal{T}}, w \rangle_{\mathcal{T}}}{\|w\|_{H^1(\Gamma)}} = \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_{\mathcal{T}}, \check{P}_{\mathcal{T}} w \rangle_{\mathcal{T}}}{\|w\|_{H^1(\Gamma)}} \\ &\leq \|\check{P}_{\mathcal{T}}\|_{\mathcal{L}(H_{0,\gamma}^1(\Gamma), H_{0,\gamma}^1(\Gamma))} \sup_{0 \neq w_{\mathcal{T}} \in \mathcal{W}_{\mathcal{T}}} \frac{\langle v_{\mathcal{T}}, w_{\mathcal{T}} \rangle_{\mathcal{T}}}{\|w_{\mathcal{T}}\|_{H^1(\Gamma)}} \lesssim \sup_{0 \neq w_{\mathcal{T}} \in \mathcal{W}_{\mathcal{T}}} \frac{\langle v_{\mathcal{T}}, w_{\mathcal{T}} \rangle_{\mathcal{T}}}{\|w_{\mathcal{T}}\|_{H^1(\Gamma)}} \end{aligned}$$

by $\sup_{\mathcal{T} \in \mathbb{T}} \|\check{P}_{\mathcal{T}}\|_{\mathcal{L}(H_{0,\gamma}^1(\Gamma), H_{0,\gamma}^1(\Gamma))} < \infty$. $\square$

5. CONSTRUCTION OF $B_{\mathcal{T}} \in \mathcal{L}\text{is}_c(\mathcal{W}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')$.

Having established $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')} < \infty$, $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})} < \infty$, for the construction of uniform preconditioners it remains to find $B_{\mathcal{T}} \in \mathcal{L}\text{is}_c(\mathcal{W}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')$ with $\sup_{\mathcal{T} \in \mathbb{T}} \|B_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')} < \infty$, $\sup_{\mathcal{T} \in \mathbb{T}} \|\Re(B_{\mathcal{T}})^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}', \mathcal{W}_{\mathcal{T}})} < \infty$.

We will make use of the following two properties of the collection of ‘bubbles’ $\Theta_{\mathcal{T}}$ and its span $\mathcal{B}_{\mathcal{T}} := \text{span } \Theta_{\mathcal{T}}$. It holds that

$$(5.1) \quad \left\| \sum_{T \in \mathcal{T}} c_T \theta_T \right\|_{H^k(\Omega)}^2 \approx \sum_{T \in \mathcal{T}} h_T^{d-2k} |c_T|^2, \quad (k \in \{0, 1\}),$$

(here the selected scaling of the functions θ_T entered, cf. footnote 4), and

$$(5.2) \quad \|u + v\|_{H^k(\Omega)}^2 \gtrsim \|u\|_{H^k(\Omega)}^2 + \|v\|_{H^k(\Omega)}^2 \quad (u \in \mathcal{S}_{\mathcal{T}}^{0,1}, v \in \mathcal{B}_{\mathcal{T}}).$$

Both properties are easily verified by a standard homogeneity argument, for (5.2) using that $\theta_{\hat{T}} \notin P_1(\hat{T})$. (Here and in the following, Ω should be read as Γ in the manifold case).

Below we give a construction of suitable $B_{\mathcal{T}}$ that is *independent* of the particular bubble $\theta_{\hat{T}}$ being chosen. Like $\mathcal{W}_{\mathcal{T}}$, we equip $\mathcal{S}_{\mathcal{T}}^{0,1}$, $\mathcal{B}_{\mathcal{T}}$, and $\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}}$ with $\|\cdot\|_{\mathcal{W}}$.

Proposition 5.1. *Given $B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}} \in \mathcal{L}\text{is}_c(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')$ and $B_{\mathcal{T}}^{\mathcal{B}} \in \mathcal{L}\text{is}_c(\mathcal{B}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}')$, let $B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}} : \mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}} \rightarrow (\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}})'$ be defined by*

$$(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}}(u + v))(\tilde{u} + \tilde{v}) = (B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}} u)(\tilde{u}) + (B_{\mathcal{T}}^{\mathcal{B}} v)(\tilde{v}).$$

Then $B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}} \in \mathcal{L}\text{is}_c(\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}}, (\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}})'),$ and

$$\begin{aligned} &\|\Re(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}})^{-1}\|_{\mathcal{L}((\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}})', \mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}})} \\ &\leq 2 \max(\|\Re(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}})^{-1}\|_{\mathcal{L}((\mathcal{S}_{\mathcal{T}}^{0,1})', \mathcal{S}_{\mathcal{T}}^{0,1})}, \|\Re(B_{\mathcal{T}}^{\mathcal{B}})^{-1}\|_{\mathcal{L}(\mathcal{B}_{\mathcal{T}}', \mathcal{B}_{\mathcal{T}})}), \\ &\|B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}}\|_{\mathcal{L}(\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}}, (\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}})')} \lesssim \max(\|B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}}\|_{\mathcal{L}(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')}, \|B_{\mathcal{T}}^{\mathcal{B}}\|_{\mathcal{L}(\mathcal{B}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}')}). \end{aligned}$$

Proof. One has

$$\begin{aligned} |(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}}(u + v))(u + v)| &\geq \min(\|\Re(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}})^{-1}\|_{\mathcal{L}((\mathcal{S}_{\mathcal{T}}^{0,1})', \mathcal{S}_{\mathcal{T}}^{0,1})}^{-1}, \|\Re(B_{\mathcal{T}}^{\mathcal{B}})^{-1}\|_{\mathcal{L}(\mathcal{B}_{\mathcal{T}}', \mathcal{B}_{\mathcal{T}})}^{-1}) \\ &\quad \times (\|u\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2), \end{aligned}$$

and $\|u\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2 \geq \frac{1}{2}\|u + v\|_{\mathcal{W}}^2$. Secondly,

$$\begin{aligned} |(B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}}(u + v))(\tilde{u} + \tilde{v})| &\leq \max(\|B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}}\|_{\mathcal{L}(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')}, \|B_{\mathcal{T}}^{\mathcal{B}}\|_{\mathcal{L}(\mathcal{B}_{\mathcal{T}}, \mathcal{B}_{\mathcal{T}}')}) \\ &\quad \times \sqrt{\|u\|_{\mathcal{W}}^2 + \|v\|_{\mathcal{W}}^2} \sqrt{\|\tilde{u}\|_{\mathcal{W}}^2 + \|\tilde{v}\|_{\mathcal{W}}^2}, \end{aligned}$$

which by (5.2), combined with an interpolation argument, completes the proof. $\square$

Since the splitting of a $u \in \mathcal{W}_T \subset \mathcal{S}_T^{0,1} \oplus \mathcal{B}_T$, given in terms of its basis $\{\psi_{T,T} : T \in \mathcal{T}\}$, into its components in $\mathcal{S}_T^{0,1}$ and $\mathcal{B}_T$ w.r.t. the nodal- or bubble-basis can be easily determined in linear complexity, a suitable definition of $B_T : \mathcal{W}_T \rightarrow \mathcal{W}'_T$ is given by $(B_T w)(\tilde{w}) = (B_T^{\mathcal{S}_T^{0,1} \oplus \mathcal{B}} w)(\tilde{w})$. Obviously,

$$\begin{aligned} \|B_T\|_{\mathcal{L}(\mathcal{W}_T, \mathcal{W}'_T)} &\leq \|B_T^{\mathcal{S}_T^{0,1} \oplus \mathcal{B}}\|_{\mathcal{L}(\mathcal{S}_T^{0,1} \oplus \mathcal{B}_T, (\mathcal{S}_T^{0,1} \oplus \mathcal{B}_T)'),} \\ \|\Re(B_T)^{-1}\|_{\mathcal{L}(\mathcal{W}'_T, \mathcal{W}_T)} &\leq \|\Re(B_T^{\mathcal{S}_T^{0,1} \oplus \mathcal{B}})^{-1}\|_{\mathcal{L}((\mathcal{S}_T^{0,1} \oplus \mathcal{B}_T)', \mathcal{S}_T^{0,1} \oplus \mathcal{B}_T)}. \end{aligned}$$

A choice for $B_T^{\mathcal{B}} \in \mathcal{L}is_c(\mathcal{B}_T, (\mathcal{B}_T)')$ such that

$$\sup_{T \in \mathbb{T}} \|B_T^{\mathcal{B}}\|_{\mathcal{L}(\mathcal{B}_T, \mathcal{B}'_T)} < \infty, \quad \sup_{T \in \mathbb{T}} \|\Re(B_T^{\mathcal{B}})^{-1}\|_{\mathcal{L}(\mathcal{B}'_T, \mathcal{B}_T)} < \infty,$$

is, in view (5.1), given by

$$(5.3) \quad (B_T^{\mathcal{B}} \sum_{T \in \mathcal{T}} c_T \theta_T) \left(\sum_{T \in \mathcal{T}} d_T \theta_T \right) := \beta_0 \sum_{T \in \mathcal{T}} h_T^{d-2s} c_T d_T.$$

for some constant $\beta_0 > 0$.

Possible choices for $B_T^{\mathcal{S}_T^{0,1}} \in \mathcal{L}is_c(\mathcal{S}_T^{0,1}, (\mathcal{S}_T^{0,1})')$ with

$$\sup_{T \in \mathbb{T}} \|B_T^{\mathcal{S}_T^{0,1}}\|_{\mathcal{L}(\mathcal{S}_T^{0,1}, (\mathcal{S}_T^{0,1})')} < \infty, \quad \sup_{T \in \mathbb{T}} \|\Re(B_T^{\mathcal{S}_T^{0,1}})^{-1}\|_{\mathcal{L}((\mathcal{S}_T^{0,1})', \mathcal{S}_T^{0,1})} < \infty$$

include $(B_T^{\mathcal{S}_T^{0,1}} u)(v) := (Bu)(v)$ ($u, v \in \mathcal{S}_T^{0,1}$) for some $B \in \mathcal{L}is_c(\mathcal{W}, \mathcal{W}')$. For $d \in \{2, 3\}$ and $\mathcal{W} = H_{00}^{\frac{1}{2}}(\Gamma) := [L_2(\Gamma), H_0^1(\Gamma)]_{\frac{1}{2}, 2}$, one may take the hypersingular integral operator for B , whereas for $\partial\Gamma \neq \emptyset$, and $\mathcal{W} = H^{\frac{1}{2}}(\Gamma) = [L_2(\Gamma), H^1(\Gamma)]_{\frac{1}{2}, 2}$ the recently introduced modified hypersingular integral operator can be applied (see [HJHUT18]). (Note that $H_0^1(\Gamma) = H^1(\Gamma)$ when $\partial\Gamma = \emptyset$.)

For a family of quasi-uniform partitions generated by a repeated application of *uniform refinements* starting from some given initial partition, a computationally attractive alternative is provided by multi-level preconditioner from [BPV00], whose application can be performed in *linear complexity*.

5.1. Implementation. For both the domain case and the construction in the manifold case in Subsection 4.1, the matrix representation $\mathbf{G}_T = \mathcal{F}_{\Xi_T}^{-1} G_T (\mathcal{F}'_{\Xi_T})^{-1}$ of our preconditioner G_T reads as $\boxed{\mathbf{G}_T = D_T^{-1} B_T D_T^{-\top}}$ with

$$\boxed{D_T} = \text{diag}\{|T| : T \in \mathcal{T}\},$$

and

$$\begin{aligned} \boxed{B_T} &:= \mathcal{F}'_{\Psi_T} B_T \mathcal{F}_{\Psi_T} \\ &= \mathcal{F}'_{\Psi_T} (I'_T B_T^{\mathcal{S}_T^{0,1}} I_T + (\text{Id} - I_T)' B_T^{\mathcal{B}} (\text{Id} - I_T)) \mathcal{F}_{\Psi_T} \\ &= \boxed{p_T^\top B_T^{\mathcal{S}_T^{0,1}} p_T + q_T^\top B_T^{\mathcal{B}} q_T}, \end{aligned}$$

where $I_{\mathcal{T}}$ is the projector from $\mathcal{S}_{\mathcal{T}}^{0,1} \oplus \mathcal{B}_{\mathcal{T}}$ onto $\mathcal{S}_{\mathcal{T}}^{0,1}$ with $\text{ran}(\text{Id} - I_{\mathcal{T}}) = \mathcal{B}_{\mathcal{T}}$

$$\begin{aligned} B_{\mathcal{T}}^{\mathcal{S}^{0,1}} &:= \mathcal{F}'_{\Phi_{\mathcal{T}}} B_{\mathcal{T}}^{\mathcal{S}^{0,1}} \mathcal{F}_{\Phi_{\mathcal{T}}}, \quad p_{\mathcal{T}} := \mathcal{F}_{\Phi_{\mathcal{T}}}^{-1} I_{\mathcal{T}} \mathcal{F}_{\Psi_{\mathcal{T}}}, \\ B_{\mathcal{T}}^{\mathcal{B}} &:= \mathcal{F}'_{\Theta_{\mathcal{T}}} B_{\mathcal{T}}^{\mathcal{B}} \mathcal{F}_{\Theta_{\mathcal{T}}}, \quad q_{\mathcal{T}} := \mathcal{F}_{\Theta_{\mathcal{T}}}^{-1} (\text{Id} - I_{\mathcal{T}}) \mathcal{F}_{\Psi_{\mathcal{T}}}. \end{aligned}$$

By substituting the definition of $B_{\mathcal{T}}^{\mathcal{B}}$ from (5.3), the definition of the basis $\Psi_{\mathcal{T}} = \{\psi_{\mathcal{T},T}\}_{T \in \mathcal{T}}$ for $\mathcal{W}_{\mathcal{T}}$ from (3.7) and (4.1), and that of the bases $\Phi_{\mathcal{T}} = \{\phi_{\mathcal{T},\nu}\}_{\nu \in N_{\mathcal{T}}}$ and $\Theta_{\mathcal{T}} = \{\theta_{\mathcal{T}}\}_{T \in \mathcal{T}}$ for $\mathcal{S}_{\mathcal{T}}^{0,1}$ and $\mathcal{B}_{\mathcal{T}}$, respectively, we find that

$$\begin{aligned} \boxed{B_{\mathcal{T}}^{\mathcal{B}}} &= \beta_0 D_{\mathcal{T}}^{1-\frac{2s}{d}}, \quad \boxed{(p_{\mathcal{T}})_{\nu T}} = \begin{cases} d_{\mathcal{T},\nu}^{-1} & \text{if } \nu \in N_{\mathcal{T},T}, \\ 0 & \text{if } \nu \notin N_{\mathcal{T},T}, \end{cases} \\ \boxed{(q_{\mathcal{T}})_{T'T}} &= \delta_{T'T} - \frac{1}{d+1} \sum_{\nu \in N_{\mathcal{T},T} \cap N_{\mathcal{T},T'}} d_{\mathcal{T},\nu}^{-1}, \end{aligned}$$

whereas $\boxed{B_{\mathcal{T}}^{\mathcal{S}^{0,1}}}$ depends on $B_{\mathcal{T}}^{\mathcal{S}^{0,1}} \in \mathcal{L}\text{is}_c(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')$ being chosen.

Remark 5.2. So far we determined a suitable constant β_0 by comparing different choices numerically. A possible choice for β_0 is to pick it such that $\rho(q_{\mathcal{T}}^{\top} B_{\mathcal{T}}^{\mathcal{B}} q_{\mathcal{T}} A_{\mathcal{T}}) = \rho(p_{\mathcal{T}}^{\top} B_{\mathcal{T}}^{\mathcal{S}^{0,1}} p_{\mathcal{T}} A_{\mathcal{T}})$. Another option would be to replace $\beta_0 |T|^{1-\frac{2s}{d}}$ by (an approximation for) $\|\theta_{\mathcal{T}}\|_{\mathcal{W}}^2$. In that case, however, the non-trivial question arises which choice of the bubbles $\theta_{\mathcal{T}}$ would give the best results.

6. HIGHER ORDER CASE

In this section we denote the space $\mathcal{S}_{\mathcal{T}}^{-1,0}$ as $\mathcal{V}_{\mathcal{T}}^0$, we write its basis $\Xi_{\mathcal{T}}$ and biorthogonal collection $\Psi_{\mathcal{T}}$ as $\Xi_{\mathcal{T}}^0$ and $\Psi_{\mathcal{T}}^0$, respectively, and the preconditioner $G_{\mathcal{T}}$ that we have developed as $G_{\mathcal{T}}^0$.

Let $(\mathcal{V}_{\mathcal{T}})_{\mathcal{T} \in \mathbb{T}}$ be a family of finite dimensional spaces with $\mathcal{V}_{\mathcal{T}}^0 \subseteq \mathcal{V}_{\mathcal{T}} \subset \mathcal{V}$, that satisfies the following *inverse inequality*

$$(6.1) \quad \|h_{\mathcal{T}} \cdot\|_{L_2(\Omega)} \lesssim \|\cdot\|_{H_{0,\gamma}^1(\Omega)'} \quad \text{on } \mathcal{V}_{\mathcal{T}},$$

i.e., the inequality proven for $\mathcal{V}_{\mathcal{T}}^0$ in Lemma 4.1. For $\mathcal{T} \in \mathbb{T}$, let $A_{\mathcal{T}} \in \mathcal{L}\text{is}(\mathcal{V}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}')$ with $\sup_{\mathcal{T} \in \mathbb{T}} \kappa_{\mathcal{V}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}'}(A_{\mathcal{T}}) < \infty$.

Remark 6.1. For the most relevant example where for some (fixed) $\ell \in \{1, 2, \dots\}$

$$(6.2) \quad \mathcal{V}_{\mathcal{T}} \subseteq \mathcal{S}_{\mathcal{T}}^{-1,\ell} := \{u \in L_2(\Omega) : u|_T \in \mathcal{P}_{\ell}(T)\},$$

a proof of (6.1) for $d \in \{2, 3\}$ ($d = 1$ causes no difficulties) can be found in [GHS05, Thm. 3.6 (and Rem. 3.8 when $\gamma \neq \emptyset$)], which applies under our minimal assumptions imposed on the family $\mathbb{T}$. Alternatively, it is not difficult to see that $\Psi_{\mathcal{T}}^0$ can be enlarged to a ‘local’ collection that is biorthogonal to an enlargement of $\Xi_{\mathcal{T}}^0$ to a basis for $\{u \in L_2(\Omega) : u|_T \in \mathcal{P}_{\ell}(T)\}$. Then the same proof as for Lemma 4.1 shows (6.1).

In order to construct a preconditioner $G_{\mathcal{T}} \in \mathcal{L}\text{is}(\mathcal{V}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})$ we will follow the classical approach of *subspace correction methods*. We are going to decompose, in a uniformly ‘stable’ way, $\mathcal{V}_{\mathcal{T}}$ into $\mathcal{V}_{\mathcal{T}}^0$ and a complement space $\mathcal{V}_{\mathcal{T}}^1$. On $\mathcal{V}_{\mathcal{T}}^0$ we will apply our preconditioner $G_{\mathcal{T}}^0 \in \mathcal{L}\text{is}_c((\mathcal{V}_{\mathcal{T}}^0)', \mathcal{V}_{\mathcal{T}}^0)$, whereas on the complement space a diagonal preconditioner will suffice.

To do so, an obvious option would be to decompose $\mathcal{V}_{\mathcal{T}}$ using the biorthogonal projector $P'_{\mathcal{T}}$ onto $\mathcal{V}_{\mathcal{T}}^0$ that we know is uniformly bounded on $\mathcal{V}$ (since its adjoint

$P_{\mathcal{T}}$ is unif. bounded on $\mathcal{W}$, see Thm. 3.3). A computationally more efficient implementation, however, will be yielded by using the $L_2(\Omega)$ -orthogonal projector $Q_{\mathcal{T}}^0$ onto $\mathcal{V}_{\mathcal{T}}^0$ instead. Although for $s \geq \frac{1}{2}$ it is known *not* to be uniformly bounded on $\mathcal{V}$, restricted to $\mathcal{V}_{\mathcal{T}}$ the projector $Q_{\mathcal{T}}^0$ is uniformly bounded in $\mathcal{V}$:

Lemma 6.2. *It holds that*

$$\sup_{\mathcal{T} \in \mathbb{T}} \|Q_{\mathcal{T}}^0|_{\mathcal{V}_{\mathcal{T}}}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})} < \infty.$$

Furthermore, there exist constants $M, m > 0$ s.t. with $\mathcal{V}_{\mathcal{T}}^1 := \text{ran}((\text{Id} - Q_{\mathcal{T}}^0)|_{\mathcal{V}_{\mathcal{T}}})$,

$$m\|\cdot\|_{\mathcal{V}}^2 \leq \sum_{T \in \mathcal{T}} h_T^{2s} \|\cdot\|_{L_2(T)}^2 \leq M\|\cdot\|_{\mathcal{V}}^2 \quad \text{on } \mathcal{V}_{\mathcal{T}}^1.$$

Proof. For $u \in \mathcal{V}_{\mathcal{T}}$, thanks to (6.1) we have

$$\begin{aligned} \|(\text{Id} - Q_{\mathcal{T}}^0)u\|_{H_{0,\gamma}^1(\Omega)'} &= \sup_{v \in H_{0,\gamma}^1(\Omega)} \frac{\langle u, (\text{Id} - Q_{\mathcal{T}}^0)v \rangle_{L_2(\Omega)}}{\|v\|_{H^1(\Omega)}} \lesssim \sqrt{\sum_{T \in \mathcal{T}} \|h_T u\|_{L_2(T)}^2} \\ &\lesssim \|u\|_{H_{0,\gamma}^1(\Omega)'}, \end{aligned}$$

so that the first statement follows by interpolation.

Again by interpolation, the inequalities need only to be proven for $s = 1$, and thus with $\mathcal{W}'$ reading as $H_{0,\gamma}^1(\Omega)'$. In that case the right inequality follows from (6.1), whereas the left inequality follows from the arguments applied at the beginning of this proof. $\square$

The following abstract result concerns so-called *additive subspace correction methods*. The result is well-known in the Hilbert space setting and with self-adjoint coercive preconditioners on the subspaces (e.g. [Xu92, Wid92, NS18, GO95], [Osw94, HX07]). **Present result is better than [HX07] for multiple subspaces, cf. [HX07, (2.14)]. In [Osw94] the E_i are trivial embeddings. Nowhere is the Banach space case handled.**

Proposition 6.3. *Let $\mathcal{U}$ and, for some set of i , $\mathcal{U}_i$ be reflexive Banach spaces. Let $E_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{U})$ be such that $\sum_i \text{ran } E_i = \mathcal{U}$, and let $G_i \in \mathcal{L}(\mathcal{U}_i', \mathcal{U}_i)$. Then $G := \sum_i E_i G_i E_i' \in \mathcal{L}(\mathcal{U}', \mathcal{U})$ with*

$$(6.3) \quad \|G\|_{\mathcal{L}(\mathcal{U}', \mathcal{U})} \leq \max_i \|G_i\|_{\mathcal{L}(\mathcal{U}_i', \mathcal{U}_i)} \left(\sup_{0 \neq u \in \mathcal{U}} \frac{\|u\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}'}} \right)^2,$$

$$(6.4) \quad \|\mathfrak{R}(G)^{-1}\|_{\mathcal{L}(\mathcal{U}, \mathcal{U}')} \leq \max_i \|\mathfrak{R}(G_i)^{-1}\|_{\mathcal{L}(\mathcal{U}_i, \mathcal{U}_i')} \left(\sup_{0 \neq u \in \mathcal{U}} \frac{\|u\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}'}} \right)^2,$$

where

$$\|u\|_{\mathcal{U}} := \inf_{\{(u_i)_i \in \prod_i \mathcal{U}_i : \sum_i E_i u_i = u\}} \sqrt{\sum_i \|u_i\|_{\mathcal{U}_i}^2}.$$

The proof of this proposition is given in the appendix.

Corollary 6.4. *For $k \in \{0, 1\}$, let $I_{\mathcal{T}}^k$ denote the trivial embedding of $\mathcal{V}_{\mathcal{T}}^k$ into $\mathcal{V}_{\mathcal{T}}$. Let $G_{\mathcal{T}}^0 \in \mathcal{L}(\mathcal{V}_{\mathcal{T}}^0, \mathcal{V}_{\mathcal{T}}^0)$, and let $G_{\mathcal{T}}^1 := R_{\mathcal{T}}^{-1}$ with $R_{\mathcal{T}} \in \mathcal{L}(\mathcal{V}_{\mathcal{T}}^1, (\mathcal{V}_{\mathcal{T}}^1)')$ be defined by $(R_{\mathcal{T}}u)(v) := \beta_1^{-1} \sum_{T \in \mathcal{T}} h_T^{2s} \langle u, v \rangle_{L_2(T)}$ for some constant $\beta_1 > 0$. Then*

$$G_{\mathcal{T}} := \sum_{k=0}^1 I_{\mathcal{T}}^k G_{\mathcal{T}}^k (I_{\mathcal{T}}^k)' \in \mathcal{L}(\mathcal{V}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}}),$$

and

$$\begin{aligned} \|G_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}'_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}})} &\leq 2 \max(\|G_{\mathcal{T}}^0\|_{\mathcal{L}((\mathcal{V}_{\mathcal{T}}^{-1,0})', \mathcal{V}_{\mathcal{T}}^{-1,0})}, \beta_1 m^{-1}), \\ \|\Re(G_{\mathcal{T}})^{-1}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{V}'_{\mathcal{T}})} &\leq 2\|Q_{\mathcal{T}}^0|_{\mathcal{V}_{\mathcal{T}}}\|_{\mathcal{L}(\mathcal{V}', \mathcal{V}')} \max(\|\Re(G_{\mathcal{T}}^0)^{-1}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}^{-1,0}, (\mathcal{V}_{\mathcal{T}}^{-1,0})')}, \beta_1^{-1}M). \end{aligned}$$

Proof. Using that $R_{\mathcal{T}} = R'_{\mathcal{T}}$, from Lemma 6.2 one infers that $\|G_{\mathcal{T}}^1\|_{\mathcal{L}((\mathcal{V}_{\mathcal{T}}^1)', \mathcal{V}_{\mathcal{T}}^1)} \leq \beta_1 m^{-1}$ and $\|\Re(G_{\mathcal{T}}^1)^{-1}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}^1, (\mathcal{V}_{\mathcal{T}}^1)')} \leq \beta_1^{-1}M$.

Now from $\frac{1}{2}\sqrt{2}\|u\|_{\mathcal{V}_{\mathcal{T}}} \leq \sqrt{\|Q_{\mathcal{T}}^0 u\|_{\mathcal{V}_{\mathcal{T}}}^2 + \|(\text{Id} - Q_{\mathcal{T}}^0)u\|_{\mathcal{V}_{\mathcal{T}}}^2} \leq \sqrt{2}\|Q_{\mathcal{T}}^0|_{\mathcal{V}_{\mathcal{T}}}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})}\|u\|_{\mathcal{V}_{\mathcal{T}}}$ ($u \in \mathcal{V}_{\mathcal{T}}$), where we used that $\|(\text{Id} - Q_{\mathcal{T}}^0)|_{\mathcal{V}_{\mathcal{T}}}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})} = \|Q_{\mathcal{T}}^0|_{\mathcal{V}_{\mathcal{T}}}\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})}$, the proof is completed by an application of Proposition 6.3. $\square$

Remark 6.5. Although at first sight Proposition 5.1 might look like another application of Proposition 6.3, note the reversed roles of the ‘primal’ space and its adjoint.

6.1. Implementation. With $\Xi_{\mathcal{T}}^0$, $\Xi_{\mathcal{T}}^1$, and $\Xi_{\mathcal{T}}$ being bases for $\mathcal{V}_{\mathcal{T}}^0$, $\mathcal{V}_{\mathcal{T}}^1$, and $\mathcal{V}_{\mathcal{T}}$, respectively, the matrix representation $G_{\mathcal{T}} = \mathcal{F}_{\Xi_{\mathcal{T}}}^{-1} G_{\mathcal{T}} (\mathcal{F}'_{\Xi_{\mathcal{T}}})^{-1}$ of the preconditioner from Corollary 6.4 reads as

$$\boxed{G_{\mathcal{T}}} = \sum_{k=0}^1 p_{\mathcal{T}}^k G_{\mathcal{T}}^k (p_{\mathcal{T}}^k)^{\top},$$

where

$$p_{\mathcal{T}}^k := \mathcal{F}_{\Xi_{\mathcal{T}}}^{-1} I_{\mathcal{T}}^k \mathcal{F}_{\Xi_{\mathcal{T}}^k}, \quad G_{\mathcal{T}}^k = \mathcal{F}_{\Xi_{\mathcal{T}}^k}^{-1} G_{\mathcal{T}}^k (\mathcal{F}'_{\Xi_{\mathcal{T}}^k})^{-1}.$$

We take $\Xi_{\mathcal{T}}^0$ from (3.1)-(3.2). Then $\boxed{G_{\mathcal{T}}^0}$ corresponding our preconditioner $G_{\mathcal{T}}^0 \in \mathcal{L}_{\text{isc}}((\mathcal{V}_{\mathcal{T}}^0)', \mathcal{V}_{\mathcal{T}}^0)$ has been given in Sect. 5.1.

Now let $\Xi_{\mathcal{T}} = \{\xi_{T,i}\}_{T \in \mathcal{T}, i=0, \dots, N}$ be an $L_2(\Omega)$ -orthogonal basis for $\mathcal{V}_{\mathcal{T}}$ such that

$$\xi_{T,0} = \xi_T, \quad \text{supp } \xi_{T,i} \subset T, \quad \|\xi_{T,i}\|_{L_2(T)} = |T|^{\frac{1}{2}},$$

so that $\Xi_{\mathcal{T}}^1 = \{\xi_{T,i}\}_{T \in \mathcal{T}, i=1, \dots, N}$ is an $L_2(\Omega)$ -orthogonal basis for $\mathcal{V}_{\mathcal{T}}^1$. One infers that

$$\boxed{G_{\mathcal{T}}^1} = \beta_1 \text{diag}[|T|^{-(1+\frac{2s}{d})} \text{Id}_{(N-1) \times (N-1)}]_{T \in \mathcal{T}},$$

and

$$\boxed{p_{\mathcal{T}}^0} = \text{diag}[e_1^{\top}]_{T \in \mathcal{T}}, \quad \boxed{p_{\mathcal{T}}^1} = \text{diag}[e_2^{\top}, \dots, e_N^{\top}]_{T \in \mathcal{T}}.$$

Remark 6.6. We determine a suitable constant β_1 by comparing different choices numerically (cf. Rem 5.2). A possible candidate for β_1 is found by picking it such that $\rho(p_{\mathcal{T}}^0 G_{\mathcal{T}}^0 (p_{\mathcal{T}}^0)^{\top} A_{\mathcal{T}}) = \rho(p_{\mathcal{T}}^1 G_{\mathcal{T}}^1 (p_{\mathcal{T}}^1)^{\top} A_{\mathcal{T}})$.

Remark 6.7. In the manifold case, one may prefer to avoid computing an $L_2(\Gamma)$ -orthogonal basis for $\mathcal{V}_{\mathcal{T}}$. In that case, similar to Sect. 4.1, the results in this section remain valid when on all places $\langle u, v \rangle_{L_2(T)}$ reads as $\frac{|T|}{|\kappa^{-1}(T)|} \int_{\kappa^{-1}(T)} u(\kappa(x)) v(\kappa(x)) dx$, and so $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ as $\langle \cdot, \cdot \rangle_{\mathcal{T}}$, and the $L_2(\Gamma)$ -orthogonal projector onto $\mathcal{V}_{\mathcal{T}}^0$ reads as the $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ -orthogonal projector onto $\mathcal{V}_{\mathcal{T}}^0$.

7. PRECONDITIONING AN OPERATOR OF NEGATIVE ORDER DISCRETIZED BY CONTINUOUS
PIECEWISE POLYNOMIALS.

7.1. Construction of $\mathcal{W}_T$ and D_T . Let a bounded polytopal domain $\Omega \subset \mathbb{R}^d$, $\gamma \subset \partial\Omega$, $s \in [0, 1]$, $\mathcal{W} := [L_2(\Omega), H_{0,\gamma}^1(\Omega)]_{s,2}$, $\mathcal{V} := \mathcal{W}'$, $D \in \mathcal{L}(\mathcal{V}, \mathcal{W}')$, $(T)_{T \in \mathbb{T}}$, N_T , $d_{T,\nu}$, N_T , $N_{T,T}$ and h_T be all as in Sect. 3. In addition, for $T \in \mathbb{T}$ let $\check{N}_T$ be the set of all vertices of T , and for $\nu \in \check{N}_T$, let $\omega_T(\nu) := \cup_{\{T \in \mathcal{T}: \nu \in N_T\}} \bar{T}$.

We take

$$\boxed{\mathcal{V}_T} = \mathcal{S}_T^{0,1} := \{u \in H^1(\Omega) : u|_T \in \mathcal{P}_1(T \in \mathcal{T})\} \subset \mathcal{V},$$

and, as in Sect. 3,

$$\mathcal{S}_T^{0,1} := \{u \in H_{0,\gamma}^1(\Omega) : u|_T \in \mathcal{P}_1(T \in \mathcal{T})\},$$

equipped with nodal bases $\Xi_T = \{\xi_{T,\nu} : \nu \in \check{N}_T\}$ and $\Phi_T = \{\phi_{T,\nu} : \nu \in N_T\}$, respectively, defined by

$$\xi_{T,\nu}(\nu') = \delta_{\nu,\nu'} \quad (\nu, \nu' \in \check{N}_T),$$

and $\phi_{T,\nu} = \xi_{T,\nu}$ for $\nu \in N_T$. Analogously to the case of *discontinuous* piecewise polynomial trial spaces in $\mathcal{V}$ studied in Sect. 3-6, using the framework of operator preconditioning outlined in Sect. 2 we are going to construct a family of preconditioners $G_T \in \mathcal{L}(\mathcal{V}_T', \mathcal{V}_T)$ of type $D_T^{-1} B_T (D_T')^{-1}$ with uniformly bounded norms $\|G_T\|_{\mathcal{L}(\mathcal{V}_T', \mathcal{V}_T)}$ and $\|\Re(G_T)^{-1}\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{V}_T')}$.

To construct a collection $\Psi_T = \{\psi_{T,\nu} : \nu \in \check{N}_T\} \subset H_{0,\gamma}^1(\Omega)$ that both is biorthogonal to Ξ_T and for which

$$\boxed{\mathcal{W}_T := \text{span } \Psi_T \subset \mathcal{W}}$$

has ‘approximation properties’, as in Sect. 3 we need two collections $\Sigma_T \subset \mathcal{S}_T^{0,1}$ and $\Theta_T \subset H_{0,\gamma}^1(\Omega)$, where Θ_T is biorthogonal to Ξ_T , and Σ_T has ‘approximation properties’ and $\#\Sigma_T = \#\Xi_T$.

We define $\Sigma_T = \{\sigma_{T,\nu} : \nu \in \check{N}_T\}$ by $\sigma_{T,\nu} := \phi_{T,\nu}$ when $\nu \in N_T$, and $\sigma_{T,\nu} := 0$ when $\nu \in \check{N}_T \setminus N_T$. Then, obviously, $\sum_{\nu \in \check{N}_T} \sigma_{T,\nu}$ equals $\mathbb{1}$ on $\Omega \setminus \cup_{\{T \in \mathcal{T}: \bar{T} \cap \gamma \neq \emptyset\}} \bar{T}$.

For constructing Θ_T , on a reference d -simplex $\hat{T}$, for $\varepsilon > 0$ we consider a smooth $\eta_\varepsilon \in [0, 1]$, symmetric in the barycentric coordinates, with $\eta_\varepsilon(x) = 0$ when $d(x, \partial\hat{T}) < \varepsilon$, and $\eta_\varepsilon(x) = 1$ when $d(x, \partial\hat{T}) > 2\varepsilon$. Then for some fixed $\varepsilon > 0$ small enough, it holds that

$$(7.1) \quad \inf_{0 \neq p \in \mathcal{P}_1(\hat{T})} \sup_{0 \neq q \in \mathcal{P}_1(\hat{T})} \frac{\langle p, \eta_\varepsilon q \rangle_{L_2(\hat{T})}}{\|p\|_{L_2(\hat{T})} \|\eta_\varepsilon q\|_{L_2(\hat{T})}} > 0,$$

meaning that the biorthogonal projector $P_\varepsilon \in \mathcal{L}(L_2(\hat{T}), L_2(\hat{T}))$ with $\text{ran } P_\varepsilon = \eta_\varepsilon \mathcal{P}_1(\hat{T})$ and $\text{ran}(\text{Id} - P_\varepsilon) = \mathcal{P}_1(\hat{T})^{\perp_{L_2(\hat{T})}}$ exists. Consequently, with $\Phi_{\hat{T}} = \{\phi_{\hat{T},\nu} : \nu \in N_{\hat{T}}\}$ being the nodal basis for $\mathcal{P}_1(\hat{T})$, we have that

$$\{\tilde{\phi}_{\hat{T},\varepsilon,\nu} : \nu \in N_{\hat{T}}\} := \langle \Phi_{\hat{T}}, \Phi_{\hat{T}} \rangle_{L_2(\hat{T})}^{-1} P_\varepsilon \Phi_{\hat{T}} \subset H_0^1(\hat{T})$$

is $L_2(\hat{T})$ -biorthonormal to $\{\phi_{\hat{T},\nu} : \nu \in N_{\hat{T}}\}$.

Now for $T \in \mathcal{T}$, let $F_{\hat{T},T} : \hat{T} \rightarrow T$ be an affine bijection. Then $\{\tilde{\phi}_{T,\varepsilon,\nu} : \nu \in N_T\}$ defined by

$$(7.2) \quad \tilde{\phi}_{T,\varepsilon,\nu} := \frac{|\hat{T}|}{|T|} \tilde{\phi}_{\hat{T},\varepsilon,F_{\hat{T},T}(\nu)}$$

is $L_2(T)$ -biorthonormal to the nodal basis for $P_1(T)$.

By selecting for $\nu \in \check{N}_T$, a $T(\nu) \in \mathcal{T}$ with $\nu \in N_T$, and defining $\Theta_T = \{\theta_{T,\nu} : \nu \in \check{N}_T\} \subset H_{0,\gamma}^1(\Omega)$ by

$$\theta_{T,\nu} := |\omega_T(\nu)| \tilde{\phi}_{T(\nu),\varepsilon,\nu},$$

where the specific scaling is chosen for convenience, we have for $\nu, \nu' \in \check{N}_T$,

$$(7.3) \quad \begin{aligned} \delta_{\nu\nu'} |\omega_T(\nu)| &= \langle \theta_{T,\nu}, \xi_{T,\nu'} \rangle_{L_2(\Omega)} \approx \delta_{\nu\nu'} \|\theta_{T,\nu}\|_{L_2(\Omega)} \|\xi_{T,\nu'}\|_{L_2(\Omega)}, \\ \text{supp } \theta_{T,\nu} &\subset \overline{T(\nu)}, \quad |\theta_{T,\nu}|_{H^1(\Omega)} \lesssim h_{T(\nu)}^{-1} \|\theta_{T,\nu}\|_{L_2(\Omega)}, \end{aligned}$$

i.e., properties analogous to (3.3) and (3.10).

Since furthermore Ξ_T and Σ_T satisfy properties analogous to (3.8) and (3.9), defining similarly to (3.4),

$$(7.4) \quad \begin{aligned} \psi_{T,\nu} &:= \sigma_{T,\nu} + \frac{\langle \mathbb{1} - \sigma_{T,\nu}, \xi_{T,\nu} \rangle_{L_2(\Omega)}}{\langle \theta_{T,\nu}, \xi_{T,\nu} \rangle_{L_2(\Omega)}} \theta_{T,\nu} - \sum_{\nu' \in \check{N}_T \setminus \{\nu\}} \frac{\langle \sigma_{T,\nu}, \xi_{T,\nu'} \rangle_{L_2(\Omega)}}{\langle \theta_{T,\nu'}, \xi_{T,\nu'} \rangle_{L_2(\Omega)}} \theta_{T,\nu'} \\ (7.5) \quad &= \begin{cases} \frac{\theta_{T,\nu}}{d+1} & \nu \in \check{N}_T \setminus N_T, \\ \phi_{T,\nu} + \frac{\theta_{T,\nu}}{2(d+1)} - \sum_{\nu' \in \check{N}_T \setminus \{\nu\}} \frac{|\omega_T(\nu) \cap \omega_T(\nu')|}{2d(d+1)|\omega_T(\nu')|} \theta_{T,\nu'} & \nu \in N_T, \end{cases} \end{aligned}$$

we conclude that with $(D_T v)(w) := (Dv)(w)$, we have that $\|D_T\|_{\mathcal{L}(\mathcal{V}_T, \mathcal{W}_T')} \leq 1$, and $\sup_{T \in \mathbb{T}} \|D_T^{-1}\|_{\mathcal{L}(\mathcal{W}_T', \mathcal{V}_T)} < \infty$, and

$$\boxed{D_T} = \mathcal{F}'_{\Psi_T} D_T \mathcal{F}_{\Xi_T} = \text{diag}\{\langle \mathbb{1}, \xi_{T,\nu} \rangle_{L_2(\Omega)} : \nu \in \check{N}_T\} = \text{diag}\left\{\frac{1}{d+1} |\omega_T(\nu)| : \nu \in \check{N}_T\right\}.$$

7.2. Construction of $B_T \in \mathcal{L}is_c(\mathcal{W}_T, \mathcal{W}_T')$. Since, for $k \in \{0, 1\}$, Θ_T additionally satisfies

$$\left\| \sum_{\nu \in \check{N}_T} c_\nu \theta_{T,\nu} \right\|_{H^k(\Omega)}^2 \approx \sum_{\nu \in \check{N}_T} |\omega_T(\nu)|^{1-\frac{2k}{d}} |c_\nu|^2,$$

and

$$\|u + v\|_{H^k(\Omega)}^2 \gtrsim \|u\|_{H^k(\Omega)}^2 + \|v\|_{H^k(\Omega)}^2 \quad (u \in \mathcal{S}_T^{0,1}, v \in \mathcal{B}_T := \text{span } \Theta_T).$$

(cf. (5.1)-(5.2)), we construct B_T analogously as in Section 5. Assuming that we have a $B_T^{\mathcal{S}_T^{0,1}} \in \mathcal{L}is_c(\mathcal{S}_T^{0,1}, (\mathcal{S}_T^{0,1})')$ available with $\sup_{T \in \mathbb{T}} \|B_T^{\mathcal{S}_T^{0,1}}\|_{\mathcal{L}(\mathcal{S}_T^{0,1}, (\mathcal{S}_T^{0,1})')} < \infty$, $\sup_{T \in \mathbb{T}} \|\Re(B_T^{\mathcal{S}_T^{0,1}})^{-1}\|_{\mathcal{L}((\mathcal{S}_T^{0,1})', \mathcal{S}_T^{0,1})} < \infty$, for some constant $\beta_0 > 0$ we take

$$(B_T^{\mathcal{B}} \sum_{\nu \in \check{N}_T} c_\nu \theta_{T,\nu}) \left(\sum_{\nu \in \check{N}_T} d_\nu \theta_{T,\nu} \right) := \beta_0 \sum_{\nu \in \check{N}_T} |\omega_T(\nu)|^{1-\frac{2s}{d}} c_\nu d_\nu,$$

and

$$B_T := I_T' B_T^{\mathcal{S}_T^{0,1}} I_T + (\text{Id} - I_T)' B_T^{\mathcal{B}} (\text{Id} - I_T),$$

where I_T is the projector from $\mathcal{S}_T^{0,1} \oplus \mathcal{B}_T$ onto $\mathcal{S}_T^{0,1}$ with $\text{ran}(\text{Id} - I_T) = \mathcal{B}_T$. Then $\sup_{T \in \mathbb{T}} \|B_T\|_{\mathcal{L}(\mathcal{W}_T, \mathcal{W}_T')} < \infty$ and $\sup_{T \in \mathbb{T}} \|\Re(B_T)^{-1}\|_{\mathcal{L}(\mathcal{W}_T', \mathcal{W}_T)} < \infty$.

Substituting the definition of $\psi_{T,\nu}$, one infers that $\boxed{G_T = D_T^{-1} B_T D_T^{-\top}}$, where

$$\boxed{B_T} = p_T^\top B_T^{\mathcal{S}_T^{0,1}} p_T + q_T^\top B_T^{\mathcal{B}} q_T,$$

and

$$\begin{aligned} \boxed{(q_T)_{\nu'\nu}} &:= \begin{cases} \frac{\delta_{\nu'\nu}}{d+1} & \nu \in \check{N}_T \setminus N_T, \\ \frac{1}{2(d+1)} & \nu \in N_T, \nu' = \nu, \\ -\frac{|\omega_T(\nu) \cap \omega_T(\nu')|}{2d(d+1)|\omega_T(\nu')|} & \nu \in N_T, \nu' \neq \nu, \end{cases} & \boxed{B_T^{\mathcal{S}^{0,1}}} &:= \mathcal{F}'_{\Phi_T} B_T^{\mathcal{S}^{0,1}} \mathcal{F}_{\Phi_T}, \\ \boxed{(p_T)_{\nu'\nu}} &:= \delta_{\nu'\nu} (\nu' \in N_T, \nu \in \check{N}_T), & \boxed{B_T^{\mathcal{B}}} &:= \text{diag}\{\beta_0 |\omega_T(\nu)|^{1-\frac{2s}{d}} : \nu \in \check{N}_T\}. \end{aligned}$$

7.3. Manifold case. From Sect. 4 recall the definitions of Γ , γ , $\mathcal{W}$, $\mathcal{V}$, $\kappa: \Omega \rightarrow \cup_{i=1}^p \kappa_i(\Omega_i)$, and that of the family of conforming partitions $\mathbb{T}$ of Γ .

As in the domain case discussed in Sect. 7.1, for $T \in \mathbb{T}$ let $\check{N}_T$ be the set of vertices of T , and N_T its subset of vertices not on γ , for $T \in \mathcal{T}$ let N_T be the vertices of T , $N_{T,T} := N_T \cap N_T$, and for $\nu \in \check{N}_T$ let $\omega_T(\nu) := \cup_{\{T \in \mathcal{T}: \nu \in N_T\}} \bar{T}$.

We take

$$\begin{aligned} \boxed{\mathcal{V}_T} &= \mathcal{S}_T^{0,1} := \{u \in H^1(\Gamma) : u \circ \kappa|_{\kappa^{-1}(T)} \in \mathcal{P}_1(T \in \mathcal{T})\} \subset \mathcal{V}, \\ \mathcal{S}_T^{0,1} &:= \{u \in H_{0,\gamma}^1(\Gamma) : u \circ \kappa|_{\kappa^{-1}(T)} \in \mathcal{P}_1(T \in \mathcal{T})\}, \end{aligned}$$

equipped with nodal bases $\Xi_T = \{\xi_{T,\nu} : \nu \in \check{N}_T\}$ and $\Phi_T = \{\phi_{T,\nu} : \nu \in N_T\}$, respectively, defined by

$$\xi_{T,\nu}(\nu') = \delta_{\nu,\nu'} \quad (\nu, \nu' \in \check{N}_T),$$

and $\phi_{T,\nu} = \xi_{T,\nu}$ for $\nu \in N_T$.

Actually exclusively for the deriving an inverse inequality analogous to Lemma 4.1, first we construct a collection $\Psi_T = \{\psi_{T,\nu} : \nu \in \check{N}_T\} \subset H_{0,\gamma}^1(\Gamma)$ that has ‘approximation properties’ and that is biorthogonal to Ξ_T w.r.t. the *true* $L_2(\Gamma)$ -scalar product. We define $\Sigma_T = \{\sigma_{T,\nu} : \nu \in \check{N}_T\}$ by $\sigma_{T,\nu} := \phi_{T,\nu}$ when $\nu \in N_T$, and $\sigma_{T,\nu} := 0$ when $\nu \in \check{N}_T \setminus N_T$. Then, obviously, $\sum_{\nu \in \check{N}_T} \sigma_{T,\nu}$ equals $\mathbb{1}$ on $\Gamma \setminus \cup_{\{T \in \mathcal{T}: \bar{T} \cap \gamma \neq \emptyset\}} \bar{T}$.

Given a d -simplex $T \subset \mathbb{R}^d$, by means of an affine bijection we transport the function η_ε , defined in Sect. 4 on a reference d -simplex $\hat{T}$, to a function on T and denote it by $\eta_{T,\varepsilon}$. Then for any $T \in \mathcal{T} \in \mathbb{T}$, for some $\varepsilon > 0$ small enough it holds that

$$\inf_{0 \neq p \in \mathcal{P}_1(\kappa^{-1}(T))} \sup_{0 \neq q \in \mathcal{P}_1(\kappa^{-1}(T))} \frac{\langle p \circ \kappa^{-1}, (\eta_{T,\varepsilon} q) \circ \kappa^{-1} \rangle_{L_2(T)}}{\|p \circ \kappa^{-1}\|_{L_2(T)} \|(\eta_{T,\varepsilon} q) \circ \kappa^{-1}\|_{L_2(T)}} > 0$$

Moreover, since the ‘panels’ T get increasingly flat when $\text{diam } T \rightarrow 0$, there exists an $\varepsilon > 0$ such that above inf-sup condition is satisfied *uniformly* over all $T \in \mathcal{T} \in \mathbb{T}$.

By selecting for each $\nu \in \check{N}_T$ a $T(\nu) \in \mathcal{T}$ with $\nu \in N_T$, as in Sect. 7.1 we obtain a collection $\Theta_T = \{\theta_{T,\nu} : \nu \in \check{N}_T\}$ with $\theta_{T,\nu} \subset H_0^1(T(\nu))$ that is biorthogonal to Ξ_T , in particular that satisfies (7.3), after which we define the $\psi_{T,\nu}$ by means of formula (7.4). Having constructed the biorthogonal collections Ξ_T and Ψ_T , we set the biorthogonal projector $P_T : L_2(\Gamma) \rightarrow H_{0,\gamma}^1(\Gamma) : u \mapsto \sum_{\nu \in \check{N}_T} \frac{\langle u, \xi_{T,\nu} \rangle_{L_2(\Gamma)}}{\langle \mathbb{1}, \xi_{T,\nu} \rangle_{L_2(\Gamma)}} \psi_{T,\nu}$ which satisfies $\|P_T u\|_{H^1(\Gamma)} \lesssim \|h_T^{-1} u\|_{L_2(\Gamma)}$. With the aid of this projector, as in Lemma 4.1 one infers that

$$(7.6) \quad \|h_T v_T\|_{L_2(\Gamma)} \lesssim \|v_T\|_{(H_{0,\gamma}^1)'} \quad (v_T \in \mathcal{V}_T).$$

Having established this inverse inequality, to arrive at a construction of Ψ_T that does not require the evaluation of integrals over Γ , as in Sect. 4.1 we replace $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$

by $\langle \cdot, \cdot \rangle_{\mathcal{T}}$. We redefine $\Theta_{\mathcal{T}} = \{\theta_{\mathcal{T},\nu} : \nu \in \check{N}_{\mathcal{T}}\}$ by

$$\theta_{\mathcal{T},\nu} := |\omega_{\mathcal{T}}(\nu)|^{\frac{|\kappa^{-1}(\mathcal{T})|}{|\mathcal{T}|}} \tilde{\phi}_{\kappa^{-1}(\mathcal{T}),\varepsilon,\kappa^{-1}(\nu)} \circ \kappa^{-1}$$

with the $\tilde{\phi}$'s defined in (7.2), and following (7.5), set $\Psi_{\mathcal{T}} = \{\psi_{\mathcal{T},\nu} : \nu \in \check{N}_{\mathcal{T}}\}$ and $\mathcal{W}_{\mathcal{T}} := \text{span } \Psi_{\mathcal{T}}$ by

$$\psi_{\mathcal{T},\nu} = \begin{cases} \frac{\theta_{\mathcal{T},\nu}}{d+1} & \nu \in \check{N}_{\mathcal{T}} \setminus N_{\mathcal{T}}, \\ \phi_{\mathcal{T},\nu} + \frac{\theta_{\mathcal{T},\nu}}{2(d+1)} - \sum_{\nu' \in \check{N}_{\mathcal{T}} \setminus \{\nu\}} \frac{|\omega_{\mathcal{T}}(\nu) \cap \omega_{\mathcal{T}}(\nu')|}{2d(d+1)|\omega_{\mathcal{T}}(\nu')|} \theta_{\mathcal{T},\nu'} & \nu \in N_{\mathcal{T}}. \end{cases}$$

As in Sect. 4.1, we set $(D_{\mathcal{T}} v_{\mathcal{T}})(w_{\mathcal{T}}) := \langle v_{\mathcal{T}}, w_{\mathcal{T}} \rangle_{\mathcal{T}}$ ($v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$, $w_{\mathcal{T}} \in \mathcal{W}_{\mathcal{T}}$), and as in Sect. 4.1, using (7.6) one shows that $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}\|_{\mathcal{L}(\mathcal{V}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')} < \infty$. Similarly as in Lemma 4.3, one proves that

$$\|h_{\mathcal{T}} v_{\mathcal{T}}\|_{L_2(\Gamma)} \lesssim \sup_{0 \neq w \in H_{0,\gamma}^1(\Gamma)} \frac{\langle v_{\mathcal{T}}, w_{\mathcal{T}} \rangle_{\mathcal{T}}}{\|w\|_{H^1(\Gamma)}},$$

and with that $\sup_{\mathcal{T} \in \mathbb{T}} \|D_{\mathcal{T}}^{-1}\|_{\mathcal{L}(\mathcal{W}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})} < \infty$.

Constructing $B_{\mathcal{T}} \in \mathcal{L}is_c(\mathcal{W}_{\mathcal{T}}, \mathcal{W}_{\mathcal{T}}')$ as in Sect. 7.2, one arrives at the same expressions for $D_{\mathcal{T}}$, $G_{\mathcal{T}}$, $B_{\mathcal{T}}$, $q_{\mathcal{T}}$, $B_{\mathcal{T}}^{\mathcal{S}^{0,1}}$, $p_{\mathcal{T}}$, and $B_{\mathcal{T}}^{\mathcal{B}}$ as in Sect. 7.1-7.2 in the domain case.

7.4. Higher order case. For ease of presentation only, let us confine the discussion to the domain case and

$$\boxed{\mathcal{V}_{\mathcal{T}}} := \{u \in H^1(\Omega) : u|_T \in \mathcal{P}_2(T \in \mathcal{T})\} \subset \mathcal{V},$$

To construct a preconditioner, i.e., a $G_{\mathcal{T}} \in \mathcal{L}is(\mathcal{V}_{\mathcal{T}}', \mathcal{V}_{\mathcal{T}})$ one can re-use the preconditioner constructed for the continuous piecewise linears by following the method of subspace correction methods discussed in Sect. 6. It requires a stable splitting of the current $\mathcal{V}_{\mathcal{T}}$ into the space of continuous piecewise linears and a complement space.

An alternative is to apply the ‘operator preconditioning’ approach directly. Let $\Xi_{\mathcal{T}} = \{\xi_{\mathcal{T},\nu} : \nu \in \check{N}_{\mathcal{T}} \cup M_{\mathcal{T}}\}$ be the usual nodal basis of $\mathcal{V}_{\mathcal{T}}$, where $M_{\mathcal{T}}$ is the set of midpoints of edges of elements in $\mathcal{T}$, and, as in Sect. 7.1, let $\Phi_{\mathcal{T}} = \{\phi_{\mathcal{T},\nu} : \nu \in N_{\mathcal{T}}\}$ be the nodal basis for the space $\mathcal{S}_{\mathcal{T}}^{0,1}$ of continuous piecewise linears that vanish at γ . We set $\Sigma_{\mathcal{T}} = \{\sigma_{\mathcal{T},\nu} : \nu \in \check{N}_{\mathcal{T}} \cup M_{\mathcal{T}}\}$ by $\sigma_{\mathcal{T},\nu} := \phi_{\mathcal{T},\nu}$ when $\nu \in N_{\mathcal{T}}$, and $\sigma_{\mathcal{T},\nu} := 0$ when $\nu \in (\check{N}_{\mathcal{T}} \setminus N_{\mathcal{T}}) \cup M_{\mathcal{T}}$. Then $\sum_{\nu \in \check{N}_{\mathcal{T}}} \sigma_{\mathcal{T},\nu}$ equals $\mathbb{1}$ on $\Omega \setminus \cup_{\{T \in \mathcal{T} : \bar{T} \cap \gamma \neq \emptyset\}} \bar{T}$. It remains to construct $\Theta_{\mathcal{T}} = \{\theta_{\mathcal{T},\nu} : \nu \in \check{N}_{\mathcal{T}} \cup M_{\mathcal{T}}\}$ that is biorthogonal to $\Xi_{\mathcal{T}}$. Using that, similar to (7.1),

$$\inf_{0 \neq p \in \mathcal{P}_2(\hat{T})} \sup_{0 \neq q \in \mathcal{P}_2(\hat{T})} \frac{\langle p, \eta_{\varepsilon} q \rangle_{L_2(\hat{T})}}{\|p\|_{L_2(\hat{T})} \|\eta_{\varepsilon} q\|_{L_2(\hat{T})}} > 0,$$

such a $\Theta_{\mathcal{T}}$ can be constructed as in Sect. 7.1. Having $\Xi_{\mathcal{T}}$, $\Sigma_{\mathcal{T}}$, and $\Theta_{\mathcal{T}}$, a collection $\Psi_{\mathcal{T}}$ biorthogonal to $\Xi_{\mathcal{T}}$ and with ‘approximation properties’ is given by the meanwhile familiar explicit formula. The resulting $D_{\mathcal{T}}$ is diagonal, and the optimal preconditioner follows assuming that we have a $B_{\mathcal{T}}^{\mathcal{S}^{0,1}} \in \mathcal{L}is_c(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')$ available with $\sup_{\mathcal{T} \in \mathbb{T}} \|B_{\mathcal{T}}^{\mathcal{S}^{0,1}}\|_{\mathcal{L}(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')} < \infty$, $\sup_{\mathcal{T} \in \mathbb{T}} \|\Re(B_{\mathcal{T}}^{\mathcal{S}^{0,1}})^{-1}\|_{\mathcal{L}((\mathcal{S}_{\mathcal{T}}^{0,1})', \mathcal{S}_{\mathcal{T}}^{0,1})} < \infty$.

8. USING A DIFFERENT PARTITION FOR THE CONSTRUCTION OF THE PRECONDITIONER

For ease of presentation, let us restrict ourselves to the case considered in Sect. 3: Ω is a domain in $\mathbb{R}^d$, and $\mathcal{W}' = \mathcal{V} \supset \mathcal{V}_\mathcal{T} := \mathcal{S}_\mathcal{T}^{-1,0}$, being the space of piecewise constants w.r.t. $\mathcal{T}$. We have seen how to build an optimal preconditioner for an $A_\mathcal{T} \in \mathcal{L}(\mathcal{S}_\mathcal{T}^{-1,0}, (\mathcal{S}_\mathcal{T}^{-1,0})')$ assuming we have available an operator $B_\mathcal{T}^{\mathcal{S}_\mathcal{T}^{0,1}} \in \mathcal{L}(\mathcal{S}_\mathcal{T}^{0,1}, (\mathcal{S}_\mathcal{T}^{0,1})')$, where $\mathcal{S}_\mathcal{T}^{0,1} \subset \mathcal{W}$ is the space of continuous linears w.r.t. $\mathcal{T}$ that vanish on γ . Apart from the application of $B_\mathcal{T}^{\mathcal{S}_\mathcal{T}^{0,1}}$, the other ingredients needed for building this preconditioner require $\mathcal{O}(\#\mathcal{T})$ operations.

In the current section we will show that the construction can be generalized to the situation that partitions $\mathcal{T}$ and $\tilde{\mathcal{T}}$ underlying $A_\mathcal{T}$ and $B_{\tilde{\mathcal{T}}}^{\mathcal{S}_{\tilde{\mathcal{T}}}^{0,1}}$ are *unequal*, although related. This can be useful in the situation that $\mathcal{T}$ is ‘unstructured’, whereas in order to construct $B_{\tilde{\mathcal{T}}}^{\mathcal{S}_{\tilde{\mathcal{T}}}^{0,1}}$ of multi-level type see e.g [BPV00], one would like to have $\tilde{\mathcal{T}}$ to be the result of recursive refinements starting from a coarse initial partition.

As before, let $\mathbb{T}$ be some family of *conforming, uniformly shape regular* partitions of Ω into (open) d -simplices. Let $\tilde{\mathbb{T}}$ be a family of *uniformly shape regular* partitions of Ω into (open) d -simplices, thus not necessarily conforming, such that for $\tilde{T} \in \tilde{\mathbb{T}}$, γ is the (possibly empty) union of $(d-1)$ -faces of $\tilde{T} \in \tilde{\mathcal{T}}$. Moreover, we assume that for any $\mathcal{T} \in \mathbb{T}$, there exists a $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}(\mathcal{T})$ such that for any $(T, \tilde{T}) \in \mathcal{T} \times \tilde{\mathcal{T}}$ with $T \cap \tilde{T} \neq \emptyset$, it holds that $|h_T| \approx |h_{\tilde{T}}|$.

With $\Xi_\mathcal{T}$ being the basis of piecewise constants for $\mathcal{S}_\mathcal{T}^{-1,0}$, we need to construct a biorthogonal collection $\Psi_\mathcal{T}$ consisting of functions each of them being the sum of a function from $\Sigma_\mathcal{T} = \Sigma_{\mathcal{T}, \tilde{\mathcal{T}}} \subset \mathcal{S}_{\tilde{\mathcal{T}}}^{0,1}$, which collection should have ‘approximation properties’, and a linear combinations of ‘bubble functions’ from $\Theta_\mathcal{T}$. Compared to Sect. 3 we slightly adapt the construction of $\Theta_\mathcal{T} = \{\theta_T : T \in \mathcal{T}\}$ in the sense that we choose the bubble $\theta_T = \theta_{T, \tilde{T}} \in H_0^1(T \cap \tilde{T})$ for some $\tilde{T} \in \tilde{\mathcal{T}}$. Thanks to our assumptions on shape regularity and on the connection between $\mathcal{T}$ and $\tilde{\mathcal{T}}$ this can be done whilst retaining property (3.3).

To define $\Sigma_{\mathcal{T}, \tilde{\mathcal{T}}}$, let $N_{\tilde{\mathcal{T}}}$ be the set of ‘non-hanging’ vertices of $\tilde{\mathcal{T}}$ that are not on γ , let $\Phi_{\tilde{\mathcal{T}}} = \{\phi_{\tilde{T}, \tilde{\nu}} : \tilde{\nu} \in N_{\tilde{\mathcal{T}}}\}$ be the nodal basis for $\mathcal{S}_{\tilde{\mathcal{T}}}^{0,1}$, and for $\tilde{\nu} \in N_{\tilde{\mathcal{T}}}$, let $d_{\mathcal{T}, \tilde{\nu}} := \#\{T \in \mathcal{T} : \tilde{\nu} \in \bar{T}\}$. Now defining $\Sigma_{\mathcal{T}, \tilde{\mathcal{T}}} = \{\sigma_{\mathcal{T}, \tilde{\mathcal{T}}, T} : T \in \mathcal{T}\}$ by

$$\sigma_{\mathcal{T}, \tilde{\mathcal{T}}, T} := \sum_{\{\tilde{\nu} \in N_{\tilde{\mathcal{T}}} : \nu \in \bar{T}\}} d_{\mathcal{T}, \tilde{\nu}}^{-1} \phi_{\tilde{T}, \tilde{\nu}},$$

the whole preceding construction that was developed for $\tilde{\mathcal{T}} = \mathcal{T}$ goes through. That is, defining $\Psi_\mathcal{T}$ by (3.4), Lemma 3.1 is valid (but the convenient expression (3.7) for $\psi_{\mathcal{T}, T}$ is not) so in particular $\mathbf{D}_\mathcal{T} = \text{diag}\{|T| : T \in \mathcal{T}\}$, properties (3.8), (3.9), (3.10) hold true, and so does Theorem 3.3. Equation (5.1) holds true and, thanks to the modified construction of $\Theta_\mathcal{T}$, so does (5.2) concerning the stable splitting of $\mathcal{S}_{\tilde{\mathcal{T}}}^{0,1} \oplus \text{span } \Theta_\mathcal{T}$ into $\mathcal{S}_{\tilde{\mathcal{T}}}^{0,1}$ and $\text{span } \Theta_\mathcal{T}$.

Given a collection $\mathbb{T}$ as above, $\tilde{\mathbb{T}}$ can be chosen as the collection of partitions that can be generated by *newest vertex bisection* starting from some coarse initial conforming partition of Ω that satisfies a matching condition ([BDD04, Ste08]). Given $\mathcal{T} \in \mathbb{T}$, the partition $\tilde{\mathcal{T}} \in \tilde{\mathbb{T}}$ can then be constructed by recursive refinements

of those $\tilde{T} \in \tilde{\mathbb{T}}$ for which $h_{\tilde{T}}$ is larger than some fixed constant multiple of h_T for any $T \in \mathcal{T}$ with $T \cap \tilde{T} \neq \emptyset$. The resulting $\tilde{\mathcal{T}} \in \tilde{\mathbb{T}}$ will generally be nonconforming, but for a convenient construction of $B_{\tilde{\mathcal{T}}}^{\mathcal{S}^{0,1}}$ it can be refined to a conforming partition $\tilde{\mathcal{T}}_c \in \tilde{\mathbb{T}}$ at the cost of inflating the total number of d -simplices by not more than a constant factor ([BDD04, Ste08]). The latter partition $\tilde{\mathcal{T}}_c$ might be locally more refined than $\mathcal{T}$ in the sense that $h_{\tilde{T}_c} \gtrsim h_T$ for all $(T, \tilde{T}_c) \in \mathcal{T} \times \tilde{\mathcal{T}}_c$ with $T \cap \tilde{T}_c \neq \emptyset$ cannot be guaranteed. Since this condition is needed to guarantee an inverse inequality for the ‘ σ -functions’ assumed in (3.9), the intermediate nonconforming partition $\tilde{\mathcal{T}}$ remains needed for the construction of $\Sigma_{\mathcal{T}, \tilde{\mathcal{T}}}$.

9. NUMERICAL EXPERIMENTS

Let $\Gamma = \partial[0, 1]^3 \subset \mathbb{R}^3$ be the two-dimensional manifold without boundary given as the boundary of the unit cube, $\mathcal{W} := H^{1/2}(\Gamma)$, $\mathcal{V} := H^{-1/2}(\Gamma)$, and $\mathcal{V}_{\mathcal{T}} = \mathcal{S}_{\mathcal{T}}^{-1, \ell} \subset \mathcal{V}$ the trial space of discontinuous piecewise polynomials of degree ℓ w.r.t. a partition $\mathcal{T}$. In this section we will evaluate preconditioning of the discretized single layer operator $A_{\mathcal{T}} \in \mathcal{L}is_c(\mathcal{V}_{\mathcal{T}}, \mathcal{V}_{\mathcal{T}}')$.

The role of the opposite order operator $B_{\mathcal{T}}^{\mathcal{S}^{0,1}} \in \mathcal{L}is_c(\mathcal{S}_{\mathcal{T}}^{0,1}, (\mathcal{S}_{\mathcal{T}}^{0,1})')$ from Section 5 will be fulfilled by $(B_{\mathcal{T}}^{\mathcal{S}^{0,1}} u)(v) := (Bu)(v)$ for an adapted hypersingular operator $B \in \mathcal{L}is_c(\mathcal{W}, \mathcal{W}')$. The hypersingular operator $\tilde{B} \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$ itself is only semi-coercive, but there are various options to change it into a coercive operator ([SW98]). We consider $B \in \mathcal{L}is_c(\mathcal{W}, \mathcal{W}')$ given by $(Bu)(v) = (\tilde{B}u)(v) + \alpha \langle u, \mathbb{1} \rangle_{L_2(\Gamma)} \langle v, \mathbb{1} \rangle_{L_2(\Gamma)}$ for some $\alpha > 0$. By comparing different values numerically, we find $\alpha = 0.05$ to give good results in our examples.

Equipping $\mathcal{V}_{\mathcal{T}}$ with the usual $L_2(\Gamma)$ -orthogonal basis (where any basis function supported on T has norm $|T|^{\frac{1}{2}}$), and $\mathcal{S}_{\mathcal{T}}^{0,1}$ with the nodal basis, the matrix representation of the preconditioned system reads, when $\ell = 0$, as

$$\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}} = \mathbf{D}_{\mathcal{T}}^{-1} (\mathbf{p}_{\mathcal{T}}^{\top} \mathbf{B}_{\mathcal{T}}^{\mathcal{S}^{0,1}} \mathbf{p}_{\mathcal{T}} + \beta_0 \mathbf{q}_{\mathcal{T}}^{\top} \mathbf{D}_{\mathcal{T}}^{1/2} \mathbf{q}_{\mathcal{T}}) \mathbf{D}_{\mathcal{T}}^{-1} \mathbf{A}_{\mathcal{T}},$$

with $\mathbf{D}_{\mathcal{T}} = \text{diag}\{|T| : T \in \mathcal{T}\}$ and uniformly sparse $\mathbf{p}_{\mathcal{T}}$ and $\mathbf{q}_{\mathcal{T}}$ as given in Sect. 5.1; and when $\ell > 0$, writing $\mathbf{G}_{\mathcal{T}}^0 = \mathbf{G}_{\mathcal{T}}$ for the above preconditioner on piecewise constants, as

$$\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}} = \left(\mathbf{p}_{\mathcal{T}}^0 \mathbf{G}_{\mathcal{T}}^0 (\mathbf{p}_{\mathcal{T}}^0)^{\top} + \mathbf{p}_{\mathcal{T}}^1 \mathbf{G}_{\mathcal{T}}^1 (\mathbf{p}_{\mathcal{T}}^1)^{\top} \right) \mathbf{A}_{\mathcal{T}},$$

with $\mathbf{G}_{\mathcal{T}}^1 = \beta_1 \text{diag}\{|T|^{-3/2} \text{Id}_{(N-1) \times (N-1)} : T \in \mathcal{T}\}$ where $N = \binom{d+\ell}{\ell}$, and uniformly sparse $\mathbf{p}_{\mathcal{T}}^0$ and $\mathbf{p}_{\mathcal{T}}^1$ as given in Sect. 6. The (full) matrix representations of the discretized singular integral operators $\mathbf{A}_{\mathcal{T}}$ and $\mathbf{B}_{\mathcal{T}}^{\mathcal{S}^{0,1}}$ are calculated using the BETL2 software package [HK12] (alternatively, one may apply low rank approximations in a hierarchical format). Condition numbers are determined using Lanczos iteration with respect to $\|\cdot\| := \|\mathbf{A}_{\mathcal{T}}^{\frac{1}{2}} \cdot\|$. The constants β_i are selected using the strategy from Remark 5.2 and 6.6.

We will compare our preconditioner to the diagonal preconditioner $\text{diag}(\mathbf{A}_{\mathcal{T}})^{-1}$, and in the piecewise constant case, also to the related preconditioner $\hat{\mathbf{G}}_{\mathcal{T}}$ from [HUT16], where $\hat{\mathbf{G}}_{\mathcal{T}} = \hat{\mathbf{D}}_{\mathcal{T}}^{-1} \mathbf{E}_{\mathcal{T}}^{\top} \mathbf{B}_{\mathcal{T}}^{\mathcal{S}^{0,1}} \mathbf{E}_{\mathcal{T}} \hat{\mathbf{D}}_{\mathcal{T}}^{-\top}$ is defined as follows. With $\hat{\mathcal{T}}$ being the barycentric refinement of $\mathcal{T}$, a collection $\hat{\Psi}_{\mathcal{T}} \subset \mathcal{S}_{\hat{\mathcal{T}}}^{0,1}$ is constructed in [BC07] such that the Fortin projector $\hat{P}_{\mathcal{T}}$ with $\text{ran } \hat{P}_{\mathcal{T}} = \hat{\mathcal{W}}_{\mathcal{T}} := \text{span } \hat{\Psi}_{\mathcal{T}}$ and $\text{ran}(\text{Id} - \hat{P}_{\mathcal{T}}) = \mathcal{V}_{\mathcal{T}}^{\perp, L_2(\Gamma)}$ exists, and, under an additional sufficiently mild grading

TABLE 1. Spectral condition numbers of the preconditioned single layer system, using uniform refinements, discretized by piecewise constants $\mathcal{S}_T^{-1,0}$. Both matrices G_T and $\hat{G}_T$ are constructed using the adapted hypersingular operator with $\alpha = 0.05$; and $\beta_0 = 1.25$ in G_T .

dofs	$\kappa_S(\text{diag}(\mathbf{A}_T)^{-1} \mathbf{A}_T)$	$\kappa_S(\mathbf{G}_T \mathbf{A}_T)$	$\kappa_S(\hat{\mathbf{G}}_T \mathbf{A}_T)$
12	14.56	2.51	1.29
48	29.30	2.52	1.58
192	58.25	2.66	1.77
768	116.3	2.71	1.89
3072	230.0	2.74	1.94
12288	444.8	2.79	

condition on the partition, has a uniformly bounded norm $\|\hat{P}_T\|_{\mathcal{L}(\mathcal{W}, \mathcal{W})}$ (cf. Theorem 3.3); $\hat{\mathbf{D}}_T := \langle \Xi_T, \hat{\Psi}_T \rangle_{L_2(\Gamma)}$; $\mathbf{E}_T$ is the representation of the embedding $\mathcal{W}_T \hookrightarrow \mathcal{S}_T^{0,1}$ equipped with $\hat{\Psi}_T$ and the nodal basis of $\mathcal{S}_T^{0,1}$, respectively; and $B_{\hat{T}}^{\mathcal{S}_T^{0,1}} \in \mathcal{L}is_c(\mathcal{S}_T^{0,1}, (\mathcal{S}_T^{0,1})')$ is an opposite order operator that we take as $(B_{\hat{T}}^{\mathcal{S}_T^{0,1}} u)(v) := (Bu)(v)$, with B the adapted hypersingular operator.

Compared to our $G_T = G_T^0$, the preconditioner $\hat{G}_T$ has the disadvantages that, besides the aforementioned mildly grading condition, the matrix $\hat{\mathbf{D}}_T$, although uniformly sparse, is not diagonal, so that the (sufficiently accurate) application of its inverse cannot be performed in linear complexity; furthermore that it requires evaluating the adapted hypersingular operator on the larger space $\mathcal{S}_{\hat{T}}^{0,1} \supset \mathcal{S}_T^{0,1}$ ($\#\hat{T} = 6\#T$); and finally that the non-standard barycentric refinement $\hat{T}$ has to be generated.

9.1. Uniform refinements. Consider a conforming triangulation $\mathcal{T}_1$ of Γ consisting of 2 triangles per side, so 12 triangles in total. We let $\mathbb{T}$ be the sequence $\{\mathcal{T}_k\}_{k \geq 1}$ of uniform red-refinements, where $\mathcal{T}_k \succ \mathcal{T}_{k-1}$ is found by subdividing each triangle from $\mathcal{T}_{k-1}$ into 4 congruent subtriangles.

For $\mathcal{V}_T = \mathcal{S}_T^{-1,\ell}$, Tables 1 and 2 show the condition numbers of the preconditioned system for $\ell = 0$ and $\ell = 2$, respectively. Aside from being uniformly bounded, the condition numbers of our preconditioner G_T are of modest size. In the constant case, $\ell = 0$, Table 1 reveals that the preconditioner $\hat{G}_T$ from [BC07, HUT16] gives better condition numbers. As described above, this quantitative gain comes at a price. In the result of $\dim \mathcal{S}_T^{-1,0} = 3072$, using full matrices for the discretized adapted hypersingular operator, we found a setup and application time of 1816s and 0.0971s for $\hat{G}_T$, compared to 385s and 0.00284s for G_T . These differences are due to numerical inversion of $\hat{\mathbf{D}}_T$ by LU factorization with partial pivoting, and the enlargement $\mathcal{S}_{\hat{T}}^{0,1} \supset \mathcal{S}_T^{0,1}$, also causing our test machine to go out of memory in calculation $\hat{G}_T$ for the last refinement. Although we expect them to be in any case significant, these differences can be made smaller when the exact inversion of $\hat{\mathbf{D}}_T$ is avoided, and $B_{\hat{T}}^{\mathcal{S}_T^{0,1}}$ and $B_T^{\mathcal{S}_T^{0,1}}$ are replaced by suitable low rank approximations.

TABLE 2. Spectral condition numbers of the preconditioned single layer system, using uniform refinements, discretized by discontinuous piecewise quadratics $\mathcal{S}_{\mathcal{T}}^{-1,2}$. The matrix $\mathbf{G}_{\mathcal{T}}$ is constructed using the adapted hypersingular operator, with $\alpha = 0.05$, and $\beta_0 = \beta_1 = 1.25$.

dofs	$\kappa_S(\text{diag}(\mathbf{A}_{\mathcal{T}})^{-1} \mathbf{A}_{\mathcal{T}})$	$\kappa_S(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}})$
72	167.16	9.58
288	309.12	10.4
1152	616.03	11.1
4608	1211.3	11.3
18432	2337.2	11.4

TABLE 3. Spectral condition numbers of the preconditioned single layer system discretized by piecewise constants $\mathcal{S}_{\mathcal{T}}^{-1,0}$ using local refinements at each of the eight cube corners. Both matrices $\mathbf{G}_{\mathcal{T}}$ and $\hat{\mathbf{G}}_{\mathcal{T}}$ are constructed using the adapted hypersingular operator with $\alpha = 0.05$; and $\beta_0 = 1.2$ in $\mathbf{G}_{\mathcal{T}}$. The second column is defined by $h_{\mathcal{T},\min} := \min_{T \in \mathcal{T}} h_T$.

dofs	$h_{\mathcal{T},\min}$	$\kappa_S(\text{diag}(\mathbf{A}_{\mathcal{T}})^{-1} \mathbf{A}_{\mathcal{T}})$	$\kappa_S(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}})$	$\kappa_S(\hat{\mathbf{G}}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}})$
12	$7.0 \cdot 10^{-1}$	14.56	2.61	1.29
432	$2.2 \cdot 10^{-2}$	68.66	2.64	2.91
912	$6.9 \cdot 10^{-4}$	73.15	2.64	3.14
1872	$6.7 \cdot 10^{-7}$	73.70	2.64	3.25
2352	$2.1 \cdot 10^{-8}$	73.80	2.64	3.26
2976	$2.3 \cdot 10^{-10}$	73.66	2.64	

9.2. Local refinements. Here we take $\mathbb{T}$ to be the sequence $\{\mathcal{T}_k\}_{k \geq 1}$ of locally refined triangulations, where $\mathcal{T}_k \succ \mathcal{T}_{k-1}$ is constructed using conforming newest vertex bisection to refine all triangles in $\mathcal{T}_{k-1}$ that touch a corner of the cube.

As noted before, the preconditioner $\hat{\mathbf{G}}_{\mathcal{T}}$ provides uniformly bounded condition numbers if the family $\mathbb{T}$ satisfies some sufficiently mildly-grading condition on the partition [Ste03, HUT16]. It is not directly clear whether $\mathbb{T}$ satisfies this condition, but we included the results nonetheless.

Table 3 gives the results for the preconditioned single layer operator discretized by piecewise constants $\mathcal{S}_{\mathcal{T}}^{-1,0}$. The condition numbers $\kappa_S(\mathbf{G}_{\mathcal{T}} \mathbf{A}_{\mathcal{T}})$ are nicely bounded under local refinements. In this case our preconditioner gives condition numbers slightly smaller than the ones found with $\hat{\mathbf{G}}_{\mathcal{T}}$. The calculation of the LU decomposition with partial pivoting of $\hat{\mathbf{D}}_{\mathcal{T}}$ turns out to break down in the last result ($\dim \mathcal{S}_{\mathcal{T}}^{-1,0} = 2976$).

10. CONCLUSION

In this paper, we have seen how a boundedly invertible operator $B_{\mathcal{T}}^{\mathcal{S}^{0,1}}$ from the space of continuous piecewise linears $\mathcal{S}_{\mathcal{T}}^{0,1}$ w.r.t. any conforming shape regular

partition $\mathcal{T}$, equipped with the norm of $H^s(\Omega)$ (or $H^s(\Gamma)$) for some $s \in [0, 1]$, to its dual $(\mathcal{S}_{\mathcal{T}}^{0,1})'$ can be used to optimally precondition a boundedly invertible operator of opposite order discretized by discontinuous or continuous polynomials of some fixed degree w.r.t. $\mathcal{T}$ (or even w.r.t. some partition close to $\mathcal{T}$). The cost of the resulting preconditioner is the sum of a cost that scales linearly in $\#\mathcal{T}$ and the cost of the application of $B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}}$. In any case for $\mathcal{T}$ being member of a nested sequence of quasi-uniform partitions, $B_{\mathcal{T}}^{\mathcal{S}_{\mathcal{T}}^{0,1}}$ can be constructed so that it requires linear cost.

APPENDIX A. PROOF OF PROPOSITION 6.3

Lemma A.1. *For reflexive Banach spaces $\mathcal{Y}$ and $\mathcal{Z}$, let $E \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ with $\text{ran } E = \mathcal{Z}$. Then $\|z\|_{\mathcal{Z}} := \inf_{\{y \in \mathcal{Y} : Ey=z\}} \|y\|_{\mathcal{Y}}$ defines an alternative norm on the linear space $\mathcal{Z}$, and*

$$\sup_{0 \neq g \in \mathcal{Z}'} \frac{\|E'g\|_{\mathcal{Y}'}}{\|g\|_{\mathcal{Z}'}} = \sup_{0 \neq z \in \mathcal{Z}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}}, \quad \inf_{0 \neq g \in \mathcal{Z}'} \frac{\|E'g\|_{\mathcal{Y}'}}{\|g\|_{\mathcal{Z}'}} \geq \inf_{0 \neq z \in \mathcal{Z}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}}.$$

Proof. The verification that $\|\cdot\|_{\mathcal{Z}}$ defines a norm is easy. Obviously the supremum at the left hand side of the next statement equals

$$\begin{aligned} \|E'\|_{\mathcal{L}(\mathcal{Z}', \mathcal{Y}')} &= \|E\|_{\mathcal{L}(\mathcal{Y}, \mathcal{Z})} = \sup_{0 \neq y \in \mathcal{Y}} \frac{\|Ey\|_{\mathcal{Z}}}{\|y\|_{\mathcal{Y}}} = \sup_{0 \neq z \in \mathcal{Z}} \sup_{\{y \in \mathcal{Y} : Ey=z\}} \frac{\|z\|_{\mathcal{Z}}}{\|y\|_{\mathcal{Y}}} \\ &= \sup_{0 \neq z \in \mathcal{Z}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}}. \end{aligned}$$

The last statement follows from

$$\begin{aligned} \|E'g\|_{\mathcal{Y}'} &= \sup_{0 \neq y \in \mathcal{Y}} \frac{(E'g)(y)}{\|y\|_{\mathcal{Y}}} = \sup_{0 \neq y \in \mathcal{Y}} \frac{g(Ey)}{\|y\|_{\mathcal{Y}}} = \sup_{0 \neq z \in \mathcal{Z}} \frac{g(z)}{\|z\|_{\mathcal{Z}}} \sup_{\{y \in \mathcal{Y} : Ey=z\}} \frac{\|z\|_{\mathcal{Z}}}{\|y\|_{\mathcal{Y}}} \\ &= \sup_{0 \neq z \in \mathcal{Z}} \frac{g(z)}{\|z\|_{\mathcal{Z}}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}} \geq \inf_{0 \neq z \in \mathcal{Z}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}} \sup_{0 \neq z \in \mathcal{Z}} \frac{g(z)}{\|z\|_{\mathcal{Z}}} = \|g\|_{\mathcal{Z}'}, \quad \inf_{0 \neq z \in \mathcal{Z}} \frac{\|z\|_{\mathcal{Z}}}{\|z\|_{\mathcal{Z}}} = 1. \quad \square \end{aligned}$$

Proof of Proposition 6.3. We equip $\prod_i \mathcal{U}_i$ with $\sqrt{\sum_i \|\cdot\|_{\mathcal{U}_i}^2}$, and $(\prod_i \mathcal{U}_i)'$ with the corresponding dual norm, and equip $\prod_i \mathcal{U}_i'$ with $\sqrt{\sum_i \|\cdot\|_{\mathcal{U}_i'}^2}$.

We start by showing that $(\prod_i \mathcal{U}_i)' \simeq \prod_i \mathcal{U}_i'$. We set

$$S: \prod_i \mathcal{U}_i' \rightarrow (\prod_i \mathcal{U}_i)': (f_i)_i \mapsto ((u_i)_i \mapsto \sum_i f_i(u_i)).$$

Given $f \in (\prod_i \mathcal{U}_i)'$, taking $f_i = f(\cdot e_i)$ we see that $S(f_i)_i = f$, whereas S is clearly also injective. From $\|S(f_i)_i\|_{(\prod_i \mathcal{U}_i)'} = \sup_{0 \neq (u_i)_i \in \prod_i \mathcal{U}_i} \frac{|\sum_i f_i(u_i)|}{\sqrt{\sum_i \|u_i\|_{\mathcal{U}_i}^2}}$, we infer that $\|S(f_i)_i\|_{(\prod_i \mathcal{U}_i)'} \leq \sqrt{\sum_i \|f_i\|_{\mathcal{U}_i'}^2}$, whereas taking u_i such that $f_i(u_i) = \|f_i\|_{\mathcal{U}_i'} \|u_i\|_{\mathcal{U}_i}$ and $\|u_i\|_{\mathcal{U}_i} = \|f_i\|_{\mathcal{U}_i'}$ gives $|\sum_i f_i(u_i)| = \sum_i \|u_i\|_{\mathcal{U}_i}^2$, showing that S is an *isometry*. We will further use that

$$(A.1) \quad (f_i)_i (S'(u_i)_i) = (S(f_i))(u_i)_i = \sum_i f_i(u_i) \quad ((u_i) \in \prod_i \mathcal{U}_i, (f_i) \in \prod_i \mathcal{U}_i').$$

We set $E := (u_i)_i \mapsto \sum_i E_i u_i \in \mathcal{L}(\prod_i \mathcal{U}_i, \mathcal{U})$. For $f \in \mathcal{U}$ and $(u_i)_i \in \prod_i \mathcal{U}_i$, (A.1) shows that

$$\begin{aligned} (S(E'_i f)_i)(u_i)_i &= (E'_i f)_i(S'(u_i)_i) = \sum_i (E'_i f)(u_i) \\ &= \sum_i f(E_i u_i) = f(E(u_i)_i) = (E' f)(u_i)_i, \end{aligned}$$

that is, $S^{-1} E' f = (E' f)_i$.

The definition of G shows that for $f_1, f_2 \in \mathcal{U}'$,

$$f_2(G f_1) = \sum_i E'_i f_2(G_i E'_i f_1) \leq \max_i \|G_i\|_{\mathcal{L}(\mathcal{U}'_i, \mathcal{U}_i)} \prod_{k=1}^2 \sqrt{\sum_i \|E'_i f_k\|_{\mathcal{U}'_i}^2}.$$

The proof of the first bound (6.3) is completed by

$$\sqrt{\sum_i \|E'_i f_k\|_{\mathcal{U}'_i}^2} = \|S^{-1} E' f_k\|_{\prod_i \mathcal{U}'_i} = \|E' f_k\|_{(\prod_i \mathcal{U}_i)'} \leq \|E'\|_{\mathcal{L}(\mathcal{U}', (\prod_i \mathcal{U}_i)')} \|f_k\|_{\mathcal{U}'},$$

and $\|E'\|_{\mathcal{L}(\mathcal{U}', (\prod_i \mathcal{U}_i)')} = \sup_{0 \neq u \in \mathcal{U}} \frac{\|u\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}'}}$ by an application of Lemma A.1.

The second bound (6.4) follows by, for $f \in \mathcal{U}'$,

$$f(G f) = \sum_i E'_i f(G_i E'_i f) \geq \min_i \|\Re(G_i)^{-1}\|_{\mathcal{L}(\mathcal{U}_i, \mathcal{U}'_i)}^{-1} \sum_i \|E'_i f\|_{\mathcal{U}'_i}^2,$$

and

$$\sum_i \|E'_i f\|_{\mathcal{U}'_i}^2 = \|E' f\|_{(\prod_i \mathcal{U}_i)'}^2 \geq \|f\|_{\mathcal{U}'}^2 \left(\inf_{0 \neq u \in \mathcal{U}} \frac{\|u\|_{\mathcal{U}}}{\|u\|_{\mathcal{U}'}} \right)^2$$

by an application of Lemma A.1. □

REFERENCES

- [BC07] A. Buffa and S.H. Christiansen. A dual finite element complex on the barycentric refinement. *Math. Comp.*, 76(260):1743–1769, 2007.
- [BDD04] P. Binev, W. Dahmen, and R. DeVore. Adaptive finite element methods with convergence rates. *Numer. Math.*, 97(2):219–268, 2004.
- [BPV00] J. H. Bramble, J. E. Pasciak, and P. S. Vassilevski. Computational scales of Sobolev norms with application to preconditioning. *Math. Comp.*, 69(230):463–480, 2000.
- [Bra01] D. Braess. *Finite Elements*. Cambridge University Press, 2001. Second edition.
- [Dah96] W. Dahmen. Stability of multiscale transformations. *J. Fourier Anal. Appl.*, 2(4):341–362, 1996.
- [DFG⁺04] W. Dahmen, B. Faermann, I.G. Graham, W. Hackbusch, and S.A. Sauter. Inverse inequalities on non-quasiuniform meshes and application to the mortar element method. *Math. Comp.*, 73:1107–1138, 2004.
- [For77] M. Fortin. An analysis of the convergence of mixed finite element methods. *RAIRO Anal. Numér.*, 11(4):341–354, iii, 1977.
- [GHS05] I. G. Graham, W. Hackbusch, and S. A. Sauter. Finite elements on degenerate meshes: inverse-type inequalities and applications. *IMA J. Numer. Anal.*, 25(2):379–407, 2005.
- [GO95] M. Griebel and P. Oswald. On the abstract theory of additive and multiplicative Schwarz algorithms. *Numer. Math.*, 70:163–180, 1995.
- [Hip06] R. Hiptmair. Operator preconditioning. *Comput. Math. Appl.*, 52(5):699–706, 2006.
- [HJHUT18] R. Hiptmair, C. Jerez-Hanckes, and C. Urzúa-Torres. Closed-form inverses of the weakly singular and hypersingular operators on disks. *Integral Equations Operator Theory*, 90(1):Art. 4, 14, 2018.
- [HK12] R. Hiptmair and L. Kielhorn. Betl – a generic boundary element template library. Technical Report 2012-36, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2012.

- [HUT16] R. Hiptmair and C. Urzúa-Torres. Dual mesh operator preconditioning on 3d screens: Low-order boundary element discretization. Technical Report 2016-14, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2016.
- [HX07] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$ spaces. *SIAM J. Numer. Anal.*, 45(6):2483–2509, 2007.
- [NS18] M. Neumüller and I. Smears. Time-parallel iterative solvers for parabolic evolution equations. Technical report, 2018. To appear in *SIAM J. Sci. Comput.*
- [Osw94] P. Oswald. *Multilevel finite element approximation: Theory and applications*. B.G. Teubner, Stuttgart, 1994.
- [Osw98] P. Oswald. Multilevel norms for $H^{-1/2}$. *Computing*, 61(3):235–255, 1998.
- [Ste02] O. Steinbach. On a generalized L_2 projection and some related stability estimates in Sobolev spaces. *Numer. Math.*, 90(4):775–786, 2002.
- [Ste03] O. Steinbach. *Stability estimates for hybrid coupled domain decomposition methods*, volume 1809 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.
- [Ste08] R.P. Stevenson. The completion of locally refined simplicial partitions created by bisection. *Math. Comp.*, 77:227–241, 2008.
- [SW98] O. Steinbach and W. L. Wendland. The construction of some efficient preconditioners in the boundary element method. *Adv. Comput. Math.*, 9(1-2):191–216, 1998. Numerical treatment of boundary integral equations.
- [SZ90] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.
- [Wid92] O. B. Widlund. Some Schwarz methods for symmetric and nonsymmetric elliptic problems. In *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991)*, pages 19–36. SIAM, Philadelphia, PA, 1992.
- [Xu92] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Rev.*, 34:581–613, 1992.

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