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First Order System Least Squares with inhomogeneous boundary conditions

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We derive well-posed first order system least squares formulations of second order elliptic boundary value problems, in which homogeneous essential boundary conditions are appended by means of additional equations, rather than by their incorporation in the trial space. This approach has the advantage that it applies equally well to inhomogeneous boundary conditions.

Keywords: First order system least squares, inhomogeneous boundary conditions, trace theorems

1. Introduction

Let us start with briefly recalling the principle of a least squares formulation of an operator equation, e.g., a boundary value problem on a domain Ω. Although the approach can be extended to nonlinear equations, here we consider *linear* equations only. For some Hilbert spaces X and Y, let $G \in \mathscr{B}(X, Y')$ be a *homeomorphism onto its range*, i.e.,

$$\|u\|_X \eqsim \|Gu\|_{Y'} \quad (u \in X). \tag{1.1}$$

Here and on other places, with $a \eqsim b$ we mean $a \lesssim b$ and $a \gtrsim b$, with the first relation meaning that a can be bounded by some absolute multiple of b, and the second one being defined as $b \lesssim a$.

For given $f \in Y'$, let us consider the least-squares problem to find $u = \operatorname{argmin}_{v \in X} \frac{1}{2}\|Gv - f\|_{Y'}^2$. Necessarily this u is a solution of the corresponding Euler-Lagrange equations

$$\langle Gu, Gv \rangle_{Y'} = \langle f, Gv \rangle_{Y'} \quad (v \in X). \tag{1.2}$$

Thanks to (1.1), the bilinear form on $X \times X$ at the left hand side is *bounded*, *symmetric*, and *elliptic*, and the right-hand side defines a *bounded* functional on X. From the Lax-Milgram lemma, we conclude that (1.2), and so the least-squares problem, has a unique solution $u \in X$ that depends continuously on $f \in Y'$. Whenever the equation $Gu = f$ has a solution, i.e., $f \in \Im G$ (consistency), it is the unique solution of the least-squares problem.

For a closed subspace $X_h \subset X$, the Galerkin solution $u_h \in X_h$ of (1.2), i.e., $\operatorname{argmin}_{v_h \in X_h} \frac{1}{2}\|Gv_h - f\|_{Y'}^2$, satisfies $\|u - u_h\|_X \lesssim \inf_{v_h \in X_h} \|u - v_h\|_X$, only dependent on the hidden constants absorbed by the $\eqsim$ symbol in (1.1). Because of the symmetry of the bilinear form, the Galerkin solution is conveniently computed, assuming $\langle \cdot, \cdot \rangle_{Y'}$ can be evaluated. In view of the latter, in the setting of a boundary value problem, usually one prefers Y' to be (a multiple copy of) $L_2(\Omega)$.

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Otherwise, one may construct a computationally implementable method following the approach in Bramble *et al.* (1997, 1998), that basically requires a good preconditioner for $A_h : Y_h \to Y_h'$ defined by $(A_h y_h)(z_h) = \langle y_h, z_h\rangle_Y$ for finite dimensional subspaces $Y_h \subset Y$.

Alternatively, when (wavelet) Riesz bases Ψ_X and Ψ_Y for X and Y are available, one has $\|g\|_{Y'} \eqsim \|[g(\psi_Y)]_{\psi_Y\in\Psi_Y}\|_{\ell_2(\Psi_Y)}$ ($g \in Y'$). With Y' equipped with the latter norm, (1.2) is equivalent to the infinite system of normal equations $\mathbf{G}^\top(\mathbf{G}\mathbf{u}-\mathbf{f}) = 0$, where $\mathbf{f} = [f(\psi_Y)]_{\psi_Y\in\Psi_Y}$, $\mathbf{G} = [(G(\psi_X)(\psi_Y)]_{\psi_Y\in\Psi_Y,\psi_X\in\Psi_X}$, and $u = \sum_{\psi_X\in\Psi_X}\mathbf{u}_{\psi_X}\psi_X$, cf. Dahmen *et al.* (2002). This system can be solved with an optimally converging adaptive wavelet scheme, cf. Stevenson (2011).

Finally for an inconvenient $\langle\cdot,\cdot\rangle_{Y'}$, an approximate Galerkin solver can be constructed by realizing that the exact Galerkin solution is the first component of $(u_h, y) \in U_h \times Y$ that solves

$$\begin{aligned} \langle y, v\rangle_Y + (Gu_h)(v) &= f(v) \quad && (v \in Y),\\ (Gw_h)(y) &= 0 && (w_h \in X_h), \end{aligned}$$

cf. (Cohen *et al.*, 2012, §2.2). Now the remaining task is to replace Y by a sufficiently large finite dimensional subspace, such that the error in the resulting solution is of the same order as that in the exact Galerkin solution.

Forming a least squares functional essentially means doubling the order of the equation, which can be expected to have a quantitative harmful effect. In view of this, the least squares approach is commonly applied to boundary value problems that are (re)formulated as *systems* of *first order*. We refer to Bochev & Gunzburger (2009) for an comprehensive overview of the least squares approach.

In this paper, in Sect. 2 a general mechanism will be presented with which an essential *homogeneous* boundary condition, that is incorporated in the space X of a well-posed least-squares problem, is alternatively imposed by appending a squared norm of the corresponding residual to the quadratic functional, and by simultaneously removing the boundary condition from X. Our initial motivation for doing so was to allow for an easy construction of wavelet Riesz bases for the arising spaces, but perhaps more importantly, it extends the applicability of the least squares approach to *inhomogeneous boundary conditions*.

The arising additional norms are typically norms of fractional Sobolev spaces, of the form $H^{\pm\frac{1}{2}}(\Gamma)$ for some $\Gamma \subset \partial\Omega$. In a wavelet setting, the occurrence of such spaces does not give rise to problems. In publications about the finite element solution of least squares problems, often the use of these spaces is avoided, thus restricting them to homogeneous boundary conditions, or the fractional Sobolev norms are replaced by weighted L_2-norms which gives rise to suboptimal results. There is, however, no real need to do so (cf. the discussion in (Bochev & Gunzburger, 2009, §12.1)). In Starke (1999), it was demonstrated how to replace the arising fractional norms by equivalent, efficiently computable quantities in terms of multi-level preconditioners. In this respect, the results of this paper are equally well applicable in a finite element setting.

As a first application of our approach, in Sect. 3 we consider the usual div-grad first order reformulation of an elliptic second order boundary value problem with Neumann and Dirichlet boundary conditions on parts of the boundary. We consider three possible choices for the norms in which the residuals of both PDEs in the system are measured.

With the first, common choice, both residuals are measured in L_2-norms (we call it the *mild-formulation*). With this choice, both Neumann and Dirichlet boundary conditions are essential boundary conditions, and we show how to impose possibly inhomogeneous boundary conditions by appending appropriate squared norms of the corresponding residuals to the quadratic functional.

With the second choice (*mild-weak formulation*), first proposed in Bramble *et al.* (1997), one residual is measured in a dual norm, which turns the Neumann boundary condition into a natural one. A possibly inhomogeneous Dirichlet boundary will be imposed by appending a squared norm of the corresponding residual to the quadratic functional.

With the third choice (*ultra-weak formulation*), both residuals will be measured in dual norms, turning both Neumann and Dirichlet boundary conditions into natural ones.

In view of the application of an adaptive wavelet scheme for solving the least squares problem, we prefer to avoid (components of) the spaces X and Y to be vector spaces like $H(\mathrm{div};\Omega)$ or $H(\mathbf{curl};\Omega)$, since wavelet bases for such spaces are not easily constructed for non-rectangular domains. The aforementioned mild-weak formulation satisfies this requirement. Although a wavelet method can deal with the dual norm in which, with this formulation, a part of the residual is measured, it is more convenient to work with L_2-norms. This holds in particular true for the extension to semi-linear equations (see (Stevenson, 2011, §4)). For this reason returning to the mild formulation, it is known (see Cai *et al.* (1997)) that the $H(\mathrm{div};\Omega)$ space, that arises there, can be replaced by $H^1(\Omega)^n$ by adding to the system redundant equations that involve the curl operator ("extended div-grad" system). The latter is valid for domains Ω for which the intersection of $H(\mathrm{div};\Omega)$ and $H(\mathbf{curl};\Omega)$ is $H^1(\Omega)^n$, with all three space incorporating appropriate boundary conditions. A sufficient, and as we will see, necessary condition on Ω is that the Laplacian is $H^2(\Omega)$-regular.

The boundary conditions incorporated in the resulting $H^1(\Omega)^n$ space read as vanishing normal traces on the Neumann part, and vanishing tangential traces on the Dirichlet part. As a second application of the approach introduced in Sect. 2, in Sect. 4 we will identify spaces of functions on the boundary with respect to which the corresponding trace operator is bounded and surjective. Consequently, by appending squared norms of the boundary residuals to the quadratic functional, we arrive at a well-posed least-squares problem on the full space $H^1(\Omega)^n$. With this, we have generalized the extended div-grad first order least squares method to inhomogeneous boundary conditions. All the arising spaces in this formulation can be equipped with wavelet bases.

2. Appending a constraint as a residual to a least squares problem

The following theorem will be the key to extend a well-posed least squares problem, with homogeneous essential boundary conditions incorporated in the trial space, to a well-posed least squares problem for possibly inhomogeneous boundary conditions.

THEOREM 2.1 Let X, Y_1, and Y_2 be Hilbert spaces. Let $G \in \mathscr{B}(X, Y_1')$ and $T \in \mathscr{B}(X, Y_2')$, with $\Im T$ being closed. With

$$X_0 := \{u \in X : Tu = 0\},$$

let $G \in \mathscr{B}(X_0, Y_1')$ be a *homeomorphism onto its range*, then $(G,T) \in \mathscr{B}(X, Y_1' \times Y_2')$ is a *homeomorphism onto its range*.

Proof. Since $\Im T$ is closed, the *open mapping theorem* shows that T has a right-inverse $E \in \mathscr{B}(\Im T, X)$. For $u \in X$, from $(\mathrm{Id} - ET)u \in X_0$, and $GE \in \mathscr{B}(\Im T, Y_1')$, we have

$$\begin{aligned}\|u\|_X^2 &\lesssim \|ETu\|_X^2 + \|(\mathrm{Id}-ET)u\|_X^2 \lesssim \|Tu\|_{Y_2'}^2 + \|G(\mathrm{Id}-ET)u\|_{Y_1'}^2 \\ &\lesssim \|Tu\|_{Y_2'}^2 + \|Gu\|_{Y_1'}^2 + \|GETu\|_{Y_1'}^2 \lesssim \|Tu\|_{Y_2'}^2 + \|Gu\|_{Y_1'}^2 \lesssim \|u\|_X^2.\end{aligned}$$

□

So in the situation of Thm. 2.1, for $f \in Y_1'$ and $g \in Y_2'$, both problems $\operatorname{argmin}_{v \in X_0} \frac{1}{2}\|Gv - f\|_{Y_1'}^2$ and $\operatorname{argmin}_{v \in X} \frac{1}{2}(\|Gv - f\|_{Y_1'}^2 + \|Tv - g\|_{Y_2'}^2)$ are well-posed. For $g = 0$, both are least squares formulations of the homogeneous problem $\begin{cases} Gu = f, \\ Tu = 0, \end{cases}$ whereas for $g \neq 0$, the second one is a least squares formulation of the inhomogeneous problem $\begin{cases} Gu = f, \\ Tu = g. \end{cases}$

3. Second order elliptic boundary value problem as div-grad system

On a domain $\Omega \subset \mathbb{R}^n$, we consider the boundary value problem

$$\begin{cases} -\operatorname{div} A\nabla p + Bp = f & \text{on } \Omega, \\ p = g & \text{on } \Gamma_D, \\ \mathbf{n} \cdot A\nabla p = h & \text{on } \Gamma_N, \end{cases} \tag{3.1}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\mathbf{n}$ is the outward pointing unit vector normal to the boundary, B is a bounded linear partial differential operator of at most first order, i.e.,

$$\|Bq\|_{L_2(\Omega)} \lesssim \|q\|_{H^1(\Omega)} \quad (q \in H^1(\Omega)), \tag{C.1}$$

and $A(\cdot) \in L_\infty(\Omega)^{n \times n}$ is real, symmetric with

$$\xi^\top A(\cdot)\xi \eqsim \|\xi\|^2 \quad (\xi \in \mathbb{R}^n). \tag{C.2}$$

We assume that the standard variational formulation of (3.1) in case of homogeneous Dirichlet boundary conditions is well-posed, i.e., that with

$$V := H^1(\Omega),$$

or in case $\Gamma_D = \emptyset$, possibly

$$V := H^1(\Omega)/\mathbb{R},$$

and $V_{0,\Gamma_D} := \{v \in V : v = 0 \text{ on } \Gamma_D\}$, the mapping

$$V_{0,\Gamma_D} \to (V_{0,\Gamma_D})' : p \mapsto (q \mapsto \int_\Omega A\nabla p \cdot \nabla q + qBp) \text{ is boundedly invertible.} \tag{C.3}$$

Introducing $\mathbf{u} = \nabla A p$, we consider the reformulation of (3.1) as the first order div-grad system

$$\begin{cases} \mathbf{u} - A\nabla p = 0 & \text{on } \Omega, \\ Bp - \operatorname{div} \mathbf{u} = f & \text{on } \Omega, \\ p = g & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = h & \text{on } \Gamma_N. \end{cases} \tag{3.2}$$

3.1 *The mild formulation*

Let

$$H_{0,\Gamma_N}(\mathrm{div};\Omega) := \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_N\}.$$

Under the assumption of having homogeneous boundary conditions, i.e., $g = 0 = h$, by measuring the residuals of the first two equations from (3.2) in the mild, $L_2(\Omega)$-sense, one arrives at a well-posed least squares problem on $H_{0,\Gamma_N}(\mathrm{div};\Omega) \times V_{0,\Gamma_D}$ in the sense of (1.1):

THEOREM 3.1 Under conditions (C.1)–(C.3), for $(\mathbf{u}, p) \in H_{0,\Gamma_N}(\mathrm{div};\Omega) \times V_{0,\Gamma_D}$,

$$\|\mathbf{u}\|^2_{H(\mathrm{div};\Omega)} + \|p\|^2_{H^1(\Omega)} \eqsim \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \|Bp - \mathrm{div}\,\mathbf{u}\|^2_{L_2(\Omega)}.$$

Proof. For convenience we include a short proof of the even stronger statement that $G : H_{0,\Gamma_N}(\mathrm{div};\Omega) \times V_{0,\Gamma_D} \to L_2(\Omega)^n \times L_2(\Omega) : (\boldsymbol{u}, p) \mapsto (\mathbf{u} - A\nabla p, Bp - \mathrm{div}\,\mathbf{u})$ is boundedly invertible. Earlier proofs of G being a homeomorphism onto its range can be found in (Cai *et al.*, 1994, Thm. 3.1) (for 2 and 3-dimensional Lipschitz domains) and (Bochev & Gunzburger, 2009, Thms. 5.14-5.15) (for Lipschitz domains).

Boundedness of G follows directly from (C.1) and (C.2). Using the open mapping theorem, it now suffices to show that G is surjective and injective, the latter being equivalent to G' being surjective. Given $(\mathbf{f}, g) \in L_2(\Omega)^n \times L_2(\Omega)$, consider the problem of finding $(\mathbf{u}, p) \in H_{0,\Gamma_N}(\mathrm{div};\Omega) \times V_{0,\Gamma_D}$ such that

$$\mathbf{u} - A\nabla p = \mathbf{f}, \quad Bp - \mathrm{div}\,\mathbf{u} = g.$$

Using (C.3), we define p as the solution in V_{0,Γ_D} of

$$\int_\Omega A\nabla p \cdot \nabla\tilde{p} + \tilde{p}Bp = \int_\Omega g\tilde{p} - \nabla\tilde{p}\cdot\mathbf{f}, \quad (\tilde{p} \in V_{0,\Gamma_D}),$$

and take $\mathbf{u} = A\nabla p + \mathbf{f} \in L_2(\Omega)$. From $\int_\Omega \mathbf{u}\cdot\nabla\tilde{p} = \int_\Omega (g - Bp)\tilde{p}$ $(\tilde{p} \in V_{0,\Gamma_D})$, we find that $\mathrm{div}\,\mathbf{u} = Bp - g \in L_2(\Omega)$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ_N.

To show that G' is surjective, given $(\mathbf{f}, g) \in H_{0,\Gamma_N}(\mathrm{div};\Omega)' \times V'_{0,\Gamma_D}$, consider the problem of finding $(\mathbf{v}, q) \in L_2(\Omega)^n \times L_2(\Omega)$ such that

$$\int_\Omega \mathbf{u}\cdot\mathbf{v} - q\,\mathrm{div}\,\mathbf{u} = \mathbf{f}(\mathbf{u}) \quad (\mathbf{u} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)),$$
$$\int_\Omega -A\nabla p\cdot\mathbf{v} + qBp = g(p) \quad (p \in V_{0,\Gamma_D}(\Omega)).$$

Using Riesz' representation theorem, there exists an $\mathbf{r} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)$ such that $\mathbf{f}(\mathbf{u}) = \int_\Omega \mathbf{r}\cdot\mathbf{u} + \mathrm{div}\,\mathbf{r}\,\mathrm{div}\,\mathbf{u}$. Introducing $\underline{\mathbf{v}} = \mathbf{v} - \mathbf{r}$ and $\underline{q} = q + \mathrm{div}\,\mathbf{r}$, the above system reads as

$$\int_\Omega \mathbf{u}\cdot\underline{\mathbf{v}} - \underline{q}\,\mathrm{div}\,\mathbf{u} = 0 \qquad (\mathbf{u} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)),$$
$$\int_\Omega -A\nabla p\cdot\underline{\mathbf{v}} + \underline{q}Bp = g(p) + \int_\Omega A\nabla p\cdot\mathbf{r} + \mathrm{div}\,\mathbf{r}\,Bp \quad (p \in V_{0,\Gamma_D}(\Omega)).$$

Since bounded invertibility of the mapping guaranteed by (C.3) implies bounded invertibility of the adjoint mapping, we may define $\underline{q}$ as the solution in V_{0,Γ_D} of

$$\int_\Omega A\nabla p\cdot\nabla\underline{q} + \underline{q}Bp = g(p) + \int_\Omega A\nabla p\cdot\mathbf{r} + \mathrm{div}\,\mathbf{r}Bp, \quad (p \in V_{0,\Gamma_D}),$$

and take $\underline{\mathbf{v}} = -\nabla \underline{q}$. Then both equations are satisfied, $\mathbf{v} = \underline{\mathbf{v}} + \mathbf{r} \in L_2(\Omega)^n$, and $q = \underline{q} - \operatorname{div} \mathbf{r} \in L_2(\Omega)$, with which the proof is completed. $\square$

Next, we consider the case of having possibly *inhomogeneous boundary conditions*. It is well-known, e.g. see (Girault & Raviart, 1979, Corollary 2.4), that

$$T : (\mathbf{u}, p) \mapsto (\mathbf{n} \cdot \mathbf{u}|_{\Gamma_N}, p|_{\Gamma_D}) \in \mathscr{B}(H(\mathrm{div};\Omega) \times V, H^{-\frac{1}{2}}(\Gamma_N) \times H^{\frac{1}{2}}(\Gamma_D)) \text{ is surjective,}$$

so that in particular its image is closed. Since the inequality "$\gtrsim$" from Theorem 3.1 obviously also holds for $(\mathbf{u}, p) \in H(\mathrm{div};\Omega) \times V$, using the result from Theorem 3.1 an application of Theorem 2.1 gives the following:

COROLLARY 3.1 Under conditions (C.1)–(C.3), for $(\mathbf{u}, p) \in H(\mathrm{div};\Omega) \times V$,

$$\begin{aligned} &\|\mathbf{u}\|^2_{H(\mathrm{div};\Omega)} + \|p\|^2_{H^1(\Omega)} \eqsim \\ &\quad \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \|Bp - \operatorname{div} \mathbf{u}\|^2_{L_2(\Omega)} + \|p|_{\Gamma_D}\|^2_{H^{\frac{1}{2}}(\Gamma_D)} + \|\mathbf{n} \cdot \mathbf{u}|_{\Gamma_N}\|^2_{H^{-\frac{1}{2}}(\Gamma_N)}. \end{aligned}$$

A direct proof of the latter result can be found in Starke (1999) (for $B = 0$ and $A = \mathrm{Id}$).

3.2 *The mild-weak formulation*

In Bramble *et al.* (1997, 1998), an alternative least squares functional for the div-grad system was proposed, in which the second equation $Bp - \operatorname{div} \mathbf{u} = f$ from (3.2) is imposed only weakly. As shown by the next theorem, under the assumption of the Dirichlet boundary condition being homogeneous, it leads to a well-posed least squares problem on $L_2(\Omega)^n \times V_{0,\Gamma_D}$.

THEOREM 3.2 Under conditions (C.1)–(C.3), for $(\mathbf{u}, p) \in L_2(\Omega)^n \times V_{0,\Gamma_D}$ we have

$$\|\mathbf{u}\|^2_{L_2(\Omega)^n} + \|p\|^2_{H^1(\Omega)} \eqsim \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \left\| q \mapsto \int_\Omega qBp + \mathbf{u} \cdot \nabla q \right\|^2_{V'_{0,\Gamma_D}}.$$

Proof. For convenience we give the proof using arguments from Bramble *et al.* (1998). Condition (C.3) shows that for $(\mathbf{u}, p) \in L_2(\Omega)^n \times V_{0,\Gamma_D}$,

$$\begin{aligned} \|p\|_{H^1(\Omega)} &\lesssim \sup_{0 \neq q \in V_{0,\Gamma_D}} \frac{\int_\Omega A\nabla p \cdot \nabla q + qBp}{\|q\|_{H^1(\Omega)}} \\ &= \sup_{0 \neq q \in V_{0,\Gamma_D}} \frac{\int_\Omega (A\nabla p - \mathbf{u}) \cdot \nabla q + qBp + \mathbf{u} \cdot \nabla q}{\|q\|_{H^1(\Omega)}} \\ &\leqslant \|\mathbf{u} - A\nabla p\|_{L_2(\Omega)^n} + \left\| q \mapsto \int_\Omega qBp + \mathbf{u} \cdot \nabla q \right\|_{V'_{0,\Gamma_D}}. \end{aligned}$$

From $\|\mathbf{u}\|_{L_2(\Omega)^n} \lesssim \|\mathbf{u} - A\nabla p\|_{L_2(\Omega)^n} + \|p\|_{H^1(\Omega)}$ by Condition (C.2), we infer that $\|u\|^2_{L_2(\Omega)^n} + \|p\|^2_{H^1(\Omega)} \lesssim \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \left\| q \mapsto \int_\Omega qBp + \mathbf{u} \cdot \nabla q \right\|^2_{V'_{0,\Gamma_D}}$. The reversed inequality follows easily from (C.1)-(C.2). $\square$

With a least squares formulation of (3.2) corresponding to Thm 3.2, the Neumann boundary condition on Γ_N is *a natural boundary condition*. Indeed, with $G : L_2(\Omega)^n \times V_{0,\Gamma_D} \to L_2(\Omega)^n \times V'_{0,\Gamma_D}$ defined by

$$G(\mathbf{u}, p)(\mathbf{v}, q) = \int_\Omega (\mathbf{u} - A\nabla p) \cdot \mathbf{v} + qBp + \mathbf{u} \cdot \nabla q,$$

a consistent variational formulation of (3.2) with $g = 0$ reads as

$$G(\mathbf{u},p)(\mathbf{v},q) = \int_\Omega fq + \int_{\Gamma_N} hq \quad ((\mathbf{v},q) \in L_2(\Omega)^n \times V_{0,\Gamma_D}).$$

Note that $q \mapsto \int_\Omega fq + \int_{\Gamma_N} hq \in V'_{0,\Gamma_D}$ when $f \in V'_{0,\Gamma_D}$ and $h \in H^{-\frac{1}{2}}(\Gamma_N) := H^{\frac{1}{2}}_{00}(\Gamma_N)'$.

To append an inhomogeneous Dirichlet boundary condition, we use that $L_2(\Omega)^n \times V \to H^{\frac{1}{2}}(\Gamma_D) : (\mathbf{u},p) \mapsto p|_{\Gamma_D}$ is surjective, so that in particular its image is closed. Since the inequality "$\gtrsim$" from Theorem 3.2 obviously also holds for $(\mathbf{u},p) \in L_2(\Omega)^n \times V$, using the result from Theorem 3.2 an application of Theorem 2.1 gives the following:

COROLLARY 3.2 Under conditions (C.1)–(C.3), for $(\mathbf{u},p) \in L_2(\Omega)^n \times V$,

$$\|\mathbf{u}\|^2_{L_2(\Omega)^n} + \|p\|^2_{H^1(\Omega)} \eqsim \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \left\| q \mapsto \int_\Omega qBp + \mathbf{u}\cdot\nabla q \right\|^2_{V'_{0,\Gamma_D}} + \|p|_{\Gamma_D}\|^2_{H^{\frac{1}{2}}(\Gamma_D)}.$$

3.3 *The ultra-weak formulation*

In order to derive this formulation, we specify Condition (C.1) to

$$Bq = \mathbf{b}\cdot\nabla q + cq \quad \text{for some } \mathbf{b} \in L_\infty(\Omega)^n,\ c \in L_\infty(\Omega). \tag{C.4}$$

By testing the first and second equation in (3.2) for $\mathbf{v} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)$ and $q \in H^1_{0,\Gamma_D}(\Omega)$, applying integration by parts, and substituting the desired boundary conditions, we arrive at the system

$$\begin{aligned}
\int_\Omega A^{-1}\mathbf{u}\cdot\mathbf{v} + p\,\mathrm{div}\,\mathbf{v} &= \int_{\Gamma_D} g\,\mathbf{v}\cdot\mathbf{n} && (\mathbf{v} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)),\\
\int_\Omega (\mathbf{b}\cdot A^{-1}\mathbf{u} + cp)q + \mathbf{u}\cdot\nabla q &= \int_\Omega fq + \int_{\Gamma_N} hq && (q \in V_{0,\Gamma_D}(\Omega)).
\end{aligned}$$

Here we have substituted $\nabla p = A^{-1}\mathbf{u}$ in the second equation in order to avoid additional smoothness conditions on $\mathbf{b}$. Note that in this system *both* Neumann and Dirichlet boundary conditions enter as *natural boundary conditions*. For $g \in H^{\frac{1}{2}}(\Gamma_D)$ and $h \in H^{-\frac{1}{2}}(\Gamma_N)$, we have $\mathbf{v} \mapsto \int_{\Gamma_D} g\,\mathbf{v}\cdot\mathbf{n} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)'$, and $q \mapsto \int_{\Gamma_N} hq \in V_{0,\Gamma_D}(\Omega)'$.

As shown in the next theorem, the above system leads to a well-posed least-squares problem.

THEOREM 3.3 Under Conditions (C.2)-(C.4), for $(\mathbf{u},p) \in L_2(\Omega)^n \times L_2(\Omega)$,

$$\begin{aligned}
&\|\mathbf{u}\|^2_{L_2(\Omega)^n} + \|p\|^2_{L_2(\Omega)} \eqsim \\
&\left\|\mathbf{v} \mapsto \int_\Omega A^{-1}\mathbf{u}\cdot\mathbf{v} + p\,\mathrm{div}\,\mathbf{v}\right\|^2_{H_{0,\Gamma_N}(\mathrm{div};\Omega)'} + \left\|\mathbf{q} \mapsto \int_\Omega (\mathbf{b}\cdot A^{-1}\mathbf{u} + cp)q + \mathbf{u}\cdot\nabla q\right\|^2_{V_{0,\Gamma_D}(\Omega)'}
\end{aligned}$$

Proof. This proof is similar to that of Thm. 3.2. With $G : L_2(\Omega)^n \times L_2(\Omega) \to H_{0,\Gamma_N}(\mathrm{div};\Omega)' \times V_{0,\Gamma_D}(\Omega)'$ defined by

$$(G(\mathbf{u},p))(\mathbf{v},q) = \int_\Omega A^{-1}\mathbf{u}\cdot\mathbf{v} + p\,\mathrm{div}\,\mathbf{v} + (\mathbf{b}\cdot A^{-1}\mathbf{u} + cp)q + \mathbf{u}\cdot\nabla q,$$

we even show that G is boundedly invertible. The boundedness of G follows from (C.2) and (C.4). Using the open mapping theorem, it now suffices to show that G and G' are surjective.

Given $(\mathbf{f},g)\in H_{0,\Gamma_N}(\operatorname{div};\Omega)'\times V_{0,\Gamma_D}(\Omega)'$, consider the problem of finding $(\mathbf{u},p)\in L_2(\Omega)^n\times L_2(\Omega)$ such that

$$\int_\Omega A^{-1}\mathbf{u}\cdot\mathbf{v}+p\operatorname{div}\mathbf{v}=\mathbf{f}(\mathbf{v})\quad (\mathbf{v}\in H_{0,\Gamma_N}(\operatorname{div};\Omega)),$$
$$\int_\Omega(\mathbf{b}\cdot A^{-1}\mathbf{u}+cp)q+\mathbf{u}\cdot\nabla q=g(q)\quad (q\in V_{0,\Gamma_D}(\Omega)).$$

Since, thanks to (C.2), $\sqrt{\|A^{-\frac{1}{2}}\cdot\|^2_{L_2(\Omega^n)}+\|\operatorname{div}\cdot\|^2_{L_2(\Omega)}}$ defines a Hilbertian, equivalent norm on $H(\operatorname{div};\Omega)$, an application of Riesz' representation theorem shows that there exists an $\mathbf{r}\in H(\operatorname{div};\Omega)$ such that $\mathbf{f}(\mathbf{v})=\int_\Omega A^{-1}\mathbf{r}\cdot\mathbf{v}+\operatorname{div}\mathbf{r}\operatorname{div}\mathbf{v}$. Introducing $\underline{\mathbf{u}}=\mathbf{u}-\mathbf{r}$, and $\underline{p}=p-\operatorname{div}\mathbf{r}$, the above system reads as

$$\int_\Omega A^{-1}\underline{\mathbf{u}}\cdot\mathbf{v}+\underline{p}\operatorname{div}\mathbf{v}=0\qquad (\mathbf{v}\in H_{0,\Gamma_N}(\operatorname{div};\Omega)),$$
$$\int_\Omega(\mathbf{b}\cdot A^{-1}\underline{\mathbf{u}}+c\underline{p})q+\underline{\mathbf{u}}\cdot\nabla q=g(q)-\int_\Omega(\mathbf{b}\cdot A^{-1}\mathbf{r}+c\operatorname{div}\mathbf{r})q+\mathbf{r}\cdot\nabla q\quad (q\in V_{0,\Gamma_D}(\Omega)).$$

Thanks to (C.3), we may $\underline{p}$ define as the solution in V_{0,Γ_D} of

$$\int_\Omega(\mathbf{b}\cdot\nabla\underline{p}+c\underline{p})q+A\nabla\underline{p}\cdot\nabla q=g(q)-\int_\Omega(\mathbf{b}\cdot A^{-1}\mathbf{r}+c\operatorname{div}\mathbf{r})q+\mathbf{r}\cdot\nabla q\quad (q\in V_{0,\Gamma_D}(\Omega)),$$

and take $\underline{\mathbf{u}}=A\nabla\underline{p}$. Then both equations are satisfied, $\mathbf{u}=\underline{\mathbf{u}}+\mathbf{r}\in L_2(\Omega)^n$, and $p=\underline{p}+\operatorname{div}\mathbf{r}\in L_2(\Omega)$.

To show surjectivity of G', given $(\mathbf{f},g)\in L_2(\Omega^n)\times L_2(\Omega)$, consider the problem of finding $(\mathbf{v},q)\in H_{0,\Gamma_N}(\operatorname{div};\Omega)\times V_{0,\Gamma_D}(\Omega)$ such that

$$A^{-1}\mathbf{v}+A^{-1}q\mathbf{b}+\nabla q=\mathbf{f},\quad \operatorname{div}\mathbf{v}+cq=g.$$

Since bounded invertibility of the mapping guaranteed by (C.3) implies bounded invertibility of the adjoint mapping, we may define q as the solution in V_{0,Γ_D} of

$$\int_\Omega(A\nabla q+q\mathbf{b})\cdot\nabla\tilde{q}+cq\tilde{q}=\int_\Omega g\tilde{q}+A\mathbf{f}\cdot\nabla\tilde{q}\quad (\tilde{q}\in V_{0,\Gamma_D}).$$

and take $\mathbf{v}=A\mathbf{f}-q\mathbf{b}-A\nabla q\in L_2(\Omega)^n$. From $-\int_\Omega\mathbf{v}\cdot\nabla\tilde{p}=\int_\Omega(g-cq)\tilde{q}$ $(\tilde{q}\in V_{0,\Gamma_D})$, we find that $\operatorname{div}\mathbf{v}=g-cq\in L_2(\Omega)$ and $\mathbf{v}\cdot\mathbf{n}=0$ on Γ_N, with which the proof is completed.

□

4. The extended div-grad system

With the aim, as exposed in the introduction, to avoid the vector space $H(\operatorname{div};\Omega)$ from the mild variational formulation from Corollary 3.1, and simultaneously to measure the residual of the equation $Bp-\operatorname{div}\mathbf{u}=f$ in $L_2(\Omega)$, we consider the augmentation of the div-grad system (3.2) with redundant equations. Restricting to bounded domains

$$\Omega\subset\mathbb{R}^n\quad\text{for } n\in\{2,3\},$$

we set $\mathbf{curl}\,\mathbf{v}:=\nabla\times\mathbf{v}$, and following Buffa & Ciarlet (2001), define the *tangential components trace* and *tangential gradient* operators

$$\pi_{\boldsymbol{\tau}}\mathbf{v}:=\mathbf{v}|_{\partial\Omega}-(\mathbf{n}\cdot\mathbf{v}|_{\partial\Omega})\mathbf{n},\quad \nabla_{\partial\Omega}\,q:=\pi_{\boldsymbol{\tau}}\nabla q.$$

In two space dimensions, these operators should be interpreted by means of $\mathbf{a}\times\mathbf{b}=a_1b_2-a_2b_1$ and $\pi_{\boldsymbol{\tau}}\mathbf{v}:=\mathbf{n}\times\mathbf{v}|_{\partial\Omega}$. Noting that the **curl** of a gradient vanishes, from (3.2) we arrive at the *extended* div-grad system

$$\begin{cases} \mathbf{u}-A\nabla p=0 & \text{on } \Omega,\\ Bp-\operatorname{div}\mathbf{u}=f & \text{on } \Omega,\\ \mathbf{curl}A^{-1}\mathbf{u}=0 & \text{on } \Omega,\\ p=g & \text{on } \Gamma_D,\\ \pi_{\boldsymbol{\tau}}(A^{-1}\mathbf{u})=\nabla_{\partial\Omega}\,g & \text{on } \Gamma_D,\\ \mathbf{n}\cdot\mathbf{u}=h & \text{on } \Gamma_N. \end{cases} \tag{4.1}$$

REMARK 4.1 In three space dimensions, one has that $\pi_{\boldsymbol{\tau}}\mathbf{v}=\mathbf{n}\times(\mathbf{v}|_{\partial\Omega}\times\mathbf{n})$. Consequently, an alternative form of the second boundary condition on Γ_D is given by $\mathbf{n}\times A^{-1}\mathbf{u}|_{\Gamma_D}=\mathbf{n}\times\nabla g$. The difference between the mappings $\mathbf{u}\mapsto\pi_{\boldsymbol{\tau}}(A^{-1}\mathbf{u})$ and $\mathbf{u}\mapsto\mathbf{n}\times A^{-1}\mathbf{u}|_{\partial\Omega}$ only concerns a harmless rotation over $\pi/2$ in the tangential plane.

As in Subsection 3.1, first we study well-posedness of this system in least squares sense for homogeneous boundary conditions. With

$$H(\mathbf{curl}A;\Omega):=\{\mathbf{v}\in L_2(\Omega)^n:\mathbf{curl}A^{-1}\mathbf{v}\in L_2(\Omega)^{2n-3}\},$$

equipped with $\|\mathbf{v}\|^2_{H(\mathbf{curl}A;\Omega)}:=\|\mathbf{v}\|^2_{L_2(\Omega)^n}+\|\mathbf{curl}A^{-1}\mathbf{v}\|^2_{L_2(\Omega)^{2n-3}}$, and

$$H_{0,\Gamma_D}(\mathbf{curl}A;\Omega):=\{\mathbf{v}\in H(\mathbf{curl}A;\Omega):\pi_{\boldsymbol{\tau}}(A^{-1}\mathbf{v})=0\text{ on }\Gamma_D\},$$

as an immediate consequence of Theorem 3.1, we have

COROLLARY 4.1 Under conditions (C.1)–(C.3), for $(\mathbf{u},p)\in H_{0,\Gamma_N}(\operatorname{div};\Omega)\cap H_{0,\Gamma_D}(\mathbf{curl}A;\Omega)\times V_{0,\Gamma_D}$ we have

$$\begin{aligned}&\|\mathbf{u}\|^2_{H(\operatorname{div};\Omega)}+\|\mathbf{u}\|^2_{H(\mathbf{curl}A;\Omega)}+\|p\|^2_{H^1(\Omega)}\\ &\eqsim\|\mathbf{u}-A\nabla p\|^2_{L_2(\Omega)^n}+\|Bp-\operatorname{div}\mathbf{u}\|^2_{L_2(\Omega)}+\|\mathbf{curl}A^{-1}\mathbf{u}\|^2_{L_2(\Omega)^{2n-3}}.\end{aligned}$$

The next step will be to replace the squared norm $\|\cdot\|^2_{H(\operatorname{div};\Omega)}+\|\cdot\|^2_{H(\mathbf{curl}A;\Omega)}$ on $H_{0,\Gamma_N}(\operatorname{div};\Omega)\cap H_{0,\Gamma_D}(\mathbf{curl}A;\Omega)$ by $\|\cdot\|^2_{H^1(\Omega)^n}$. For this goal, we impose the following additional conditions:

$A\in C^{1,1}(\bar{\Omega})$; (C.5)

$\Gamma:=\partial\Omega$ consists of a finite number of disjoint, simple and closed curves ($n=2$) or surfaces ($n=3$) $\Gamma_0,\ldots,\Gamma_L$, where Γ_0 is the outer boundary of Ω, and, for $i\geqslant 1$, Γ_i is the boundary of a hole in Ω. Each Γ_i is assumed to be the boundary of a $C^{1,1}$-curved polytope, with edges or faces $\Gamma_i^{(1)},\ldots,\Gamma_i^{(K_i)}$, meaning that for any $x\in\Gamma_i$, there is a $C^{1,1}$ diffeomorphism from a neighborhood of x onto a neighborhood of a point on the boundary of a polytope, that maps Γ_i onto the boundary of this polytope; (C.6)

For $n=3$, each Γ_i is either part of Γ_D or of Γ_N. For $n=2$, each $\Gamma_i^{(k)}$ is either part of Γ_D or of Γ_N, and if $\mathbf{x}\in\Gamma$ separates Γ_D and Γ_N, then $\mathbf{n}_-^\top A\mathbf{n}_+\neq 0$, where $\mathbf{n}_-$ and $\mathbf{n}_+$ are outward normal vectors on the adjacent edges at $\mathbf{x}$; (C.7)

The (variational formulation) of the boundary value problem (3.1) with homogeneous boundary data is $H^2(\Omega)$*-regular.* (C.8)

Sufficient conditions for (C.8) are (cf. Cai *et al.* (1997) and references cited there):

> For $i \geqslant 1$, $\Gamma_i \in C^{1,1}$, i.e., $K_i = 1$. For $n = 3$, $\Gamma_0 \in C^{1,1}$, i.e, $K_0 = 1$, or Γ_0 is a convex polyhedron. For $n = 2$, Γ_0 has no re-entrant corners, and $\mathbf{n}_-^\top A\mathbf{n}_+ < 0$ at $x \in \Gamma$ that separates Γ_D and Γ_N.

The main result from Cai *et al.* (1997) reads as follows:

LEMMA 4.1 ((Cai *et al.*, 1997, Thm. 2.2)) Under conditions (C.2), and (C.5)–(C.8), for $\mathbf{u} \in H_{0,\Gamma_N}(\mathrm{div};\Omega) \cap H_{0,\Gamma_D}(\mathbf{curl} A;\Omega)$ it holds that

$$\|\mathbf{u}\|^2_{H(\mathrm{div};\Omega)} + \|\mathbf{u}\|^2_{H(\mathbf{curl}A;\Omega)} \eqsim \|\mathbf{u}\|^2_{H^1(\Omega)^n}.$$

Together with Corollary 4.1, this lemma leads to

THEOREM 4.1 For $\mathbf{u} \in \{\mathbf{v} \in H^1(\Omega)^n : \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_N, \pi_{\boldsymbol{\tau}}(A^{-1}\mathbf{v}) = 0 \text{ on } \Gamma_D\}$, $p \in V_{0,\Gamma_D}$, under conditions (C.1)–(C.3), and (C.5)–(C.8), it holds that

$$\begin{aligned}&\|\mathbf{u}\|^2_{H^1(\Omega)^n} + \|p\|^2_{H^1(\Omega)}\\ &\eqsim \|\mathbf{u} - A\nabla p\|^2_{L_2(\Omega)^n} + \|Bp - \mathrm{div}\,\mathbf{u}\|^2_{L_2(\Omega)} + \|\mathbf{curl} A^{-1}\mathbf{u}\|^2_{L_2(\Omega)^{2n-3}}.\end{aligned}$$

Before continuing, we note that the restrictive condition (C.8) is *necessary* for Lemma 4.1 to hold. Indeed, for $f \in L_2(\Omega) \cap (V_{0,\Gamma_D})'$, let $p \in V_{0,\Gamma_D}$ solve

$$\int_\Omega A\nabla p \cdot \nabla q + Bpq\,dx = \int_\Omega fq\,dx \quad (q \in V_{0,\Gamma_D}).$$

Setting $\mathbf{u} = A\nabla p \in L_2(\Omega)$, we have

$$\int_\Omega \mathbf{u}\cdot\nabla q\,dx = \int_\Omega fq - Bpq\,dx \quad (q \in V_{0,\Gamma_D}),$$

or $\mathrm{div}\,\mathbf{u} = Bp - f \in L_2(\Omega)$, $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ_N, and so $\mathbf{u} \in H_{0,\Gamma_N}(\mathrm{div};\Omega)$. Since $\mathbf{u} = A\nabla p$, we have $\mathbf{curl} A^{-1}\mathbf{u} = 0$, whereas $p = 0$ on Γ_D shows that $\pi_{\boldsymbol{\tau}}(A^{-1}\mathbf{u}) = 0$ on Γ_D, or $\mathbf{u} \in H_{0,\Gamma_D}(\mathbf{curl};\Omega)$. By (C.2) and (C.5), the validity of Lemma 4.1, (C.1), and (C.3), we have

$$\|p\|_{H^2(\Omega)} \lesssim \|\mathbf{u}\|_{H^1(\Omega)^n} \lesssim \|\mathbf{u}\|_{H(\mathrm{div};\Omega)} \lesssim \|p\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)} \lesssim \|f\|_{L_2(\Omega)},$$

i.e., (C.8) is valid.

Next, we consider the case of having possibly *inhomogeneous boundary conditions*. Similar to the vector space setting discussed in the previous subsection, in order to do so we will need function spaces on Γ_D and Γ_N which are the images of V or $H^1(\Omega)^n$ under the trace mappings $p \mapsto p|_{\Gamma_D}$, $\mathbf{v} \mapsto (\mathbf{n}\cdot\mathbf{v}|_{\Gamma_N}, \pi_{\boldsymbol{\tau}}(\mathbf{v})|_{\Gamma_D})$.

For $n = 3$, $0 \leqslant i \leqslant L$ with $\Gamma_i \subset \Gamma_D$, and $1 \leqslant k \leqslant K_i$, let $\mathscr{O}_i^{(k)}$ the collection of $1 \leqslant \ell \neq k \leqslant K_i$ for which $\Gamma_i^{(\ell)}$ shares an edge, denoted as $e_i^{(k,\ell)}$, with $\Gamma_i^{(k)}$. The tangential trace mapping will require special attention. The point is that for $\mathbf{v} \in H^1(\Omega)^3$, the component of $\pi_{\boldsymbol{\tau}}(\mathbf{v})|_{\Gamma_i^{(\ell)} \cup \Gamma_i^{(k)}}$ in the direction parallel to $e_i^{(k,\ell)}$ will be in $H^{\frac{1}{2}}(\Gamma_i^{(k)} \cup \Gamma_i^{(\ell)})$, whereas the remaining non-zero component of $\pi_{\boldsymbol{\tau}}(\mathbf{v})$ on $\Gamma_i^{(k)}$ and $\Gamma_i^{(\ell)}$ will generally have not any smoothness over $e_i^{(k,\ell)}$.

With $\mathbf{n}_i^{(k)}$ and $\mathbf{n}_i^{(\ell)}$ denoting the outward unit normal vectors at $\Gamma_i^{(k)}$ and $\Gamma_i^{(\ell)}$, let $\boldsymbol{\tau}_i^{(k,\ell)}$ be a Lipschitz continuous unit tangent vector on $\Gamma_i^{(k)} \cup \Gamma_i^{(\ell)}$ with

$$\boldsymbol{\tau}_i^{(k,\ell)} = \frac{\mathbf{n}_i^{(k)} \times \mathbf{n}_i^{(\ell)}}{\|\mathbf{n}_i^{(k)} \times \mathbf{n}_j^{(\ell)}\|} \quad \text{on } e_i^{(k,\ell)}.$$

Because of $\Gamma_i^{(k)} \in C^{1,1}$, one has that $\mathbf{n}_i^{(k)}$ is Lipschitz, and so indeed $\boldsymbol{\tau}_{ij}$ is Lipschitz on $e_i^{(k,\ell)}$. It can be extended to a Lipschitz unit tangent vector on $\Gamma_i^{(k)}$, and similar on $\Gamma_i^{(\ell)}$, by means of, e.g., $\boldsymbol{\tau}_i^{(k,\ell)}(\mathbf{x}) := \frac{\mathbf{n}_i^{(k)}(\mathbf{x}) \times (\boldsymbol{\tau}_i^{(k,\ell)}(\mathbf{x}') \times \mathbf{n}_i^{(k)}(\mathbf{x}'))}{\|\mathbf{n}_i^{(k)}(\mathbf{x}) \times (\boldsymbol{\tau}_i^{(k,\ell)}(\mathbf{x}') \times \mathbf{n}_i^{(k)}(\mathbf{x}'))\|}$, where $\mathbf{x}'$ is the point on $e_i^{(k,\ell)}$ whose pull-back is nearest to the pull-back of $\mathbf{x}$ in the parameter plane. Note that with this definition, $\boldsymbol{\tau}_i^{(k,\ell)}(\mathbf{x}) = \boldsymbol{\tau}_i^{(k,\ell)}(\mathbf{x}')$ when $\mathbf{n}_i^{(k)}(\mathbf{x}) = \mathbf{n}_i^{(k)}(\mathbf{x}')$, and so in particular that $\boldsymbol{\tau}_i^{(k,\ell)}$ is constant on $\Gamma_i^{(k)}$ when $\Gamma_i^{(k)}$ is planar.

Following Buffa & Ciarlet (2001), we set the Hilbert space

$$\begin{aligned}\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma_i) = \Big\{ \mathbf{w} \in L_2(\Gamma_i)^3 : \mathbf{n} \cdot \mathbf{w} = 0 \wedge \forall 1 \leqslant k \leqslant K_i, \ell \in \mathscr{O}_i^{(k)}, \mathbf{w}|_{\Gamma_i^{(k)}} \in H^{\frac{1}{2}}(\Gamma_i^{(k)})^3, \\ \int_{\Gamma_i^{(k)}} \int_{\Gamma_i^{(\ell)}} \frac{|(\mathbf{w} \cdot \boldsymbol{\tau}_i^{(k,\ell)})(\mathbf{x}) - (\mathbf{w} \cdot \boldsymbol{\tau}_i^{(k,\ell)})(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) < \infty \Big\},\end{aligned}$$

equipped with the squared norm

$$\sum_{k=1}^{K_i} \Big\{ \|\mathbf{w}\|^2_{H^{\frac{1}{2}}(\Gamma_i^{(k)})^3} + \sum_{\ell \in \mathscr{O}_i^{(k)}} \int_{\Gamma_i^{(k)}} \int_{\Gamma_i^{(\ell)}} \frac{|(\mathbf{w} \cdot \boldsymbol{\tau}_i^{(k,\ell)})(\mathbf{x}) - (\mathbf{w} \cdot \boldsymbol{\tau}_i^{(k,\ell)})(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^3} d\sigma(\mathbf{x}) d\sigma(\mathbf{y}) \Big\}.$$

REMARK 4.2 Arguments that will be applied in the second paragraph of the proof of Theorem 4.2 show that different extensions of $\boldsymbol{\tau}_i^{(k,\ell)}|_{e_i^{(k,\ell)}}$ to a Lipschitz continuous unit tangent vector on $\Gamma_i^{(k)} \cup \Gamma_i^{(\ell)}$ lead to the same space $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma_i)$, with equivalent norms. It is even allowed that outside $e_i^{(k,\ell)}$, $\boldsymbol{\tau}_i^{(k,\ell)}$ is not a tangent or a unit vector.

REMARK 4.3 As we have seen, if Γ_i is the boundary of a polyhedron, then the obvious choice for $\boldsymbol{\tau}_i^{(k,\ell)}$ on $\Gamma_i^{(k)} \cup \Gamma_i^{(\ell)}$ is the constant extension of $\boldsymbol{\tau}_i^{(k,\ell)}|_{e_i^{(k,\ell)}}$. In that case, the tangent space of $\Gamma_i^{(k)}$ is spanned by $\{\boldsymbol{\tau}_i^{(k,\ell)} : \ell \in \mathscr{O}_i^{(k)}\}$, so that an equivalent squared norm on $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma_i)$ is given by

$$\sum_{k=1}^{K_i} \sum_{\ell \in \mathscr{O}_i^{(k,\ell)}} \|\mathbf{w} \cdot \boldsymbol{\tau}_i^{(k,\ell)}\|^2_{H^{\frac{1}{2}}(\Gamma_i^{(k)} \cup \Gamma_i^{(\ell)})}.$$

THEOREM 4.2 Under conditions (C.1)–(C.3), and (C.5)–(C.8), for $(\mathbf{u},p)\in H^1(\Omega)^n\times V$, it holds that

$$\|\mathbf{u}\|^2_{H^1(\Omega)^n}+\|p\|^2_{H^1(\Omega)}\eqsim$$
$$\|\mathbf{u}-A\nabla p\|^2_{L_2(\Omega)^n}+\|Bp-\operatorname{div}\mathbf{u}\|^2_{L_2(\Omega)}+\|\mathbf{curl}A^{-1}\mathbf{u}\|^2_{L_2(\Omega)^{2n-3}}+\|p|_{\Gamma_D}\|^2_{H^{\frac12}(\Gamma_D)}+$$
$$\begin{cases}\displaystyle\sum_{i=0}^{L}\Big\{\sum_{\{k:\Gamma_i^{(k)}\subset\Gamma_N\}}\|\mathbf{n}\cdot\mathbf{u}|_{\Gamma_i^{(k)}}\|^2_{H^{\frac12}(\Gamma_i^{(k)})}+\sum_{\{k:\Gamma_i^{(k)}\subset\Gamma_D\}}\|\pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})|_{\Gamma_i^{(k)}}\|^2_{H^{\frac12}(\Gamma_i^{(k)})}\Big\} & (n=2)\\ \displaystyle\sum_{\{0\leqslant i\leqslant L:\Gamma_i\subset\Gamma_N\}}\sum_{k=1}^{K_i}\|\mathbf{n}\cdot\mathbf{u}|_{\Gamma_i^{(k)}}\|^2_{H^{\frac12}(\Gamma_i^{(k)})}+\sum_{\{0\leqslant i\leqslant L:\Gamma_i\subset\Gamma_D\}}\|\pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})|_{\Gamma_i}\|^2_{\mathbf{H}_{\|}^{\frac12}(\Gamma_i)} & (n=3).\end{cases}$$

REMARK 4.4 In (Bochev & Gunzburger, 2009, Thm 5.10) a similar setting is considered except that $B=0$, $\partial\Omega\in C^1$, and either $\Gamma_N=\partial\Omega$ or $\Gamma_D=\partial\Omega$. For the second, most challenging case, it seems that a proper squared norm of $\pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})$ or $\mathbf{n}\times A^{-1}\mathbf{u}|_{\partial\Omega}$ is missing in the quadratic functional. In particular in the practically relevant case of $\partial\Omega$ being the boundary of a (curved) polytope, so generally not C^1, the construction of such a squared norm turned out to be delicate.

Proof. Because of (C.1), (C.2), and (C.5), the inequality "$\gtrsim$" from Theorem 4.1 is also valid for $(\mathbf{u},p)\in H^1(\Omega)^n\times V$. Knowing the result of Theorem 4.1, and the boundedness and surjectivity of $V\to H^{\frac12}(\Gamma_D):p\mapsto p|_{\Gamma_D}$, in view of Theorem 3.1 what remains to verify is the boundedness and the closedness of the image of T defined by

$$T:H^1(\Omega)^2\to\prod_{i=0}^{L}\prod_{k=1}^{K_i}H^{\frac12}(\Gamma_i):\mathbf{u}\mapsto\begin{cases}\mathbf{n}\cdot\mathbf{u}|_{\Gamma_i^{(k)}} & \text{on }\Gamma_i^{(k)}\subset\Gamma_N,\\ \pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})|_{\Gamma_i^{(k)}} & \text{on }\Gamma_i^{(k)}\subset\Gamma_D,\end{cases}\tag{4.5}$$

when $n=2$, and, when $n=3$, defined by

$$\begin{aligned}T:H^1(\Omega)^3\to&\prod_{\{i:\Gamma_i\subset\Gamma_N\}}\prod_{k=1}^{K_i}H^{\frac12}(\Gamma_i)\times\prod_{\{i:\Gamma_i\subset\Gamma_D\}}\mathbf{H}_{\|}^{\frac12}(\Gamma_i):\\ &\mathbf{u}\mapsto\begin{cases}\mathbf{n}\cdot\mathbf{u}|_{\Gamma_i} & \text{on }\Gamma_i\subset\Gamma_N,\\ \pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})|_{\Gamma_i} & \text{on }\Gamma_i\subset\Gamma_D.\end{cases}\end{aligned}\tag{4.6}$$

For $n=3$, $\mathbf{u}\mapsto\pi_{\boldsymbol\tau}(A^{-1}\mathbf{u})|_{\Gamma_i}:H^1(\Omega)^3\to\mathbf{H}_{\|}^{\frac12}(\Gamma_i)$ is bounded. Because $\mathbf{u}\mapsto A^{-1}\mathbf{u}$ is bounded on $H^1(\Omega)^3$ by (C.2) and (C.5), to show this statement it is sufficient to consider $A=\mathrm{Id}$. For $1\leqslant k\leqslant K_i$ and $\ell\in\mathscr{O}_i^{(k)}$, one has $\pi_{\boldsymbol\tau}(\mathbf{u})\cdot\boldsymbol\tau_i^{(k,\ell)}=\mathbf{u}\cdot\boldsymbol\tau_i^{(k,\ell)}$ on $\Gamma_i^{(k)}\cup\Gamma_i^{(\ell)}$. Since $\boldsymbol\tau_i^{(k,\ell)}$ is Lipschitz on $\Gamma_i^{(k)}\cup\Gamma_i^{(\ell)}$, for $\mathbf{x}\in\Gamma_i^{(\ell)}$, $\mathbf{y}\in\Gamma_i^{(k)}$,

$$\begin{aligned}|(\mathbf{u}\cdot\boldsymbol\tau_i^{(k,\ell)})(\mathbf{x})-&(\mathbf{u}\cdot\boldsymbol\tau_i^{(k,\ell)})(\mathbf{y})|\\ &\leqslant|(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot\boldsymbol\tau_i^{(k,\ell)}(\mathbf{x})|+|\mathbf{u}(\mathbf{y})\cdot(\boldsymbol\tau_i^{(k,\ell)}(\mathbf{x})-\boldsymbol\tau_i^{(k,\ell)}(\mathbf{y}))|\\ &\lesssim\|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\|+\|\mathbf{u}(\mathbf{y})\|\|\mathbf{x}-\mathbf{y}\|,\end{aligned}$$

where actually the second term can be dropped when Γ_i is the boundary of a polyhedron, since in that case $\boldsymbol\tau_i^{(k,\ell)}$ can be taken to constant on $\Gamma_i^{(k)}\cup\Gamma_i^{(\ell)}$. Since

$$\int_{\Gamma_i^{(k)}}\int_{\Gamma_i^{(\ell)}}\frac{\|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\|^2}{\|\mathbf{x}-\mathbf{y}\|^3}d\sigma(\mathbf{x})d\sigma(\mathbf{y})\leqslant|\mathbf{u}|^2_{H^{\frac12}(\Gamma_i^{(k)}\cup\Gamma_i^{(\ell)})^3}$$

and, as follows from (the proof of) (Buffa & Ciarlet, 2001, Prop. 2.2), for any $\mathbf{x} \in \Gamma_i^{(\ell)}$

$$\int_{\Gamma_i^{(k)}} \frac{\|\mathbf{u}(\mathbf{y})\|^2}{\|\mathbf{x}-\mathbf{y}\|} d\sigma(\mathbf{y}) \lesssim \|\mathbf{u}\|^2_{H^{\frac{1}{2}}(\Gamma_i^{(k)})^3}, \quad \text{uniformly in } \mathbf{x},$$

a standard application of the trace theorem confirms our statement about the boundedness of the tangential components trace mapping.

Knowing this result, the boundedness of T for both $n=2$ and $n=3$ follows by standard applications of the trace theorem. The surjectivity of T, and so the closedness of its image, will be verified in Propositions 4.3 and 4.4, respectively. $\square$

PROPOSITION 4.3 The mapping T from (4.5) is surjective.

Proof. Since the Γ_i are disjoint, it is sufficient to prove the surjectivity of T followed by the restriction to some Γ_i. By means of a partition of unity, to show this it is sufficient to consider a $v \in \prod_{k=1}^{K_i} H^{\frac{1}{2}}(\Gamma_i^{(k)})$ with a support in a sufficiently small neighborhood of a corner $\mathbf{x}$ of Γ_i. Let $e_\pm$ denote the parts of the boundary that meet at $\mathbf{x}$, with outward unit normals $\mathbf{n}_\pm$. Let $v_\pm$ denote the restrictions of v to $e_\pm$, and let $\bar{v}_\pm$ denote their extensions to H^1-functions with supports in a sufficiently small neighborhood of the corner. It suffices to show the existence of an solution $\mathbf{u} \in (H^1)^2$ of the equations $\mathbf{n}_\pm \cdot \mathbf{u} = \bar{v}_\pm$, $\mathbf{n}_\pm \times A^{-1}\mathbf{u} = \bar{v}_\pm$, or $\mathbf{n}_+ \cdot \mathbf{u} = \bar{v}_+$ and $\mathbf{n}_- \times A^{-1}\mathbf{u} = \bar{v}_-$ in case of a switch from Neumann-to-Neumann, Dirichlet-to-Dirichlet, or from Neumann-to-Dirichlet boundary conditions, respectively. This existence follows in the first two cases from $\mathbf{n}_\pm(\mathbf{x})$ being independent, and in the latter case from the assumption (C.7) saying that $\mathbf{n}_-(\mathbf{x})^\top A(\mathbf{x})\mathbf{n}_+(\mathbf{x}) \neq 0$, together with the Lipschitz continuity of A and that of (extensions of) $\mathbf{n}_\pm$. $\square$

PROPOSITION 4.4 The mapping T from (4.6) is surjective.

Proof. Since the Γ_i are disjoint, it is sufficient to prove the surjectivity of T followed by the restriction to any Γ_i. First we consider the case that $\Gamma_i \subset \Gamma_N$.

Let $v \in \prod_{k=1}^{K_i} H^{\frac{1}{2}}(\Gamma_i^{(k)})$. By means of a partition of unity, it is sufficient to consider the cases that v has its support in a sufficiently small neighborhood of either a vertex, or an edge where the neighborhood does not contain a vertex, or a point on a face where the neighborhood does not contain a vertex or an edge. The last two cases are easy, and so we focus on the first case.

We consider a vertex $\mathbf{x}$ where $M \geqslant 3$ faces meet, that w.l.o.g. we call $\Gamma_i^{(1)}, \ldots, \Gamma_i^{(M)}$, ordered as illustrated in Figure 1(left). For $1 \leqslant k \leqslant M$, consider a subdivision of $\Gamma_i^{(k)}$ in 3 non-empty sectors $\Gamma_i^{(k,q)}$,

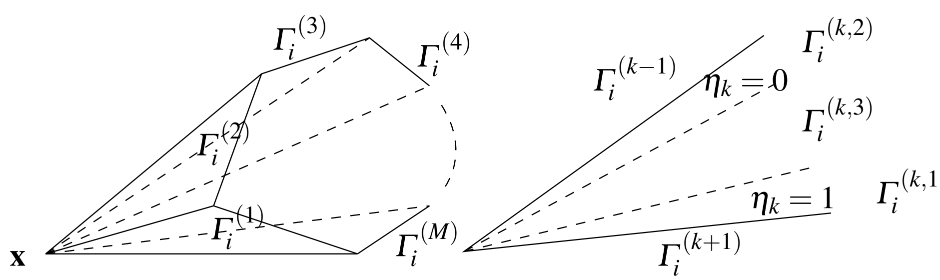

FIG. 1. Faces near a vertex (left), and the spitting of $\Gamma_i^{(k)}$ (right).

$q = 1,2,3$ as illustrated in Figure 1(right). As follows from (Buffa & Ciarlet, 2001, Prop. 2.8), for

$1 \leqslant k \leqslant M$ there exists an η_k on $\Gamma_i^{(k)}$ that is smooth outside any neighborhood of $\mathbf{x}$, with $\eta_k = 0$ on $\Gamma_i^{(k,2)}$ and $\eta_k = 1$ on $\Gamma_i^{(k,1)}$, such that multiplication with η_k is a mapping on $H^{\frac{1}{2}}(\Gamma_i^{(k)})$.

Setting $\Gamma_i^{(M+1)} := \Gamma_i^{(1)}$ and $\eta_{M+1} := \eta_1$, let

$$\xi_k := \begin{cases} \eta_k & \text{on } \Gamma_i^{(k)}, \\ 1 - \eta_{k+1} & \text{on } \Gamma_i^{(k+1)}, \\ 0 & \text{on } \Gamma_i \backslash (\Gamma_i^{(k)} \cup \Gamma_i^{(k+1)}). \end{cases}$$

Then

(1). $\operatorname{supp} \xi_k \subset \Gamma_i^{(k)} \cup \Gamma_i^{(k+1)}$,

(2). $\sum_{k=1}^{M} \xi_k = 1$ on $\cup_{k=1}^{M} \Gamma_i^{(k)}$,

(3). multiplication with ξ_k is a mapping on $H^{\frac{1}{2}}(\Gamma_i)$.

To show the last property we use the fact, as proved in (Buffa & Ciarlet, 2001, Thm. 2.5), that on a (curved) polyhedron a function is in $H^{\frac{1}{2}}$ if and only if for any pair of faces that share an edge, its restriction to these two faces is in $H^{\frac{1}{2}}$. Now consider the decomposition of $\cup_{k=1}^{M} \Gamma_i^{(k)}$ into $3M$ faces by splitting each $\Gamma_i^{(k)}$ into $\Gamma_i^{(k,q)}$ for $q = 1,2,3$. Given $w \in H^{\frac{1}{2}}(\Gamma_i)$, we consider $\xi_k w$ on each pair of these $3M$ faces that share an edge. For each of such pairs, either $\xi_k = 1$ or $\xi_k = 0$ both faces, or both faces are part of the same $\Gamma_i^{(k)}$. By combining the last mentioned result from Buffa & Ciarlet (2001) with the fact that multiplication with η_k is a mapping on $H^{\frac{1}{2}}(\Gamma_i^{(k)})$, we conclude ((3)).

Now let us return to our function $v \in \prod_{k=1}^{M} H^{\frac{1}{2}}(\Gamma_i^{(k)})$ with support in a sufficiently small neighborhood of $\mathbf{x}$. Given $1 \leqslant k \leqslant M$, for $\ell = k, k+1$, let $\bar{v}_\ell \in H^1(\mathbb{R}^3)$ be an extension of $v|_{\Gamma_\ell}$ with support some small neighborhood of $\mathbf{x}$. Since the unit normals $\mathbf{n}_k$, $\mathbf{n}_{k+1}$ on $\Gamma_i^{(k)}$, $\Gamma_i^{(k+1)}$ are independent at $\mathbf{x}$, there exists a $\mathbf{w}^{(k)} \in H^1(\mathbb{R}^3)^3$ with $\mathbf{n}_\ell \cdot \mathbf{w}^{(k)} = \bar{v}_\ell$ ($\ell = k, k+1$). Now let $\mathbf{u}^{(k)} \in H^1(\mathbb{R}^3)^3$ be an extension of $\xi_k \mathbf{w}^{(k)}|_{\Gamma_i} \in H^{\frac{1}{2}}(\Gamma_i)^3$. Then $\mathbf{n} \cdot \mathbf{u}^{(k)}|_{\Gamma_i} = \xi_k v$, and so for $\mathbf{u} := \sum_{k=1}^{M} \mathbf{u}^{(k)} \in H^1(\mathbb{R}^3)^3$, we have $\mathbf{n} \cdot \mathbf{u}|_{\Gamma_i} = v$.

Since $\mathbf{u} \mapsto A\mathbf{u}$ is a bounded map on $H^1(\Omega)^n$, for the case that $\Gamma_i \subset \Gamma_D$ it is sufficient to consider $A = \mathrm{Id}$. For this case, the surjectivity of T followed by the restriction to Γ_i was proven in (Buffa & Ciarlet, 2001, Prop. 2.7). $\square$

Concluding we can say that under some conditions, most prominently the condition of H^2-regularity, the elliptic second order boundary value problem (3.1), so with possibly inhomogeneous boundary conditions, being reformulated as the extended div-grad first order system (4.1) gives rise to a least squares problem that is well-posed in the sense of (1.1). All arising spaces can be equipped with Riesz bases of wavelet type, or, in a finite element setting, all arising norms can be replaced by efficiently computable equivalent quantities in terms of multi-level preconditions.

We confined the discussion about how to append inhomogeneous boundary conditions to scalar elliptic boundary value problems. Our findings from this section, however, equally well apply to first order system least squares formulations of curl-curl or div-curl systems (cf. (Bochev & Gunzburger, 2009, Ch. 6)), where, assuming H^2-regularity, the space $\{\mathbf{v} \in H^1(\Omega)^n : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N, \mathbf{n} \times A^{-1}\mathbf{v} = 0 \text{ on } \Gamma_D\}$ arises naturally.

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