DIVERGENCE-FREE WAVELET BASES ON THE HYPERCUBE January 26, 2010

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Abstract. Given a biorthogonal pair of multi-resolution analyses on the interval, by integration or differentiation, we build a new biorthogonal pair of multi-resolution analyses. Using both pairs, isotropic or, as we focus on, anisotropic divergence-free wavelet bases on the hypercube are constructed. Our construction generalizes the one from [*Rev. Mat. Iberoamericana*, 8 (1992), pp. 221–237] by P.G. Lemarié-Rieusset for stationary multi-resolution analyses on $\mathbb{R}$. It turns out that it requires a specific choice of boundary conditions.

1. Introduction

Divergence-free wavelet bases have been advocated for solving Stokes and incompressible Navier-Stokes equations. Although divergence-free wavelet bases on *bounded* domains have been mentioned in several papers, and some papers are devoted to their construction, it is questionable whether the collections of divergence free wavelets constructed so far are appropriately called *bases*. Indeed, as we will see, the codimension of their spans in the appropriate Sobolev space is infinite. In this paper, we will construct divergence-free wavelets on the n-dimensional unit cube, necessarily subject to rather specific boundary conditions, that form a Riesz basis for the full corresponding Sobolev space of divergence free functions.

To understand the difficulties with the construction of divergence-free wavelet bases on bounded domains, we start with recalling the construction of divergence-free wavelets on $\mathbb{R}^n$ by Lemarié-Rieusset in [LR92] . For convenience, when doing so we restrict ourselves to the two-dimensional case $n = 2$.

Let ϕ, $\tilde{\phi}$ be compactly supported biorthogonal scaling functions on $\mathbb{R}$ with $\tilde{\phi} \in H^{1+\varepsilon}(\mathbb{R})$, and let ψ, $\tilde{\psi}$ be the corresponding biorthogonal "mother" wavelets. Then, as shown in [LR92], there exists another pair of compactly supported biorthogonal scaling functions ϕ^+, $\tilde{\phi}^-$, and corresponding biorthogonal "mother" wavelets ψ^+, $\tilde{\psi}^-$ such that

$$\begin{aligned} \dot{\phi}^+(x) &= \phi(x) - \phi(x-1), \qquad & \dot{\psi}^+ &= \psi, \\ \tilde{\phi}^-(x+1) - \tilde{\phi}^-(x) &= \dot{\tilde{\phi}}(x), & \tilde{\psi}^- &= -\dot{\tilde{\psi}}. \end{aligned}$$

(Our formulas are somewhat different than in [LR92], but the differences are harmless. Among other things, we found it convenient to reverse the role of the primal and dual side.) Throughout the paper, a "dot" on top of a univariate function denotes its derivative.

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Furthermore, with for $\theta \in \{\phi, \psi, \tilde{\phi}, \tilde{\psi}, \phi^+, \psi^+, \tilde{\phi}^-, \tilde{\psi}^-\}$ and $\ell, i \in \mathbb{Z}$, $\theta_{\ell,i}(x) := 2^{\ell/2}\theta(2^\ell x - i)$, in the same paper it was shown that

$$\left\{ \begin{bmatrix} \psi^+_{\ell,i} \otimes (\phi_{\ell,j+1} - \phi_{\ell,j}) \\ \psi_{\ell,i} \otimes \phi^+_{\ell,j} \end{bmatrix}, \begin{bmatrix} \phi^+_{\ell,i} \otimes \psi_{\ell,j} \\ (\phi_{\ell,i+1} - \phi_{\ell,i}) \otimes \psi^+_{\ell,j} \end{bmatrix}, \begin{bmatrix} -\psi^+_{\ell,i} \otimes \psi_{\ell,j} \\ \psi_{\ell,i} \otimes \psi^+_{\ell,j} \end{bmatrix} : \ell, i, j \in \mathbb{Z} \right\}$$

is a Riesz basis for $\mathbf{H}(\mathrm{div}0; \mathbb{R}^2) = \{\mathbf{u} \in \mathbf{H}(\mathrm{div}; \mathbb{R}^2) : \mathrm{div}\, \mathbf{u} = 0\}$, and that, after a proper scaling, it is also a Riesz basis for $\{\mathbf{u} \in H^1(\mathbb{R}^2)^2 : \mathrm{div}\, \mathbf{u} = 0\}$.

Above vector-valued wavelets are *isotropic*; the components are tensor products of wavelets on the same level. Using the biorthogonal wavelet pairs ψ, $\tilde{\psi}$ and ψ^+, $\tilde{\psi}^-$ from the Lemarié-Rieusset construction, in [DP06] by Deriaz and Perrier, a basis was constructed of divergence-free *anisotropic* wavelets.

We consider the latter construction in somewhat more detail: Since $L_2(\mathbb{R}^2) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$, the set $\left\{ \begin{bmatrix} \psi^+_{\ell,i} \otimes \psi_{m,j} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix} : \ell, m, i, j \in \mathbb{Z} \right\}$ is a Riesz basis for $L_2(\mathbb{R}^2)^2$, with dual basis $\left\{ \begin{bmatrix} \tilde{\psi}^-_{\ell,i} \otimes \tilde{\psi}_{m,j} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}^-_{m,j} \end{bmatrix} : \ell, m, i, j \in \mathbb{Z} \right\}$. Applying the orthogonal transformation $\frac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^m & -2^\ell \\ 2^\ell & 2^m \end{bmatrix}$ to the pair of basis functions $\begin{bmatrix} \psi^+_{\ell,i} \otimes \psi_{m,j} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix}$, we infer that also

$$\left\{ \tfrac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^m \psi^+_{\ell,i} \otimes \psi_{m,j} \\ -2^\ell \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix}, \tfrac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^\ell \psi^+_{\ell,i} \otimes \psi_{m,j} \\ 2^m \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix} : \ell, m, i, j \in \mathbb{Z} \right\} \tag{1.1}$$

is a Riesz basis for $L_2(\mathbb{R}^2)^2$, with dual basis

$$\left\{ \tfrac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^m \tilde{\psi}^-_{\ell,i} \otimes \tilde{\psi}_{m,j} \\ -2^\ell \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}^-_{m,j} \end{bmatrix}, \tfrac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^\ell \tilde{\psi}^-_{\ell,i} \otimes \tilde{\psi}_{m,j} \\ 2^m \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}^-_{m,j} \end{bmatrix} : \ell, m, i, j \in \mathbb{Z} \right\}.$$

From $\dot{\psi}^+_{\ell,i} = 2^\ell \psi_{\ell,i}$, we have $\mathrm{div} \begin{bmatrix} 2^m \psi^+_{\ell,i} \otimes \psi_{m,j} \\ -2^\ell \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix} = 0$. Moreover, considering the coefficients of $\mathbf{u} \in \mathbf{H}(\mathrm{div}; \mathbb{R}^2)$ with respect to the basis (1.1), i.e., the inner products of $\mathbf{u}$ with the dual basis functions, from $\dot{\tilde{\psi}}_{\ell,i} = -2^\ell \tilde{\psi}^-_{\ell,i}$ and integration by parts, we find that

$$\left\langle \mathbf{u}, \begin{bmatrix} 2^\ell \tilde{\psi}^-_{\ell,i} \otimes \tilde{\psi}_{m,j} \\ 2^m \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}^-_{m,j} \end{bmatrix} \right\rangle_{L_2(\mathbb{R}^2)^2} = -\langle \mathbf{u}, \mathbf{grad}\, \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}_{m,j} \rangle_{L_2(\mathbb{R}^2)^2} = \langle \mathrm{div}\, \mathbf{u}, \tilde{\psi}_{\ell,i} \otimes \tilde{\psi}_{m,j} \rangle_{L_2(\mathbb{R}^2)},$$

which vanishes when $\mathrm{div}\, \mathbf{u} = 0$. We conclude that

$$\left\{ \tfrac{1}{\sqrt{4^\ell + 4^m}} \begin{bmatrix} 2^m \psi^+_{\ell,i} \otimes \psi_{m,j} \\ -2^\ell \psi_{\ell,i} \otimes \psi^+_{m,j} \end{bmatrix} : \ell, m, i, j \in \mathbb{Z} \right\}$$

is a Riesz basis for $\mathbf{H}(\mathrm{div}0; \mathbb{R}^2)$, and properly scaled, also for $\{\mathbf{u} \in H^1(\mathbb{R}^2)^2 : \mathrm{div}\, \mathbf{u} = 0\}$.

The construction of both the isotropic and anisotropic divergence-free wavelet bases generalizes to arbitrary space dimensions $n \geq 2$. The generalization of the anisotropic divergence-free wavelets to $n > 2$ proposed in [DP06, DP09] does not yield stable bases. We develop a construction that yield Riesz bases of anisotropic divergence-free wavelets in any dimension. As we will demonstrate, the advantage of the anisotropic bases is that sufficiently smooth divergence-free functions can be approximated from the span of these bases with a convergence rate that is better than with isotropic wavelets, and in particular, that is independent of n. For this reason, the focus will be on the anisotropic construction.

In view of the construction on $\mathbb{R}^n$, the key to the construction of anisotropic or isotropic divergence-free wavelet bases on

$$\square := (0,1)^n,$$

is to have available on

$$\mathrm{I} := (0,1),$$

biorthogonal Riesz bases Ψ, $\tilde{\Psi}$ and Ψ^+, $\tilde{\Psi}^-$ for $L_2(\mathrm{I})$, that for some invertible diagonal matrix $\mathbf{D}$, satisfy

$$\dot{\Psi}^+ = \mathbf{D}\Psi, \quad \dot{\tilde{\Psi}} = -\mathbf{D}\tilde{\Psi}^-.$$

Here we view bases formally as column vectors. In the stationary wavelet construction on $\mathbb{R}$, the diagonal matrix $\mathbf{D}$ is the one with diagonal entry 2^ℓ corresponding to wavelet $\psi_{\ell,i}$. With such bases at hand, the construction of divergence-free wavelet bases on $\square$ follows the same lines as on $\mathbb{R}^n$.

The difficulty lies in the construction of Ψ, $\tilde{\Psi}$, Ψ^+ and $\tilde{\Psi}^-$ on I. Using the notation $\langle\Sigma,\Upsilon\rangle := [\langle\sigma,\upsilon\rangle]_{\sigma\in\Sigma,\upsilon\in\Upsilon}$, integration by parts shows that above assumptions imply that necessarily

$$\begin{aligned}\Psi^+(1)\tilde{\Psi}(1)^\top - \Psi^+(0)\tilde{\Psi}(0)^\top &= \langle\dot{\Psi}^+,\tilde{\Psi}\rangle_{L_2(\mathrm{I})} + \langle\Psi^+,\dot{\tilde{\Psi}}\rangle_{L_2(\mathrm{I})}\\ &= \langle\mathbf{D}\Psi,\tilde{\Psi}\rangle_{L_2(\mathrm{I})} - \langle\Psi^+,\mathbf{D}\tilde{\Psi}^-\rangle_{L_2(\mathrm{I})} = \mathbf{D}\cdot\mathrm{Id} - \mathrm{Id}\cdot\mathbf{D} = 0,\end{aligned}$$

i.e.,

$$\psi^+(1)\tilde{\psi}(1) - \psi^+(0)\tilde{\psi}(0) = 0 \quad (\psi^+\in\Psi^+, \tilde{\psi}\in\tilde{\Psi}). \tag{1.2}$$

To obtain such vanishing boundary terms, in [JLR93] (in our notations) the collection Ψ^+ was taken from $H_0^1(\mathrm{I})$. As a consequence, any element of $\Psi = \mathbf{D}^{-1}\dot{\Psi}^+$ has vanishing mean, so that Ψ cannot be a basis for $L_2(\mathrm{I})$ (the reason being that the mean value is a non-zero, continuous functional on $L_2(\mathrm{I})$; it is not continuous on $L_2(\mathbb{R})$, and therefore the latter space *can* be equipped with a Riesz basis of functions all having a vanishing mean).

The collections Ψ, $\tilde{\Psi}$ can be arranged, however, to be bases for $L_{2,0}(\mathrm{I})$, being the space of $L_2(\mathrm{I})$ functions with vanishing mean. In this way, divergence-free wavelet collections can be constructed. They will, however, not span a full space of divergence-free functions with vanishing normals on a part Γ of $\partial\Omega$ (i.e., a space $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ defined in (5.5)), but, for say $n = 3$, they span such a space intersected with $L_2(\mathrm{I})\otimes L_{2,0}(\mathrm{I})\otimes L_{2,0}(\mathrm{I})\times L_{2,0}(\mathrm{I})\otimes L_2(\mathrm{I})\otimes L_{2,0}(\mathrm{I})\times L_{2,0}(\mathrm{I})\otimes L_{2,0}(\mathrm{I})\otimes L_2(\mathrm{I})$. The codimension in $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ of this intersection is infinite.

To get vanishing boundary terms in (1.2), alternatively in [Urb01] (in our notations) a framework was presented where $\tilde{\Psi}$ was taken from $H_0^1(\mathrm{I})$. In this case, $\tilde{\Psi}^-$ cannot be a basis for $L_2(\mathrm{I})$, and so neither can Ψ^+. The same arguments as applied above show that at best one ends up with a collection of divergence free wavelets whose span has infinite codimension in a "full" space $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$.

To have vanishing boundary terms in (1.2), a third possibility would be to impose periodic boundary conditions for both Ψ^+ and $\tilde{\Psi}$. In this case, any element from even both Ψ and $\tilde{\Psi}^-$ has vanishing mean, giving rise to the same problems as above.

In view of the sketched difficulties to realize a vanishing expression (1.2), in this paper, we will give a general recipe for constructing biorthogonal Riesz bases Ψ, $\tilde{\Psi}$ and Ψ^+, $\tilde{\Psi}^-$ for $L_2(\mathrm{I})$, that, for some invertible diagonal matrix $\mathbf{D}$, satisfy $\dot{\Psi}^+ = \mathbf{D}\Psi$ and $\dot{\tilde{\Psi}} = -\mathbf{D}\tilde{\Psi}^-$, and for which *the elements of* $\tilde{\Psi}$ *vanish at* 1*, and those of* Ψ^+ *vanish at* 0. The key why with these boundary conditions such bases can be constructed, is that the mapping $g \mapsto \dot{g}$ is boundedly invertible from $H^1_{0,\{0\}}(\mathrm{I})$, being the space of $H^1(\mathrm{I})$ functions that vanish at 0, to $L_2(\mathrm{I})$, with inverse given by $f \mapsto (x \mapsto \int_0^x f(y)dy)$. Obviously, by symmmetry here and on all other places the roles of the left and right boundary can be interchanged.

Our recipe will not be of the type of adapting shift and scale invariant collections on the line to the interval by keeping those that are fully supported in I, and by taking suitable linear combinations of those with supports that intersect the boundary points. Instead, given *any* biorthogonal multi-resolution analyses on I, characterized by sequences of primal and dual scaling functions and wavelets $(\Phi_\ell)_\ell$, $(\tilde{\Phi}_\ell)_\ell$, $(\Psi_\ell)_\ell$ and $(\tilde{\Psi}_\ell)_\ell$, where the dual functions vanish at 1, by integration or differentiation, we explicitly construct new biorthogonal multi-resolution analyses, characterized by $(\Phi^+_\ell)_\ell$, $(\tilde{\Phi}^-_\ell)_\ell$, $(\Psi^+_\ell)_\ell$, $(\tilde{\Psi}^-_\ell)_\ell$, for which the primals vanish at 0. If, in the original multi-resolution analyses, at the primal side no boundary conditions are incorporated, and at the dual side no boundary conditions at 0, then whenever the original multi-resolution analyses satisfy Jackson estimates of order d and $\tilde{d}$ at primal and dual side, the new multi-resolution analyses satisfy these estimates with the full orders $d+1$ and $\tilde{d}-1$, respectively. We give an example (Example 4.4) of our construction for $d = 2$, $\tilde{d} = 4$.

With these biorthogonal Riesz bases Ψ, $\tilde{\Psi}$ and Ψ^+, $\tilde{\Psi}^-$ for $L_2(\mathrm{I})$ at hand, where $\Psi := \Phi_0 \cup \cup_{\ell \in \mathbb{N}_0} \Psi_\ell$ and similarly for the other collections, we construct a divergence-free anisotropic wavelet basis for the *"full"* divergence-free spaces on the n-cube, subject to vanishing normal components on $\Gamma := \cup_{m-1}^n [0,1]^{m-1} \times \{0\} \times [0,1]^{n-m}$. That is, properly scaled, this wavelet collection will be a Riesz basis for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0; \square)$ as well as for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0; \square) \cap H^1(\square)^n$. In addition, with $\tilde{\Gamma} := \partial\square \backslash \Gamma$, we construct a Riesz basis of wavelet type for the orthogonal complement $\mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$ of $\mathbf{H}_{0,\Gamma}(\mathrm{div}0; \square)$ in $L_2(\square)^n$, and show how the corresponding so-called Helmholtz decomposition of any $\mathbf{u} \in L_2(\square)^n$ can be computed. We give also expansions of **curl** and div operators in wavelet coordinates.

Our construction on $\square$ requires specific boundary conditions, and of course we rather would have presented a construction that applies to general, given boundary conditions, but we do not know whether this is possible. What we can say is that to the best of our knowledge, so far there are no other divergence free wavelet *bases* on bounded domains available.

Finally, divergence free wavelets have been used to represent the solution of the Navier-Stokes equations or to solve them, or to compute the Helmholtz decomposition of (turbulent) velocity fields (e.g. see [Urb02, DP06]). For those applications, it seems preferable to have possibly some mismatch in the boundary conditions, than not to be able to represent vector fields whose coordinates frozen in some directions do not have a vanishing mean over the remaining directions.

This paper is organized as follows: In Section 2, standard assumptions on the original biorthogonal sets of scaling functions and wavelets are formulated. In

Sections 3 and 4, by integration or differentiation, new biorthogonal sets are constructed of wavelets and scaling functions, respectively. Section 5 is devoted to the construction of divergence-free wavelet bases. Finally, in Section 6, we discuss the computation of the Helmholtz decomposition, and give expansions of div and **curl** operators in wavelet coordinates.

In this paper, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Biorthogonal scaling functions and wavelets on the interval

For $\ell \in \mathbb{N}_0$, and index sets $I_\ell = \{1, \ldots, \#I_\ell\}$, $J_\ell = \{1, \ldots, \#J_\ell\}$ with $\#I_\ell \eqsim 2^\ell$ and $\#I_\ell + \#J_\ell = \#I_{\ell+1}$, we assume collections, often viewed as column vectors, of primal and dual scaling functions

$$\Phi_\ell = [\phi_{\ell,i}]_{i \in I_\ell}, \quad \tilde{\Phi}_\ell = [\tilde{\phi}_{\ell,i}]_{i \in I_\ell},$$

and wavelets

$$\Psi_\ell = [\psi_{\ell,i}]_{i \in J_\ell}, \quad \tilde{\Psi}_\ell = [\tilde{\psi}_{\ell,i}]_{i \in J_\ell},$$

such that, with $\mathcal{S}(\Sigma)$ denoting the span of a collection Σ,

$$\mathcal{S}(\Phi_0) \subset \mathcal{S}(\Phi_1) \subset \cdots \subset L_2(\mathrm{I}), \quad \mathcal{S}(\tilde{\Phi}_0) \subset \mathcal{S}(\tilde{\Phi}_1) \subset \cdots \subset L_2(\mathrm{I}),$$

$$\mathcal{S}(\Phi_\ell) + \mathcal{S}(\Psi_\ell) = \mathcal{S}(\Phi_{\ell+1}), \quad \mathcal{S}(\tilde{\Phi}_\ell) + \mathcal{S}(\tilde{\Psi}_\ell) = \mathcal{S}(\tilde{\Phi}_{\ell+1}),$$

and

$$\left\langle \begin{bmatrix} \Phi_\ell \\ \Psi_\ell \end{bmatrix}, \begin{bmatrix} \tilde{\Phi}_\ell \\ \tilde{\Psi}_\ell \end{bmatrix} \right\rangle_{L_2(\mathrm{I})} = \mathrm{Id} \quad (\textit{biorthogonality}). \tag{2.1}$$

Furthermore, we assume *localness* and *boundedness* of primal and dual scaling functions and wavelets in the sense that

$$\begin{cases} \sup\limits_{\ell \in \mathbb{N}_0,\, i \in I_\ell} 2^\ell \mathrm{diam}(\mathrm{supp}\, \phi_{\ell,i}) < \infty, \\ \sup\limits_{\ell,\, k \in \mathbb{N}_0} \#\{i \in I_\ell : \mathrm{supp}\, \phi_{\ell,i} \cap [k2^{-\ell}, (k+1)2^{-\ell}] \neq \emptyset\} < \infty, \end{cases} \tag{2.2}$$

and

$$\sup_{\ell \in \mathbb{N}_0,\, i \in I_\ell} \|\phi_{\ell,i}\|_{L_2(\mathrm{I})} < \infty, \tag{2.3}$$

and similarly for $\cup_\ell \tilde{\Phi}_\ell$, $\cup_\ell \Psi_\ell$ and $\cup_\ell \tilde{\Psi}_\ell$.

Without loss of generality, we assume that the sets of primal and dual scaling functions are ordered in the sense that

$$i \leq j \Longrightarrow \inf \mathrm{supp}\, \phi_{\ell,i} \leq \inf \mathrm{supp}\, \phi_{\ell,j}, \tag{2.4}$$

and similarly for the dual scaling functions.

Apart from above standard conditions, for our goal we have to impose *specific boundary conditions*. We impose no boundary conditions at the primal side, and no boundary conditions at 0 at the dual side, whereas we assume that all dual scaling functions and dual wavelets vanish at 1. Together with standard Jackson and Bernstein assumptions, it means that, for some $0 < \gamma < d \in \mathbb{N}$, $1 < \tilde{\gamma} < \tilde{d} \in \mathbb{N}$, we assume

$$\inf_{v \in \mathcal{S}(\Phi_\ell)} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell d} \|u\|_{H^d(\mathrm{I})} \quad (u \in H^d(\mathrm{I})), \tag{2.5}$$

$$\inf_{v \in \mathcal{S}(\tilde{\Phi}_\ell)} \|u - v\|_{L_2(\mathrm{I})} \lesssim 2^{-\ell \tilde{d}} \|u\|_{H^{\tilde{d}}(\mathrm{I})} \quad (u \in H^{\tilde{d}}(\mathrm{I}) \cap H^1_{0,\{1\}}(\mathrm{I})), \tag{2.6}$$

and, for any $s \in [0, \gamma)$, that $\mathcal{S}(\Phi_\ell) \subset H^s(\mathrm{I})$ with

$$\|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s} \|\cdot\|_{L_2(\mathrm{I})} \quad \text{on } \mathcal{S}(\Phi_\ell),$$

and, for any $s \in [0, \tilde{\gamma})$, that $\mathcal{S}(\tilde{\Phi}_\ell) \subset H^s(\mathrm{I}) \cap H^1_{0,\{1\}}(\mathrm{I})$ with

$$\|\cdot\|_{H^s(\mathrm{I})} \lesssim 2^{\ell s} \|\cdot\|_{L_2(\mathrm{I})} \quad \text{on } \mathcal{S}(\tilde{\Phi}_\ell). \tag{2.7}$$

Here and in the following, for Ω being a domain in $\mathbb{R}^n$, and $\Sigma \subset \partial\Omega$ with positive measure, with $H^1_{0,\Sigma}(\Omega)$ we mean the subspace of $H^1(\Omega)$ consisting of the functions whose trace vanishes on Σ.

In view of (2.6), in particular in view of the boundary conditions incorporated at the dual side, a natural further assumption is that outside a $2^{-\ell}$ neighbourhood of 1, the constants are contained in the span of the dual scaling functions on level ℓ, i.e., that

$$\sup_{\ell \in \mathbb{N}_0} 2^\ell \mathrm{dist}\Big(1, \inf \mathrm{supp}\, \big(1 - \sum_{i \in I_\ell} \langle 1, \phi_{\ell,i} \rangle_{L_2(\mathrm{I})} \tilde{\phi}_{\ell,i}\big)\Big) < \infty. \tag{2.8}$$

Finally, for convenience we assume that for all $\ell \in \mathbb{N}_0$ and $j \in I_\ell$, $\langle 1, \phi_{\ell,j} \rangle_{L_2(\mathrm{I})} \neq 0$ and

$$\sup_{\ell \in \mathbb{N}_0} \sup_{\{i,j \in I_\ell : |i-j| \leq 1\}} \frac{|\langle 1, \phi_{\ell,i} \rangle_{L_2(\mathrm{I})}|}{|\langle 1, \phi_{\ell,j} \rangle_{L_2(\mathrm{I})}|} < \infty. \tag{2.9}$$

In all examples that we know of, $|\langle 1, \phi_{\ell,i} \rangle_{L_2(\mathrm{I})}| \eqsim 2^{-\ell/2}$ so that (2.9) is satisfied.

Collections of primal and dual scaling functions and wavelets that satisfy all conditions mentioned in this section can be found in [DKU99, DS98, Pri06, Dij09]. An example taken from [Dij09] will be given at the end of Section 4.

As a consequence of the boundedness (2.3), the biorthogonality (2.1) and the localness (2.2), the collections

$$\Phi_\ell,\ \tilde{\Phi}_\ell,\ \Psi_\ell,\ \tilde{\Psi}_\ell \ \text{ are Riesz systems, uniformly in } \ell,$$

meaning that the corresponding mass matrices and their inverses are bounded uniformly in ℓ. For completeness, let us recall the short argument, say for Φ_ℓ. By the boundedness and localness of the primal scaling functions, $\|\sum_{i \in I_\ell} c_i \phi_{\ell,i}\|^2_{L_2(\mathrm{I})} \lesssim \sum_{i \in I_\ell} |c_i|^2$. Writing $u = \sum_{i \in I_\ell} c_i \phi_{\ell,i}$, by the boundedness of the dual scaling functions, $|c_i| = |\langle u, \tilde{\phi}_{\ell,i} \rangle_{L_2(\mathrm{I})}| \lesssim \|u\|_{L_2(\mathrm{supp}\, \tilde{\phi}_{\ell,i})}$. From the localness of the dual scaling functions, we conclude that $\sum_{i \in I_\ell} |c_i|^2 \lesssim \|u\|^2_{L_2(\mathrm{I})}$.

Now we set $\Psi_{-1} = \Phi_0$, $\tilde{\Psi}_{-1} = \tilde{\Phi}_0$, use λ as a shorthand notation for the double index (ℓ, i), set $|\lambda| := \ell$, and define

$$\nabla := \bigcup_{\ell \in \mathbb{N}_0 \cup \{-1\}} (\ell, I_\ell),$$

and finally,

$$\Psi = \{\psi_\lambda : \lambda \in \nabla\}, \quad \tilde{\Psi} = \{\tilde{\psi}_\lambda : \lambda \in \nabla\}.$$

It is well-known (e.g. [Dah96, DS99, Coh03]) that as a consequence of the boundedness of the biorthogonal primal and dual wavelets and of the Jackson and Bernstein estimates,

$$\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\} \text{ is a Riesz basis for } H^s(\mathrm{I}),\ s \in [0,\gamma), \tag{2.10}$$

$$\{2^{-|\lambda|s}\tilde{\psi}_\lambda : \lambda \in \nabla\} \text{ is a Riesz basis for } \mathcal{H}^s_{0,\{1\}}(\mathrm{I}),\ s \in [0,\tilde{\gamma}), \tag{2.11}$$

where

$$\mathcal{H}^s_{0,\{1\}}(\mathrm{I}) = \begin{cases} H^s(\mathrm{I}) \cap H^1_{0,\{1\}}(\mathrm{I}) & \text{when } s \geq 1, \\ [L_2(\mathrm{I}), H^1_{0,\{1\}}(\mathrm{I})]_s & \text{when } s \in [0,1]. \end{cases} \tag{2.12}$$

By duality, these results extend to Sobolev spaces with negative smoothness indices. By interpreting $v \in L_2(\mathrm{I})$ as a functional by means of $v(u) = \langle u, v\rangle_{L_2(\mathrm{I})}$, we have that for $s \in (-\tilde{\gamma}, 0]$, $\{2^{-|\lambda|s}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $(\mathcal{H}^s_{0,\{1\}}(\mathrm{I}))'$, and for $s \in (-\gamma, 0]$, $\{2^{-|\lambda|s}\tilde{\psi}_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $(H^s(\mathrm{I}))'$.

Remark 2.1. With an appropriate generalization of the Jackson and Bernstein assumptions, these results for the Sobolev spaces measuring smoothness in $L_2(\mathrm{I})$ can be generalized to Sobolev or Besov spaces measuring smoothness in $L_p(\mathrm{I})$ for $p \neq 2$. Such results are particularly relevant in the context of nonlinear approximation.

3. A new pair of biorthogonal multi-resolution analyses by integration/differentiation

For $\lambda \in \nabla$, on I we define

$$\psi^+_\lambda := x \mapsto 2^{|\lambda|} \int_0^x \psi_\lambda(y)dy, \qquad \tilde{\psi}^-_\lambda := -2^{-|\lambda|}\dot{\tilde{\psi}}_\lambda. \tag{3.1}$$

and set $\Psi^+ := \{\psi^+_\lambda : \lambda \in \nabla\}$ and $\tilde{\Psi}^- := \{\tilde{\psi}^-_\lambda : \lambda \in \nabla\}$, and for $\ell \in \mathbb{N}_0 \cup \{-1\}$, $\Psi^+_\ell := \{\psi^+_\lambda : |\lambda| = \ell\}$, $\tilde{\Psi}^-_\ell := \{\tilde{\psi}^-_\lambda : |\lambda| = \ell\}$.

Proposition 3.1. *Ψ^+ and $\tilde{\Psi}^-$ are local in the sense of* (2.2).

Proof. The localness of $\tilde{\Psi}^-$ follows from the localness of $\tilde{\Psi}$. Assumption (2.8) together with $\mathcal{S}(\Psi_\ell) \perp_{L_2(\mathrm{I})} \mathcal{S}(\tilde{\Phi}_{\ell-1})$, shows that all ψ_λ that vanish in some $2^{-|\lambda|}$ neighbourhood of 1 have zero mean. From the localness of Ψ, now the localness of Ψ^+ follows. □

Proposition 3.2. *$(\Psi^+, \tilde{\Psi}^-)$ are biorthogonal Riesz bases for $L_2(\mathrm{I})$.*

Proof. Since the $\tilde{\psi}_\lambda$'s vanish at 1 by assumption, and the ψ^+_λ's vanish at 0 by definition, integration by parts shows that for $\lambda, \mu \in \nabla$,

$$\begin{aligned} \langle \psi^+_\lambda, \tilde{\psi}^-_\mu\rangle_{L_2(\mathrm{I})} = \langle \psi^+_\lambda, -2^{-|\mu|}\dot{\tilde{\psi}}_\mu\rangle_{L_2(\mathrm{I})} &= 2^{-|\mu|}\langle \dot{\psi}^+_\lambda, \tilde{\psi}_\mu\rangle_{L_2(\mathrm{I})} \\ &= 2^{|\lambda|-|\mu|}\langle \psi_\lambda, \tilde{\psi}_\mu\rangle_{L_2(\mathrm{I})} = \delta_{\lambda,\mu}, \end{aligned}$$

i.e., $(\Psi^+, \tilde{\Psi}^-)$ are biorthogonal.

The mapping $g \mapsto \dot{g}$ is boundedly invertible from $H^1_{0,\{1\}}(\mathrm{I})$ to $L_2(\mathrm{I})$ with inverse $f \mapsto (x \mapsto -\int_x^1 f(y)dy)$. So $\{2^{-|\lambda|}\tilde{\psi}_\lambda : \lambda \in \nabla\}$ being a Riesz basis for $H^1_{0,\{1\}}(\mathrm{I})$ (cf. (2.11)) is equivalent to $\{2^{-|\lambda|}\dot{\tilde{\psi}}_\lambda : \lambda \in \nabla\}$ being a Riesz basis for $L_2(\mathrm{I})$. We conclude that $\tilde{\Psi}^-$ is a Riesz basis for $L_2(\mathrm{I})$, and by biorthogonality, so is Ψ^+. □

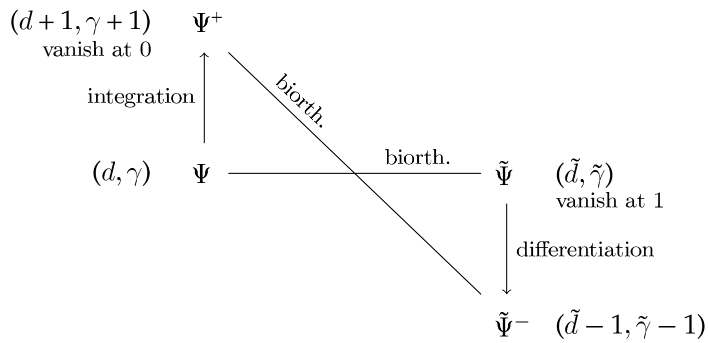

FIGURE 1. Schematic relation between Ψ, $\tilde{\Psi}$, Ψ^+ and $\tilde{\Psi}^-$

Since standard arguments lead to the statements of the following two propositions, we have omitted the proofs.

Proposition 3.3. *The following Jackson estimates are valid for Ψ^+ and $\tilde{\Psi}^-$:*

$$\inf_{v\in\operatorname{span}\{\psi^+_\lambda:|\lambda|\leq\ell\}}\|u-v\|_{L_2(\mathrm{I})}\lesssim 2^{-\ell(d+1)}\|u\|_{H^{d+1}(\mathrm{I})}\quad(u\in H^1_{0,\{0\}}(\mathrm{I})\cap H^{d+1}(\mathrm{I})),\tag{3.2}$$

$$\inf_{v\in\operatorname{span}\{\tilde{\psi}^-_\lambda:|\lambda|\leq\ell\}}\|u-v\|_{L_2(\mathrm{I})}\lesssim 2^{-\ell(\tilde{d}-1)}\|u\|_{H^{\tilde{d}-1}(\mathrm{I})}\quad(u\in H^{\tilde{d}-1}(\mathrm{I})).\tag{3.3}$$

Proposition 3.4. *The following Bernstein estimates are valid for Ψ^+ and $\tilde{\Psi}^-$: For $s\in[0,\gamma+1)$, $\Psi^+_\lambda\subset H^1_{0,\{0\}}(\mathrm{I})\cap H^s(\mathrm{I})$ with*

$$\|\cdot\|_{H^s(\mathrm{I})}\lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})}\quad\text{on}\;\operatorname{span}\{\psi^+_\lambda:|\lambda|\leq\ell\}.\tag{3.4}$$

For $s\in[0,\tilde{\gamma}-1)$, $\tilde{\Psi}^-_\lambda\subset H^s(\mathrm{I})$ with

$$\|\cdot\|_{H^s(\mathrm{I})}\lesssim 2^{\ell s}\|\cdot\|_{L_2(\mathrm{I})}\quad\text{on}\;\operatorname{span}\{\tilde{\psi}^-_\lambda:|\lambda|\leq\ell\}.$$

From Propositions 3.2, 3.3 and 3.4, we conclude that

$$\{2^{-|\lambda|s}\psi^+_\lambda:\lambda\in\nabla\}\text{ is a Riesz basis for }\mathcal{H}^s_{0,\{0\}}(\mathrm{I}),\ s\in[0,\gamma+1),\tag{3.5}$$

$$\{2^{-|\lambda|s}\tilde{\psi}^-_\lambda:\lambda\in\nabla\}\text{ is a Riesz basis for }H^s(\mathrm{I}),\ s\in[0,\tilde{\gamma}-1),\tag{3.6}$$

where

$$\mathcal{H}^s_{0,\{0\}}(\mathrm{I}):=\begin{cases}H^s(\mathrm{I})\cap H^1_{0,\{0\}}(\mathrm{I}) & \text{when } s\geq 1,\\ [L_2(\mathrm{I}),H^1_{0,\{0\}}(\mathrm{I})]_s & \text{when } s\in[0,1].\end{cases}\tag{3.7}$$

By duality, these results extend to Sobolev spaces with negative smoothness indices in a way similar as indicated at the end of Sect. 2.

In Figure 1, the relation between Ψ, $\tilde{\Psi}$, Ψ^+ and $\tilde{\Psi}^-$ is illustrated.

Remark 3.5. As outlined in the introduction, Ψ, $\tilde{\Psi}$, Ψ^+, $\tilde{\Psi}^-$ related as in Figure 1, and such that all are Riesz bases for $L_2(\mathrm{I})$ do not exist for $\tilde{\Psi}\subset H^1_0(\mathrm{I})$ or $\Psi^+\subset H^1_0(\mathrm{I})$. Yet, some additional boundary conditions can be incorporated. For example, Ψ can be taken from $H^1_0(\mathrm{I})$. In this case, (3.5) can be shown for $s\in[0,\frac{3}{2})$. Since the derivatives of all ψ^+_λ vanish at the boundary, instead of the Jackson estimate (3.2) of order $d+1$, in this case only a Jackson estimate of order $\frac{3}{2}$ is valid. Without going into details, we note that this can be compensated by building approximations from the spans of wavelet sets to which additional wavelets with supports near the boundary are added (possibly adaptively).

4. Scaling functions and two-level transforms

In view of the definition of the wavelets ψ_λ^+ and $\tilde{\psi}_\lambda^-$, an obvious definition of the collections of corresponding primal and dual scaling functions $\underline{\Phi}_\ell^+ = [\underline{\phi}_{\ell,i}^+]_{i\in I_\ell}$ and $\underline{\tilde{\Phi}}_\ell^- = [\underline{\tilde{\phi}}_{\ell,i}^-]_{i\in I_\ell}$ is by means of

$$\underline{\phi}_{\ell,i}^+ := x \mapsto 2^\ell \int_0^x \phi_{\ell,i}(y)dy, \qquad \underline{\tilde{\phi}}_{\ell,i}^- := -2^{-\ell}\dot{\tilde{\phi}}_{\ell,i}.$$

Here we underlined the symbols to distinguish them from later alternatively defined scaling functions. By linearity of integration and differentiation, indeed

$$\operatorname{span}\{\underline{\phi}_{\ell,i}^+ : i \in I_\ell\} = \operatorname{span}\{\psi_\lambda^+ : |\lambda| \le \ell\}$$

and similarly at the dual side. Furthermore, since the $\tilde{\phi}_{\ell,i}$'s vanish at 1, and the $\underline{\phi}_{\ell,i}^+$'s vanish at 0,

$$\langle \underline{\phi}_{\ell,i}^+, \underline{\tilde{\phi}}_{\ell,j}^- \rangle_{L_2(\mathrm{I})} = \langle \underline{\phi}_{\ell,i}^+, -2^{-\ell}\dot{\tilde{\phi}}_{\ell,j} \rangle_{L_2(\mathrm{I})} = 2^{-\ell}\langle \dot{\underline{\phi}}_{\ell,i}^+, \tilde{\phi}_{\ell,j} \rangle_{L_2(\mathrm{I})} = \langle \phi_{\ell,i}, \tilde{\phi}_{\ell,j} \rangle_{L_2(\mathrm{I})} = \delta_{i,j},$$

i.e., we have biorthogonality of these primal and dual scaling functions.

The disadvantage of the above definition of the scaling functions is that the primal scaling functions are not locally supported. Therefore, with

$$c_{\ell,i} := \langle 1, \phi_{\ell,i} \rangle_{L_2(\mathrm{I})},$$

which was assumed to be nonzero (in (2.9)), and the convention that any $\phi_{\ell,i}$, $\tilde{\phi}_{\ell,i}$, $\underline{\phi}_{\ell,i}^+$ or $\underline{\tilde{\phi}}_{\ell,i}^-$ for $i \notin I_\ell$ is zero, we now define *new scaling functions* $\Phi_\ell^+ = [\phi_{\ell,i}^+]_{i\in I_\ell}$ and $\tilde{\Phi}_\ell^- = [\tilde{\phi}_{\ell,i}^-]_{i\in I_\ell}$ by means of

$$(4.1)\quad \begin{cases} \phi_{\ell,i}^+ := x \mapsto 2^\ell \displaystyle\int_0^x \phi_{\ell,i}(y) - \frac{c_{\ell,i}}{c_{\ell,i+1}}\phi_{\ell,i+1}(y)dy & = \underline{\phi}_{\ell,i}^+ - \dfrac{c_{\ell,i}}{c_{\ell,i+1}}\underline{\phi}_{\ell,i+1}^+, \\ \tilde{\phi}_{\ell,i}^- := -2^{-\ell}\displaystyle\sum_{p\le i}\frac{c_{\ell,p}}{c_{\ell,i}}\dot{\tilde{\phi}}_{\ell,p} & = \displaystyle\sum_{p\le i}\frac{c_{\ell,p}}{c_{\ell,i}}\underline{\tilde{\phi}}_{\ell,p}^-. \end{cases}$$

Since compared to the earlier definition, the new definition comprises a basis transformation, the new primal and dual scaling functions span the correct spaces.

Proposition 4.1. *The collections of scaling functions Φ_ℓ^+ and $\tilde{\Phi}_\ell^-$ defined by* (4.1) *are biorthogonal, and $\cup_\ell\Phi_\ell^+$ and $\cup_\ell\tilde{\Phi}_\ell^-$ are local and bounded in the sense of* (2.2) *and* (2.3)*, respectively.*

Proof. Because $\phi_{\ell,i}^+\tilde{\phi}_{\ell,p}$ $(i,p \in I_\ell)$ vanish at $\{0,1\}$, biorthogonality of $(\Phi_\ell^+, \tilde{\Phi}_\ell^-)$ follows from

$$\begin{aligned} \langle \phi_{\ell,i}^+, \tilde{\phi}_{\ell,j}^- \rangle_{L_2(\mathrm{I})} &= \langle \phi_{\ell,i}^+, -2^{-\ell}\sum_{p\le j}\frac{c_{\ell,p}}{c_{\ell,j}}\dot{\tilde{\phi}}_{\ell,p} \rangle_{L_2(\mathrm{I})} \\ &= \langle \phi_{\ell,i} - \frac{c_{\ell,i}}{c_{\ell,i+1}}\phi_{\ell,i+1}, \sum_{p\le j}\frac{c_{\ell,p}}{c_{\ell,j}}\tilde{\phi}_{\ell,p} \rangle_{L_2(\mathrm{I})} = \delta_{i,j} \end{aligned}$$

by distinguishing between the cases $j < i$, $j = i$ and $j > i$.

Because of the Jackson assumption (2.5), in any case for ℓ sufficiently large, the supports of the $\phi_{\ell,i}$ cover $[0,1]$. Now from the ordering assumption (2.4), we infer that for such ℓ the supports of $\phi_{\ell,i}$ and $\phi_{\ell,i+1}$ overlap. From $\int_0^1 \phi_{\ell,i}(y) - \frac{c_{\ell,i}}{c_{\ell,i+1}}\phi_{\ell,i+1}(y)dy = 0$, and the localness of $\cup_\ell \Phi_\ell$, we conclude the localness of $\cup_\ell \Phi_\ell^+$.

By (2.4), we have $\sum_{p\leq i} c_{\ell,p}\tilde{\phi}_{\ell,p} = \sum_{p\in I_\ell} c_{\ell,p}\tilde{\phi}_{\ell,p}$ on $[0, \inf \operatorname{supp} \tilde{\phi}_{\ell,i}]$. By (2.8), the latter function is equal to 1 outside a $2^{-\ell}$ neighbourhood of 1. From the localness of $\cup_\ell \tilde{\Phi}_\ell$, we conclude that of $\cup_\ell \tilde{\Phi}_\ell^-$.

From (2.9) and the localness and boundedness of $\cup_\ell \Phi_\ell$, an application of Hölder's inequality shows the boundedness of $\cup_\ell \Phi_\ell^+$.

From (2.7) and the boundedness of $\cup_\ell \tilde{\Phi}_\ell$, we have $\|\dot{\tilde{\phi}}_{\ell,i}\|_{L_2(\mathrm{I})} \lesssim 2^\ell$. Now by (2.9), (2.4), the localness of $\cup_\ell \tilde{\Phi}_\ell$ and the fact that $\tilde{\phi}_{\ell,i}^-$ vanishes left from some $2^{-\ell}$ neighbourhood of $\operatorname{supp} \tilde{\phi}_{\ell,i}$, we conclude the boundedness of $\cup_\ell \tilde{\Phi}_\ell^-$. $\square$

Remark 4.2. Note that $\Psi_{-1}^+ = \underline{\Phi}_0^+ \neq \Phi_0^+$ and $\tilde{\Psi}_{-1}^- = \underline{\tilde{\Phi}}_0^- \neq \tilde{\Phi}_0^-$. Redefining Ψ_{-1}^+ as Φ_0^+ or $\tilde{\Psi}_{-1}^-$ as $\tilde{\Phi}_0^-$ would destroy the relation $\dot{\Psi}_{-1}^+ = \frac{1}{2}\Psi_{-1}$ or $\dot{\tilde{\Psi}}_{-1} = -\frac{1}{2}\tilde{\Psi}_{-1}^-$, respectively.

Finally in this section, we discuss the refinement relations at primal and dual side. For $\ell \in \mathbb{N}_0$, let $\mathbf{M}_\ell$ be the matrix with

$$[\Phi_\ell^\top \ \Psi_\ell^\top] = \Phi_{\ell+1}^\top \mathbf{M}_\ell.$$

Biorthogonality shows that

$$\mathbf{M}_\ell = \left\langle \tilde{\Phi}_{\ell+1}, \begin{bmatrix} \Phi_\ell \\ \Psi_\ell \end{bmatrix} \right\rangle_{L_2(\mathrm{I})}.$$

Since $\Phi_{\ell+1}$ and $\Phi_\ell \cup \Psi_\ell$ are Riesz systems, uniformly in ℓ, that span the same space, $\mathbf{M}_\ell$ is boundedly invertible, uniformly in ℓ. Again biorthogonality shows that

$$[\tilde{\Phi}_\ell^\top \ \tilde{\Psi}_\ell^\top] = \tilde{\Phi}_{\ell+1}^\top \mathbf{M}_\ell^{-\top}.$$

The localness of $\cup_\ell \Phi_\ell$, Ψ, $\cup_\ell \tilde{\Phi}_\ell$, $\tilde{\Psi}$ shows that both $\mathbf{M}_\ell$ and its transposed inverse $\mathbf{M}_\ell^{-\top}$ are sparse, uniformly in ℓ. By linearity of integration and differentiation, we have that

$$[(\underline{\Phi}_\ell^+)^\top \ (\Psi_\ell^+)^\top] = \tfrac{1}{2}(\underline{\Phi}_{\ell+1}^+)^\top \mathbf{M}_\ell, \quad [(\underline{\tilde{\Phi}}_\ell^-)^\top \ (\tilde{\Psi}_\ell^-)^\top] = 2(\underline{\tilde{\Phi}}_{\ell+1}^-)^\top \mathbf{M}_\ell^{-\top}. \tag{4.2}$$

Let

$$\mathbf{M}_\ell^+ := \left\langle \tilde{\Phi}_{\ell+1}^-, \begin{bmatrix} \Phi_\ell^+ \\ \Psi_\ell^+ \end{bmatrix} \right\rangle_{L_2(\mathrm{I})}.$$

As a consequence of the biorthogonality of Ψ^+ and $\tilde{\Psi}^-$, and that of Φ_ℓ^+ and $\tilde{\Phi}_\ell^-$, and the localness and boundedness of $\cup_\ell \Phi_\ell^+$, Ψ^+, $\cup_\ell \tilde{\Phi}_\ell^-$ and $\tilde{\Psi}^-$, the matrices $\mathbf{M}_\ell^+$ and its transposed inverse $(\mathbf{M}_\ell^+)^{-\top}$ are bounded and sparse, uniformly in ℓ. It holds that

$$[(\Phi_\ell^+)^\top \ (\Psi_\ell^+)^\top] = (\Phi_{\ell+1}^+)^\top \mathbf{M}_\ell^+, \quad [(\tilde{\Phi}_\ell^-)^\top \ (\tilde{\Psi}_\ell^-)^\top] = (\tilde{\Phi}_{\ell+1}^-)^\top (\mathbf{M}_\ell^+)^{-\top}. \tag{4.3}$$

Now we split

$$\mathbf{M}_\ell = [\mathbf{M}_{\ell,0} \ \mathbf{M}_{\ell,1}],$$

and define the $\#I_\ell \times \#I_\ell$ matrix $\mathbf{T}_\ell$ by

$$\mathbf{T}_\ell := \begin{bmatrix} 1 & -\frac{c_{\ell,1}}{c_{\ell,2}} & & & \\ & 1 & -\frac{c_{\ell,2}}{c_{\ell,3}} & & \\ & & \ddots & \ddots & \\ & & & 1 & -\frac{c_{\ell,\#I_\ell-1}}{c_{\ell,\#I_\ell}} \\ & & & & 1 \end{bmatrix}. \tag{4.4}$$

Note that $\Phi_\ell^+ = \mathbf{T}_\ell \underline{\Phi}_\ell^+$ and $\tilde{\Phi}_\ell^+ = \mathbf{T}_\ell^{-\top} \underline{\tilde{\Phi}}_\ell^+$. Comparing the first equation in (4.2) with (4.3) now reveals that

$$\mathbf{M}_\ell^+ = \frac{1}{2}\mathbf{T}_{\ell+1}^{-\top}[\mathbf{M}_{\ell,0}\mathbf{T}_\ell^\top \;\; \mathbf{M}_{\ell,1}].$$

The remarkable aspect of this relation is that although $\mathbf{T}_{\ell+1}^{-\top}$ is clearly not sparse, as we have seen $\mathbf{M}_\ell^+$ is, uniformly in ℓ, and so is its inverse.

Remark 4.3. Let us consider the case that instead of on the interval I, we work on the real line $\mathbb{R}$, and that for $\theta \in \{\phi, \psi, \tilde{\phi}, \tilde{\psi}\}$,

$$\theta_{\ell,i}(x) = 2^{\ell/2}\theta(2^\ell x - i).$$

Then our definitions give that for $\eta \in \{\phi^+, \psi^+, \tilde{\phi}^-, \tilde{\psi}^-\}$

$$\eta_{\ell,i}(x) = 2^{\ell/2}\eta(2^\ell x - i),$$

where

$$\phi^+(x) = \int_0^x \phi(y) - \phi(y-1)dy, \qquad \psi^+(x) = \int_0^x \psi(y)dy,$$
$$\tilde{\phi}^-(x) = -\sum_{p\in\mathbb{N}_0} \dot{\tilde{\phi}}(x+p), \qquad \tilde{\psi}^- = -\dot{\tilde{\psi}}.$$

As a consequence, it holds that

$$\dot{\phi}^+(x) = \phi(x) - \phi(x-1), \qquad \dot{\psi}^+ = \psi,$$
$$\tilde{\phi}^-(x+1) - \tilde{\phi}^-(x) = \dot{\tilde{\phi}}(x), \qquad \tilde{\psi}^- = -\dot{\tilde{\psi}}.$$

As already noted in the introduction, up to harmless differences, these relations between a pair of stationary biorthogonal multi-resolution analyses characterized by $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$, and a new pair constructed by integration/differentiation were found by Lemarié-Rieusset in [LR92]. We conclude that our procedure is a generalization of the one from [LR92] to pairs of non-stationary multi-resolution analyses and bounded intervals.

Example 4.4. With ${}_d\xi$, ${}_{d,\tilde{d}}\tilde{\xi}$ being the biorthogonal generators of the stationary multi-resolution analyses from [CDF92], and with $d = 2$, $\tilde{d} = 4$, in this example we take $\phi_{\ell,i}$ and $\tilde{\phi}_{\ell,i}$ from the span of $\{{}_d\xi_{[\ell,j]} := 2^{\ell/2}{}_d\xi(2^\ell \cdot -j)|_\mathrm{I} : j \in \mathbb{Z}\}$ or $\{{}_{d,\tilde{d}}\tilde{\xi}_{[\ell,j]} := 2^{\ell/2}{}_{d,\tilde{d}}\tilde{\xi}(2^\ell \cdot -j)|_\mathrm{I} : j \in \mathbb{Z}\}$, respectively, in such a way that all conditions imposed in Section 2 are satisfied.

We apply the general construction of biorthogonal wavelets on the interval from [Dij09], which differs from that of [DKU99, DS98] in that the resulting primal scaling functions span the standard spline space of order d with respect to a *uniform* partition of I, with an appropriate multiplicity of the knots at the endpoints to meet prescribed homogeneous Dirichlet boundary conditions. Any freedom in the construction at the dual side was employed to minimize the condition numbers of the resulting wavelet bases, which numbers, in particular for larger d, are indeed (much) smaller than those that can be found in the literature, including those from [Pri06].

For $d = 2$, $\tilde{d} = 4$, and with the specific boundary conditions needed in the current setting (no boundary conditions at the primal side, and homogeneous Dirichlet boundary conditions of order 1 at the right boundary point at the dual side), this construction yields $\Phi_\ell = \{{}_2\xi_{[\ell,j]} : j \in \mathbb{Z},\ {}_2\xi_{[\ell,j]} \neq 0\}$ with cardinality $\# I_\ell = 2^\ell + d - 1$. Applying the natural left-to-right ordering of the basis functions, the coefficients of the $\tilde{\phi}_{\ell,i}$ in terms of the non-zero ${}_{2,4}\tilde{\xi}_{[\ell,j]}$ are given, columnwise, by the following $(2^\ell + d + 2\tilde{d} - 3) \times (2^\ell + d - 1)$ matrix

$$
\begin{bmatrix}
100 & -\frac{280}{3} & \frac{178}{3} & -15 & & & & & \\
50 & -\frac{265}{6} & \frac{80}{3} & -\frac{13}{2} & & & & & \\
20 & -\frac{47}{3} & \frac{26}{3} & -2 & & & & & \\
5 & -\frac{29}{12} & \frac{7}{6} & -\frac{1}{4} & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & & & & & 1 & & \\
 & & & & & & & 1 & \\
 & & & & & -\frac{1}{448} & -\frac{67}{672} & \frac{1007}{1344} & -\frac{365}{112} \\
 & & & & & -\frac{1}{32} & -\frac{67}{48} & \frac{911}{96} & -\frac{365}{8} \\
 & & & & & -\frac{25}{224} & -\frac{2011}{336} & \frac{25175}{672} & -\frac{9125}{56} \\
 & & & & & -\frac{279}{224} & -\frac{3685}{336} & \frac{55385}{672} & -\frac{20075}{56}
\end{bmatrix}.
$$

The primal, and so the dual wavelets are determined by the $(2^{\ell+1}+d-1) \times (2^{\ell+1} + d - 1)$ refinement matrix (cf. (4.2))

$$
\boldsymbol{M}_\ell := \frac{1}{\sqrt{2}}
\left[\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & -\frac{35}{32} & \frac{15}{32} & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & & & & & \frac{875}{768} & -\frac{45}{256} & \frac{3}{64} & & & & & \\
 & 1 & & & & & & & & -\frac{241}{384} & -\frac{105}{128} & \frac{3}{32} & & & & & \\
 & \frac{1}{2} & \frac{1}{2} & & & & & & & -\frac{53}{256} & \frac{345}{256} & -\frac{1}{4} & \frac{3}{64} & & & & \\
 & & 1 & & & & & & & \frac{41}{192} & -\frac{31}{64} & -\frac{19}{32} & \frac{3}{32} & & & & \\
 & & \frac{1}{2} & \frac{1}{2} & & & & & & \frac{67}{768} & -\frac{53}{256} & \frac{45}{32} & -\frac{1}{4} & \ddots & & & \\
 & & & 1 & & & & & & -\frac{5}{128} & \frac{9}{128} & -\frac{19}{32} & -\frac{19}{32} & & & & \\
 & & & \frac{1}{2} & \ddots & & & & & -\frac{5}{256} & \frac{9}{256} & -\frac{1}{4} & \frac{45}{32} & & \frac{3}{64} & & \\
 & & & & & & & & & & & \frac{3}{32} & -\frac{19}{32} & & \frac{3}{32} & & \\
 & & & & \ddots & \frac{1}{2} & & & & & & \frac{3}{64} & -\frac{1}{4} & & -\frac{1}{4} & \frac{1341}{28672} & -\frac{23}{28672} \\
 & & & & & 1 & & & & & & & \frac{3}{32} & & -\frac{19}{32} & \frac{1341}{14336} & -\frac{23}{14336} \\
 & & & & & \frac{1}{2} & \frac{1}{2} & & & & & & \frac{3}{64} & & \frac{45}{32} & -\frac{7305}{28672} & \frac{881}{86016} \\
 & & & & & & 1 & & & & & & & & -\frac{19}{32} & -\frac{4323}{7168} & \frac{475}{21504} \\
 & & & & & & \frac{1}{2} & \frac{1}{2} & & & & & & \ddots & -\frac{1}{4} & \frac{41193}{28672} & -\frac{1819}{28672} \\
 & & & & & & & 1 & & & & & & & \frac{3}{32} & -\frac{7505}{14336} & -\frac{6407}{43008} \\
 & & & & & & & \frac{1}{2} & \frac{1}{2} & & & & & & \frac{3}{64} & -\frac{11885}{28672} & \frac{64885}{86016} \\
 & & & & & & & & 1 & & & & & & & -\frac{1095}{3584} & -\frac{8395}{3584}
\end{array}\right].
$$

Having specified $(\Phi_\ell)_\ell$, $(\Psi_\ell)_\ell$, $(\tilde{\Phi}_\ell)_\ell$, $(\tilde{\Psi}_\ell)_\ell$, the new collections of biorthogonal scaling functions and wavelets $(\Phi_\ell^+)_\ell$, $(\Psi_\ell^+)_\ell$, $(\tilde{\Phi}_\ell^-)_\ell$, $(\tilde{\Psi}_\ell^-)_\ell$ are fully determined by the definitions (3.1) and (4.1). The coefficients $c_{\ell,i} = \int_{\mathrm{I}} \phi_{\ell,i}$ are equal to $2^{-\ell/2}$, except for the left and rightmost scaling functions, for which they read as $\frac{1}{2} 2^{-\ell/2}$.

Since the $\phi_{\ell,i}$ are splines of order d with respect to a uniform partition of I with stepsize $2^{-\ell}$, the $\phi_{\ell,i}^+$ are in the span of $\{{}_3\xi_{[\ell,j]} : j \in \mathbb{Z},\ {}_3\xi_{[\ell,j]} \neq 0\}$. Knowing that ${}_{d,\tilde{d}}\dot{\tilde{\xi}} = {}_{d+1,\tilde{d}-1}\tilde{\xi}(\cdot + 1 - d \bmod 2) - {}_{d+1,\tilde{d}-1}\tilde{\xi}(\cdot - d \bmod 2)$, we conclude that the $\tilde{\phi}^-_{\ell,i}$ are in the span of $\{{}_{3,3}\tilde{\xi}_{[\ell,j]} : j \in \mathbb{Z},\ {}_{3,3}\tilde{\xi}_{[\ell,j]} \neq 0\}$. Applying the natural left-to-right ordering of the basis functions, the coefficients of the $\phi_{\ell,i}$ or $\phi^+_{\ell,i}$ in terms of the non-zero ${}_3\xi_{[\ell,j]}$ or ${}_{3,3}\tilde{\xi}_{[\ell,j]}$ are given by the $(2^\ell + d) \times (2^\ell + d - 1)$ and $(2^\ell + d + 1 + 2(\tilde{d} - 1) - 3) \times (2^\ell + d - 1)$ matrices

$$
\begin{bmatrix}
-\tfrac12 & & & & & & & & \\
\tfrac12 & & & & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & & & & & 1 & & \\
 & & & & & & & 1 & \\
 & & & & & & & -1 & 1
\end{bmatrix},
\begin{bmatrix}
50 & -\tfrac{145}{6} & \tfrac{17}{2} & & & & & & \\
30 & -\tfrac{27}{2} & \tfrac{9}{2} & & & & & & \\
15 & -\tfrac{23}{4} & \tfrac{7}{4} & & & & & & \\
5 & -\tfrac{11}{12} & \tfrac{1}{4} & & & & & & \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
 & & & 1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\
 & & & & & & 1 & & \\
 & & & & & \tfrac{1}{448} & \tfrac{137}{1344} & \tfrac{79}{224} & \tfrac{111}{28} \\
 & & & & & \tfrac{13}{448} & \tfrac{1781}{1344} & -\tfrac{1661}{224} & \tfrac{771}{28} \\
 & & & & & \tfrac{9}{112} & \tfrac{523}{112} & -\tfrac{1305}{56} & \tfrac{495}{7} \\
 & & & & & \tfrac{127}{112} & \tfrac{685}{112} & -\tfrac{2175}{56} & \tfrac{825}{7}
\end{bmatrix},
$$

respectively.

From the coefficients $c_{\ell,i}$ and the matrix $\mathbf{M}_\ell$ we obtain the $(2^{\ell+1} + d - 1) \times (2^{\ell+1} + d - 1)$ refinement matrix $\mathbf{M}^+_\ell$ that determines the primal and dual wavelet collections Ψ_ℓ and $\tilde{\Psi}^-_\ell$. It reads as

$$
\mathbf{M}^+_\ell = \frac{1}{\sqrt{2}}
\begin{bmatrix}
\tfrac12 & & & & & & & & & -\tfrac{35}{16} & \tfrac{15}{16} & & & & & & \\
\tfrac38 & \tfrac14 & & & & & & & & \tfrac{455}{384} & \tfrac{15}{128} & \tfrac{3}{32} & & & & & \\
\tfrac18 & \tfrac34 & & & & & & & & -\tfrac{9}{128} & -\tfrac{195}{128} & \tfrac{9}{32} & & & & & \\
 & \tfrac34 & \tfrac14 & & & & & & & -\tfrac{31}{64} & \tfrac{75}{64} & -\tfrac{7}{32} & \tfrac{3}{32} & & & & \\
 & \tfrac14 & \tfrac34 & & & & & & & -\tfrac{11}{192} & \tfrac{13}{64} & -\tfrac{45}{32} & \tfrac{9}{32} & & & & \\
 & & \tfrac34 & \tfrac14 & & & & & & \tfrac{15}{128} & -\tfrac{27}{128} & \tfrac{45}{32} & -\tfrac{7}{32} & \ddots & & & \\
 & & \tfrac14 & \tfrac34 & & & & & & \tfrac{5}{128} & -\tfrac{9}{128} & \tfrac{7}{32} & -\tfrac{45}{32} & & & & \\
 & & & \tfrac34 & \ddots & & & & & & & -\tfrac{9}{32} & \tfrac{45}{32} & & \tfrac{3}{32} & & \\
 & & & \tfrac14 & & & & & & & & -\tfrac{3}{32} & \tfrac{7}{32} & & \tfrac{9}{32} & & \\
 & & & & & \tfrac14 & & & & & & & -\tfrac{9}{32} & & -\tfrac{7}{32} & \tfrac{1341}{14336} & -\tfrac{23}{14336} \\
 & & & & \ddots & \tfrac34 & & & & & & & -\tfrac{3}{32} & & -\tfrac{45}{32} & \tfrac{4023}{14336} & -\tfrac{69}{14336} \\
 & & & & & \tfrac34 & \tfrac14 & & & & & & & & \tfrac{45}{32} & -\tfrac{1641}{7168} & \tfrac{337}{21504} \\
 & & & & & \tfrac14 & \tfrac34 & & & & & & & \ddots & \tfrac{7}{32} & -\tfrac{10287}{7168} & \tfrac{429}{7168} \\
 & & & & & & \tfrac34 & \tfrac14 & & & & & & & -\tfrac{9}{32} & \tfrac{20619}{14336} & -\tfrac{961}{14336} \\
 & & & & & & \tfrac14 & \tfrac34 & & & & & & & -\tfrac{3}{32} & \tfrac{5609}{14336} & -\tfrac{15697}{43008} \\
 & & & & & & & \tfrac12 & \tfrac14 & & & & & & & -\tfrac{1569}{3584} & \tfrac{4099}{3584} \\
 & & & & & & & & 1 & & & & & & & -\tfrac{333}{224} & -\tfrac{537}{224}
\end{bmatrix}.
$$

In Figure 2, for $\ell = 4$, a number of scaling functions $\phi_{\ell,i}$, $\phi^+_{\ell,i}$, and wavelets $\psi_{\ell,i}$ $\psi^+_{\ell,i}$ are illustrated. One may observe that $\dot{\psi}^+_{\ell,i}$ is (a multiple of) $\psi_{\ell,i}$, and that all $\phi^+_{\ell,i}$ and $\psi^+_{\ell,i}$ vanish at the left boundary point. All $\psi^+_{\ell,i}$ have 3 vanishing moments. The $\psi_{\ell,i}$ have 4 vanishing moments, except for the two right-most ones that are orthogonal to $x \mapsto (1-x)^k$ for $k \in \{1,2,3\}$, but not to constant functions, caused by the fact that $\mathcal{S}(\tilde{\Phi}_\ell)$ satisfies homogeneous Dirichlet boundary conditions of order 1 at the right boundary.

5. Divergence-free wavelets

For $\Sigma \subset \partial\Box$ with positive measure, and for $s \geq 0$, let

$$
\mathcal{H}^s_{0,\Sigma}(\Box) := \begin{cases} H^s(\Box) \cap H^1_{0,\Sigma}(\Box) & \text{when } s \geq 1, \\ [L_2(\Box), H^1_{0,\Sigma}(\Box)]_s & \text{when } s \in [0,1]. \end{cases}
$$

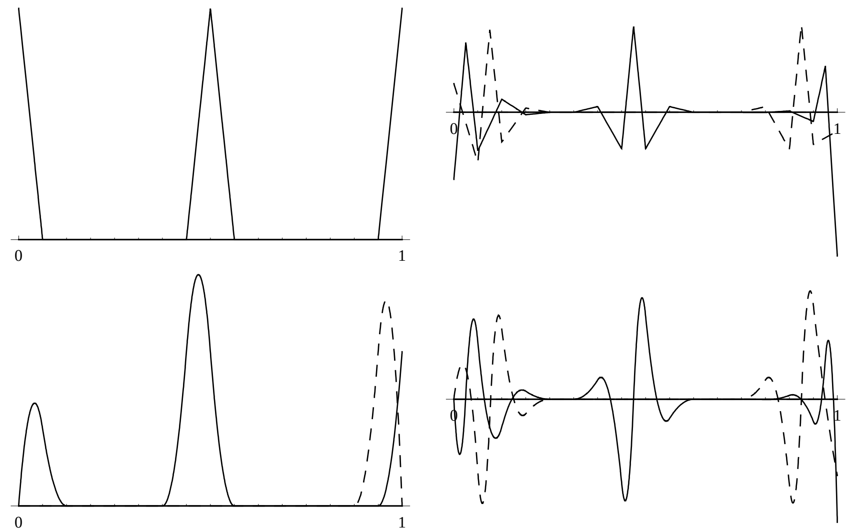

FIGURE 2. Some scaling functions $\phi_{4,i}$ (upper left), $\phi^{+}_{4,i}$ (lower left), and wavelets $\psi_{4,i}$ (upper right), $\psi^{+}_{4,i}$ (lower right). (Some basis functions are dotted to distinguish them from others with whom they have an overlapping support)

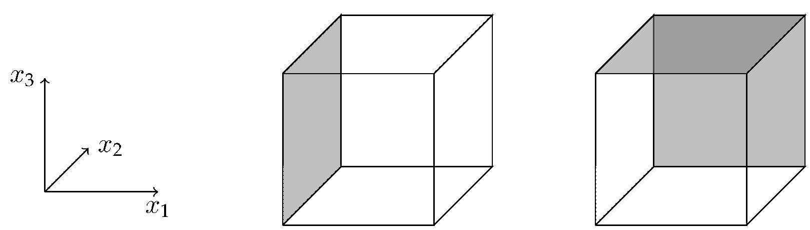

FIGURE 3. Γ_1 and $\tilde{\Gamma}_1$ for $n = 3$.

For $1 \leq k \leq n$, let

$$\Gamma_k = [0,1]^{k-1} \times \{0\} \times [0,1]^{n-k}, \qquad \tilde{\Gamma}_k = \bigcup_{m=1, m\neq k}^{n} [0,1]^{m-1} \times \{1\} \times [0,1]^{n-m},$$

see Figure 3 for an illustration. Then, following arguments as in [GO95, Example 3], for $s \geq 0$ we have

$$\mathcal{H}^s_{0,\Gamma_k}(\square) = H^s \otimes L_2 \otimes \cdots \otimes L_2 \cap \cdots \cap L_2 \otimes \cdots \otimes \overset{\downarrow k\text{th pos.}}{\mathcal{H}^s_{0,\{0\}}} \otimes \cdots \otimes L_2 \cap \cdots \cap L_2 \otimes \cdots \otimes L_2 \otimes H^s,$$

$$\mathcal{H}^s_{0,\tilde{\Gamma}_k}(\square) = \mathcal{H}^s_{0,\{1\}} \otimes L_2 \otimes \cdots \otimes L_2 \cap \cdots \cap L_2 \otimes \cdots \otimes \overset{\downarrow k\text{th pos.}}{H^s} \otimes \cdots \otimes L_2 \cap \cdots \cap L_2 \otimes \cdots \otimes L_2 \otimes \mathcal{H}^s_{0,\{1\}},$$

where the spaces on the right are spaces of functions on the unit interval. As shown in [GO95], from these characterizations, and (2.10), (2.11), (3.5), (3.6), we have the following result.

Proposition 5.1. *For $0 \leq s < \gamma$ and $0 \leq \tilde{s} < \tilde{\gamma} - 1$,*

$$\Big\{(\sum_{m=1}^n 4^{|\lambda_m|})^{-s/2} \psi_{\lambda_1} \otimes \cdots \otimes \psi^+_{\lambda_k} \otimes \cdots \otimes \psi_{\lambda_n} : \boldsymbol{\lambda} \in \boldsymbol{\nabla} := \nabla^n\Big\},$$
$$\Big\{(\sum_{m=1}^n 4^{|\lambda_m|})^{-\tilde{s}/2} \tilde{\psi}_{\lambda_1} \otimes \cdots \otimes \tilde{\psi}^-_{\lambda_k} \otimes \cdots \otimes \tilde{\psi}_{\lambda_n} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\Big\}$$

are Riesz bases of $\mathcal{H}^s_{0,\Gamma_k}(\square)$ and $\mathcal{H}^{\tilde{s}}_{0,\tilde{\Gamma}_k}(\square)$, respectively. For $s = \tilde{s} = 0$, the collections are biorthogonal.

Remark 5.2. Before proceeding to vector-valued wavelets, in this remark we briefly discuss the rates of approximation that can be realized with the anisotropic wavelet bases from Proposition 5.1. With $\psi^{(k)}_{\boldsymbol{\lambda}} := \psi_{\lambda_1} \otimes \cdots \otimes \psi^+_{\lambda_k} \otimes \cdots \otimes \psi_{\lambda_n}$, and, for $L \in \mathbb{N}$ and $\beta \geq 1$, with the (optimized) *sparse grid* index set $\boldsymbol{\nabla}_{L,\beta} := \{\boldsymbol{\lambda} \in \boldsymbol{\nabla} : \beta \||\boldsymbol{\lambda}|\|_1 + (1-\beta)\||\boldsymbol{\lambda}|\|_\infty \leq L\}$, where $|\boldsymbol{\lambda}| := (|\lambda_1|, \ldots, |\lambda_n|)$, it is known that $\#\boldsymbol{\nabla}_{L,1} \eqsim 2^L L^{n-1}$ and $\#\boldsymbol{\nabla}_{L,\beta} \eqsim 2^L$ when $\beta > 1$. Furthermore, it is known that

$$\inf_{v \in \operatorname{span}\{\psi^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{L,1}\}} \|u - v\|_{L_2(\square)} \lesssim L^{\frac{n-1}{2}} 2^{-dL} \|\partial_1^d \cdots \partial_n^d u\|_{L_2(\square)}$$

and that for $0 < s < \gamma$, $\beta \in (1, \frac{d}{d-s})$, and $q > \beta(d-s)$,

$$\inf_{v \in \operatorname{span}\{\psi^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{L,\beta}\}} \|u - v\|_{\mathcal{H}^s_{0,\Gamma_k}(\square)} \lesssim 2^{-(d-s)L} \sqrt{\sum_{m=1}^n \|\partial_1^q \cdots \partial_m^d \cdots \partial_n^q u\|^2_{L_2(\square)}},$$

cf. [GK00]. So assuming sufficient smoothness of certain mixed derivatives of the function u to be approximated, the error in $\mathcal{H}^s_{0,\Gamma_k}(\square)$ of the best approximation from the span of N suitably selected anisotropic wavelets is of order $N^{-(d-s)}$ (up to log-terms when $s = 0$), with the rate $d - s$ thus being independent of the space dimension n.

What is more, as shown in [Nit06], the regularity conditions on u for obtaining this rate $d - s$ can be largely reduced when the approximation is sought from the span of the best possible set of N wavelets depending on u (*nonlinear approximation*), instead of the aforementioned sparse grid index sets. When solving well-posed operator equations using wavelets, the rate of approximation of these so-called best N-term approximations can be realized with adaptive wavelet schemes ([CDD01, CDD02, GHS07, SS08]).

Finally, with isotropic wavelet contructions, it is well-known that the best possible rate reads as $\frac{d-s}{n}$. Analogous observations are valid at the dual side.

Setting for $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, $1 \leq k \leq n$, the vector-valued wavelets

$$\begin{aligned} \underline{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} &:= \psi_{\lambda_1} \otimes \cdots \otimes \psi^+_{\lambda_k} \otimes \cdots \otimes \psi_{\lambda_n} \mathbf{e}_k, \\ \underline{\tilde{\boldsymbol{\psi}}}^{(k)}_{\boldsymbol{\lambda}} &:= \tilde{\psi}_{\lambda_1} \otimes \cdots \otimes \tilde{\psi}^-_{\lambda_k} \otimes \cdots \otimes \tilde{\psi}_{\lambda_n} \mathbf{e}_k, \end{aligned} \tag{5.1}$$

as an immediate consequence of Proposition 5.1, we have

Corollary 5.3. *For $0 \leq s < \gamma$ and $0 \leq \tilde{s} < \tilde{\gamma} - 1$,*

$$\big\{(\sum_{m=1}^n 4^{|\lambda_m|})^{-s/2} \underline{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} : 1 \leq k \leq n, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\big\}, \quad \big\{(\sum_{m=1}^n 4^{|\lambda_m|})^{-\tilde{s}/2} \underline{\tilde{\boldsymbol{\psi}}}^{(k)}_{\boldsymbol{\lambda}} : 1 \leq k \leq n, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\big\}$$

are Riesz bases for

$$\mathcal{H}^s_{0,\Gamma_1}(\square) \times \cdots \times \mathcal{H}^s_{0,\Gamma_n}(\square), \quad \mathcal{H}^{\tilde{s}}_{0,\tilde{\Gamma}_1}(\square) \times \cdots \times \mathcal{H}^{\tilde{s}}_{0,\tilde{\Gamma}_n}(\square),$$

respectively. For $s = \tilde{s} = 0$, the collections are biorthogonal.

In order to construct divergence-free wavelets, now we are going to apply basis transformations. For any $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, let us select an orthogonal $\mathbf{A}^{\boldsymbol{\lambda}} \in \mathbb{R}^{n\times n}$ with its nth row given by

$$\mathbf{A}^{\boldsymbol{\lambda}}_{n\bullet} = \boldsymbol{\alpha}^\top \quad \text{where } \boldsymbol{\alpha}\,(=\boldsymbol{\alpha}_{\boldsymbol{\lambda}}) := [2^{|\lambda_1|} \cdots 2^{|\lambda_n|}]^\top / (\sum_{m=1}^n 4^{|\lambda_m|})^{\frac{1}{2}}. \tag{5.2}$$

The fact that necessarily the first $n-1$ rows of $\mathbf{A}^{\boldsymbol{\lambda}}$ are orthogonal to $\boldsymbol{\alpha}^\top$ will be the key why the wavelets $\boldsymbol{\psi}^{(1)}_{\boldsymbol{\lambda}}, \ldots, \boldsymbol{\psi}^{(n-1)}_{\boldsymbol{\lambda}}$ defined below are divergence-free. The mutual orthogonality of the rows of $\mathbf{A}^{\boldsymbol{\lambda}}$ gives the stability of the transformation and therefore that of the resulting basis. An example of such a matrix $\mathbf{A}^{\boldsymbol{\lambda}}$ is given by the Householder transformation

$$\mathbf{A}^{\boldsymbol{\lambda}} = \mathrm{Id} - \frac{2(\boldsymbol{\alpha} - \boldsymbol{e}_n)(\boldsymbol{\alpha} - \boldsymbol{e}_n)^\top}{(\boldsymbol{\alpha} - \boldsymbol{e}_n)^\top(\boldsymbol{\alpha} - \boldsymbol{e}_n)}, \tag{5.3}$$

that for $n = 2, 3$ reads as

$$\begin{bmatrix} -\alpha_2 & \alpha_1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \quad \begin{bmatrix} 1 - \frac{\alpha_1^2}{1-\alpha_3} & -\frac{\alpha_1\alpha_2}{1-\alpha_3} & \alpha_1 \\ -\frac{\alpha_1\alpha_2}{1-\alpha_3} & 1 - \frac{\alpha_2^2}{1-\alpha_3} & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix},$$

respectively.

We use the matrices $\mathbf{A}^{\boldsymbol{\lambda}}$ to orthogonally transform the bases from Corollary 5.3: We define $\boldsymbol{\Psi} = \{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}, 1 \le k \le n\}$, $\tilde{\boldsymbol{\Psi}} = \{\tilde{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}, 1 \le k \le n\}$ by setting for any $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$,

$$\begin{bmatrix} \boldsymbol{\psi}^{(1)}_{\boldsymbol{\lambda}} \\ \vdots \\ \boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda}} \end{bmatrix} := \mathbf{A}^{\boldsymbol{\lambda}} \begin{bmatrix} \underline{\boldsymbol{\psi}}^{(1)}_{\boldsymbol{\lambda}} \\ \vdots \\ \underline{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}} \end{bmatrix}, \qquad \begin{bmatrix} \tilde{\boldsymbol{\psi}}^{(1)}_{\boldsymbol{\lambda}} \\ \vdots \\ \tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}} \end{bmatrix} := \mathbf{A}^{\boldsymbol{\lambda}} \begin{bmatrix} \underline{\tilde{\boldsymbol{\psi}}}^{(1)}_{\boldsymbol{\lambda}} \\ \vdots \\ \underline{\tilde{\boldsymbol{\psi}}}^{(n)}_{\boldsymbol{\lambda}} \end{bmatrix}.$$

We will need some Sobolev spaces of vector valued functions, other than those that are simply Cartesian products of Sobolev spaces of scalar functions. Setting

$$\Gamma := \bigcup_{k=1}^n \Gamma_k, \quad \tilde{\Gamma} := \partial\square \backslash \Gamma \quad , \tag{5.4}$$

we define

$$\begin{aligned} \mathbf{H}(\mathrm{div};\square) &:= \big\{\mathbf{u} \in L_2(\square)^n : \mathrm{div}\,\mathbf{u} \in L_2(\square)\big\}, \\ \mathbf{H}_{0,\Gamma}(\mathrm{div};\square) &:= \big\{\mathbf{u} \in \mathbf{H}(\mathrm{div};\square) : \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\big\}, \\ \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square) &:= \big\{\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathrm{div};\square) : \mathrm{div}\,\mathbf{u} = 0\big\}. \end{aligned} \tag{5.5}$$

Since $\mathbf{n} = -\mathbf{e}_k$ on Γ_k, it holds that $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ if and only if $u_k = 0$ on Γ_k ($1 \le k \le n$). So, in particular,

$$\prod_{k=1}^n H^1_{0,\Gamma_k}(\square) = \{\mathbf{u} \in H^1(\square)^n : \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}. \tag{5.6}$$

We are going to show that $\boldsymbol{\Psi}^{\rm df}=\big\{\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}:1\leq k\leq n-1,\,\boldsymbol{\lambda}\in\boldsymbol{\nabla}\big\}$ is a Riesz basis for $\mathbf{H}_{0,\Gamma}({\rm div}0;\square)$. In order to do so, it is not sufficient to show that these $\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}$ are in $\mathbf{H}_{0,\Gamma}({\rm div}0;\square)$. Instead, we have to show that any $\mathbf{u}\in\mathbf{H}_{0,\Gamma}({\rm div}0;\square)$ has a convergent expansion in terms of these $\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}$, with the ℓ_2 norms of the sequence of coefficients being equivalent to $\|\mathbf{u}\|_{L_2(\square)}$.

Proposition 5.4. (a). *For $0\leq s<\gamma$ and $0\leq\tilde{s}<\tilde{\gamma}-1$,*

$$\big\{(\sum_{m=1}^{n}4^{|\lambda_m|})^{-s/2}\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}:1\leq k\leq n,\,\boldsymbol{\lambda}\in\boldsymbol{\nabla}\big\},\quad\big\{(\sum_{m=1}^{n}4^{|\lambda_m|})^{-\tilde{s}/2}\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(k)}:1\leq k\leq n,\,\boldsymbol{\lambda}\in\boldsymbol{\nabla}\big\}$$

are Riesz bases for

$$\mathcal{H}^s_{0,\Gamma_1}(\square)\times\cdots\times\mathcal{H}^s_{0,\Gamma_n}(\square),\quad\mathcal{H}^{\tilde{s}}_{0,\tilde{\Gamma}_1}(\square)\times\cdots\times\mathcal{H}^{\tilde{s}}_{0,\tilde{\Gamma}_n}(\square),$$

respectively. Furthermore $\langle\boldsymbol{\Psi},\tilde{\boldsymbol{\Psi}}\rangle_{L_2(\square)^n}={\rm Id}$.

(b). *For $\mathbf{u}\in\mathbf{H}_{0,\Gamma}({\rm div};\square)$,*

$$\langle\mathbf{u},\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}\rangle_{L_2(\square)^n}=\langle{\rm div}\,\mathbf{u},\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\rangle_{L_2(\square)}/(\sum_{m=1}^{n}4^{|\lambda_m|})^{\frac12}.$$

It holds that $\boldsymbol{\Psi}\subset\mathbf{H}_{0,\Gamma}({\rm div};\square)$, with

$${\rm div}\,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}=\begin{cases}0 & \text{for } 1\leq k\leq n-1,\\ (\sum_{m=1}^{n}4^{|\lambda_m|})^{\frac12}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n} & \text{for } k=n.\end{cases}$$

(c). *For $q\in H^1_{0,\tilde{\Gamma}}(\square)$,*

$$\begin{aligned}&\langle\mathbf{grad}\,q,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}\rangle_{L_2(\square)^n}\\&=\begin{cases}0 & \text{for } 1\leq k\leq n-1,\\ (\sum_{m=1}^{n}4^{|\lambda_m|})^{\frac12}\langle q,\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n}\rangle_{L_2(\square)^n} & \text{for } k=n.\end{cases}\end{aligned}$$

It holds that

$$\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}=-\mathbf{grad}\,\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}/(\sum_{m=1}^{n}4^{|\lambda_m|})^{\frac12}\in\mathbf{grad}\,H^1_{0,\tilde{\Gamma}}(\square).$$

Proof. (a). This part is a consequence of Corollary 5.3. Biorthogonality of the collections from Corollary 5.3 is preserved because $\mathbf{A}^{\boldsymbol{\lambda}}$ is orthogonal. The remainder follows from the fact that the scaling factors $(\sum_{m=1}^{n}4^{|\lambda_m|})^{-s/2}$ and $(\sum_{m=1}^{n}4^{|\lambda_m|})^{-\tilde{s}/2}$ in the statement of Corollary 5.3 are independent of k.

(b). Since for $\mathbf{u}\in\mathbf{H}_{0,\Gamma}({\rm div};\square)$ and $1\leq m\leq n$, $u_m\mathbf{n}\cdot\mathbf{e}_m\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}$ vanishes on $\partial\square$, integration by parts and the definition of the nth row of $\mathbf{A}^{\boldsymbol{\lambda}}$ show that

$$\begin{aligned}\langle\mathbf{u},\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}\rangle_{L_2(\square)^n}&=\sum_{m=1}^{n}\mathbf{A}^{\boldsymbol{\lambda}}_{nm}\langle\mathbf{u},\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}^-_{\lambda_m}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\mathbf{e}_m\rangle_{L_2(\square)^n}\\&=-\sum_{m=1}^{n}\mathbf{A}^{\boldsymbol{\lambda}}_{nm}2^{-|\lambda_m|}\langle\mathbf{u},\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\dot{\tilde{\psi}}_{\lambda_m}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\mathbf{e}_m\rangle_{L_2(\square)^n}\\&=\langle{\rm div}\,\mathbf{u},\tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\rangle_{L_2(\square)}/(\sum_{m=1}^{n}4^{|\lambda_m|})^{\frac12}.\end{aligned}$$

Since $\Psi^+ \subset H^1_{0,\{0\}}(\mathrm{I})$, for all k and $\boldsymbol{\lambda}$ we have $\underline{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} \in \mathbf{H}_{0,\Gamma}(\mathrm{div};\square)$, and so $\boldsymbol{\Psi} \subset \mathbf{H}_{0,\Gamma}(\mathrm{div};\square)$. By definition of $\mathbf{A}^{\boldsymbol{\lambda}}$,

$$\begin{aligned}\mathrm{div}\,\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} &= \sum_{m=1}^n \mathbf{A}^{\boldsymbol{\lambda}}_{km} \mathrm{div}\,\underline{\boldsymbol{\psi}}^{(m)}_{\boldsymbol{\lambda}} = \Big(\sum_{m=1}^n \mathbf{A}^{\boldsymbol{\lambda}}_{km} 2^{|\lambda_m|}\Big)\,\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n}\\ &= \begin{cases} 0 & \text{for } 1 \le k \le n-1,\\ (\sum_{m=1}^n 4^{|\lambda_m|})^{\frac12}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n} & \text{for } k=n.\end{cases}\end{aligned}$$

(c). Since $q\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}\cdot\mathbf{n} = 0$ on $\partial\square$, $\langle \mathbf{grad}\, q, \boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}\rangle_{L_2(\square)^n} = \langle q, \mathrm{div}\,\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}\rangle_{L_2(\square)}$ by integration by parts, and so the expression for $\langle \mathbf{grad}\, q, \boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}\rangle_{L_2(\square)^n}$ follows from (b). The last statement follows by definition of $\tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}}$. $\square$

Proposition 5.4(a) shows that any $\mathbf{u} \in L_2(\square)^n$ has a unique expansion $\mathbf{u} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}}\sum_{k=1}^n c^{(k)}_{\boldsymbol{\lambda}}\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}$ with $\|\mathbf{u}\|^2_{L_2(\square)^n} \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}}\sum_{k=1}^n |c^{(k)}_{\boldsymbol{\lambda}}|^2$. If $\mathbf{u}$ is in the subspace $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$, then Part (b) shows that $c^{(n)}_{\boldsymbol{\lambda}} = 0$. Since moreover, for $1 \le k \le n-1$, $\mathrm{div}\,\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} = 0$, we conclude that $\{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} : 1 \le k \le n-1, \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$ is a Riesz basis for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$, which is the first statement from the following corollary. Taking into account (5.6), in the same way the second statement of this corollary is deduced. The last statement follows from Proposition 5.4(c).

Corollary 5.5. *The collection*

$$\boldsymbol{\Psi}^{\mathrm{df}} = \{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} : 1 \le k \le n-1,\, \boldsymbol{\lambda}\in\boldsymbol{\nabla}\},$$

and, if $\gamma > 1$, its properly scaled version

$$\boldsymbol{\Psi}^{\mathrm{df}}_{\mathbf{H}^1} = \{(\sum_{m=1}^n 4^{|\lambda_m|})^{-1/2}\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} : 1 \le k \le n-1,\, \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$$

are Riesz bases for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ and $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)\cap H^1(\square)^n$, respectively.

The collection $\tilde{\boldsymbol{\Psi}}^{\mathrm{gr}} = \{\tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$ is a Riesz basis for $\mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$ equipped with $\|\cdot\|_{L_2(\square)^n}$.

Remark 5.6. Although for $k \neq m$ and any $\boldsymbol{\lambda}\in\boldsymbol{\nabla}$, $\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} \perp_{L_2(\square)^n} \boldsymbol{\psi}^{(m)}_{\boldsymbol{\lambda}}$, being a consequence of $\underline{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} \perp_{L_2(\square)^n} \underline{\boldsymbol{\psi}}^{(m)}_{\boldsymbol{\lambda}}$ and the orthogonality of $\mathbf{A}^{\boldsymbol{\lambda}}$, generally $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square) \not\perp_{L_2(\square)^n} \mathrm{span}\,\{\boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla}\}$, the reason being that generally $\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}} \not\perp_{L_2(\square)^n} \boldsymbol{\psi}^{(m)}_{\boldsymbol{\mu}}$ for $\boldsymbol{\lambda}\neq\boldsymbol{\mu}\in\boldsymbol{\nabla}$.

Remark 5.7. From Remark 5.2, we deduce that for $\mathbf{u}\in\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$,

$$\inf_{\mathbf{v}\in\mathrm{span}\{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}:\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{L,1},\,1\le k\le n-1\}} \|\mathbf{u}-\mathbf{v}\|_{L_2(\square)^n} \lesssim L^{\frac{n-1}{2}}2^{-dL}\sqrt{\sum_{\ell=1}^n \|\partial_1^d\ldots\partial_n^d u_\ell\|^2_{L_2(\square)}}$$

and, for $\beta\in(1,\frac{d}{d-1})$ and $q > \beta(d-1)$,

$$\inf_{\mathbf{v}\in\mathrm{span}\{\boldsymbol{\psi}^{(k)}_{\boldsymbol{\lambda}}:\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{L,\beta},\,1\le k\le n-1\}} \|\mathbf{u}-\mathbf{v}\|_{H^1(\square)^n} \lesssim 2^{-(d-1)L}\sqrt{\sum_{m,\ell=1}^n \|\partial_1^q\cdots\partial_m^d\cdots\partial_n^q u_\ell\|^2_{L_2(\square)}},$$

assuming $\mathbf{u}$ is such that the right hand sides are bounded. Similarly for $\mathbf{u} \in \mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$,

$$\inf_{\mathbf{v}\in\mathrm{span}\{\tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}}:\boldsymbol{\lambda}\in\boldsymbol{\nabla}_{L,1}\}} \|\mathbf{u}-\mathbf{v}\|_{L_2(\square)^n} \lesssim L^{\frac{n-1}{2}} 2^{-\tilde{d}L} \sqrt{\sum_{\ell=1}^{n} \|\partial_1^{\tilde{d}} \dots \partial_n^{\tilde{d}} u_\ell\|^2_{L_2(\square)}},$$

assuming $\mathbf{u}$ is such that the right hand side is bounded. So, possibly up to log factors, we obtain rates d, $d-1$ or $\tilde{d}$ using these anisotropic divergence-free or gradient wavelet bases. With corresponding isotropic constructions, the rates would read as $\frac{d}{n}$, $\frac{d-1}{n}$ or $\frac{\tilde{d}}{n}$, respectively. As in Remark 5.2, the required regularity conditions on $\mathbf{u}$ can be largely reduced when nonlinear approximation is applied.

Remark 5.8. The construction of anisotropic divergence-free wavelet bases (on $\mathbb{R}^n$) was first proposed in [DP06], by Deriaz and Perrier. For $n = 2$, our construction is equal to that from [DP06, DP09]. For $n \geq 3$, the constructions are different in the sense that our mappings $\mathbf{A}^{\boldsymbol{\lambda}}$ are well-conditioned, even orthogonal, uniformly in $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$, so that the transformation from the bases from Corollary 5.3 to that of Proposition 5.4(a) is a boundedly invertible, even orthogonal, mapping on $\ell_2(\boldsymbol{\nabla})$. As a consequence, we obtain divergence-free wavelets or "gradient wavelets" that are *Riesz bases*.

Example 5.9. For Ψ and Ψ^+ from Example 4.4 and $n = 2$, in Figure 4 some divergence-wavelets from the collection $\boldsymbol{\Psi}^{\mathrm{df}}$ defined in Corollary 5.5 are illustrated. Note that the normal components of these wavelets vanish at the left boundary (as they will vanish at the bottom boundary), but not at the top and right boundaries.

6. Helmholtz decomposition

Since $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ is a closed subspace of $L_2(\square)^n$, we have

$$L_2(\square)^n = \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square) \oplus \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)^\perp.$$

From $\boldsymbol{\Psi}^{\mathrm{df}}$ and $\tilde{\boldsymbol{\Psi}}^{\mathrm{gr}}$ being Riesz bases for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ and $\mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$, biorthogonality shows that $\mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square) \subset \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)^\perp$. On the other hand, since $\tilde{\boldsymbol{\Psi}}$ is a Riesz basis for $L_2(\square)^n$, any $\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)^\perp \subset L_2(\square)^n$ has an expansion $\mathbf{u} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^n c^{(k)}_{\boldsymbol{\lambda}} \tilde{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}}$. From $\boldsymbol{\Psi}^{\mathrm{df}}$ being a Riesz basis for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ and biorthogonality, we infer that for all $\boldsymbol{\lambda} \in \boldsymbol{\nabla}$ and $1 \leq k \leq n-1$, $c^{(k)}_{\boldsymbol{\lambda}} = 0$, and thus that $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)^\perp \subset \mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$ and so we conclude

Corollary 6.1.

$$L_2(\square)^n = \mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square) \oplus^{\perp_{L_2(\square)^n}} \mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$$

known as a Helmholtz decomposition*.*

In view of *computing* a Helmholtz decomposition of a given $\mathbf{u} \in L_2(\square)^n$, we realize that we do not have a dual basis for $\boldsymbol{\Psi}^{\mathrm{df}} \cup \tilde{\boldsymbol{\Psi}}^{\mathrm{gr}}$ available. Indeed, note that $\{\tilde{\boldsymbol{\psi}}^{(k)}_{\boldsymbol{\lambda}} : 1 \leq k \leq n-1,\, \boldsymbol{\lambda} \in \boldsymbol{\nabla}\} \cup \{\boldsymbol{\psi}^{(n)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}\}$ is *not* such a basis. Since orthonormal bases for $\mathbf{H}_{0,\Gamma}(\mathrm{div}0;\square)$ and $\mathbf{grad}\, H^1_{0,\tilde{\Gamma}}(\square)$ are given by $\langle \boldsymbol{\Psi}^{\mathrm{df}}, \boldsymbol{\Psi}^{\mathrm{df}} \rangle^{-\frac{1}{2}}_{L_2(\square)^n} \boldsymbol{\Psi}^{\mathrm{df}}$

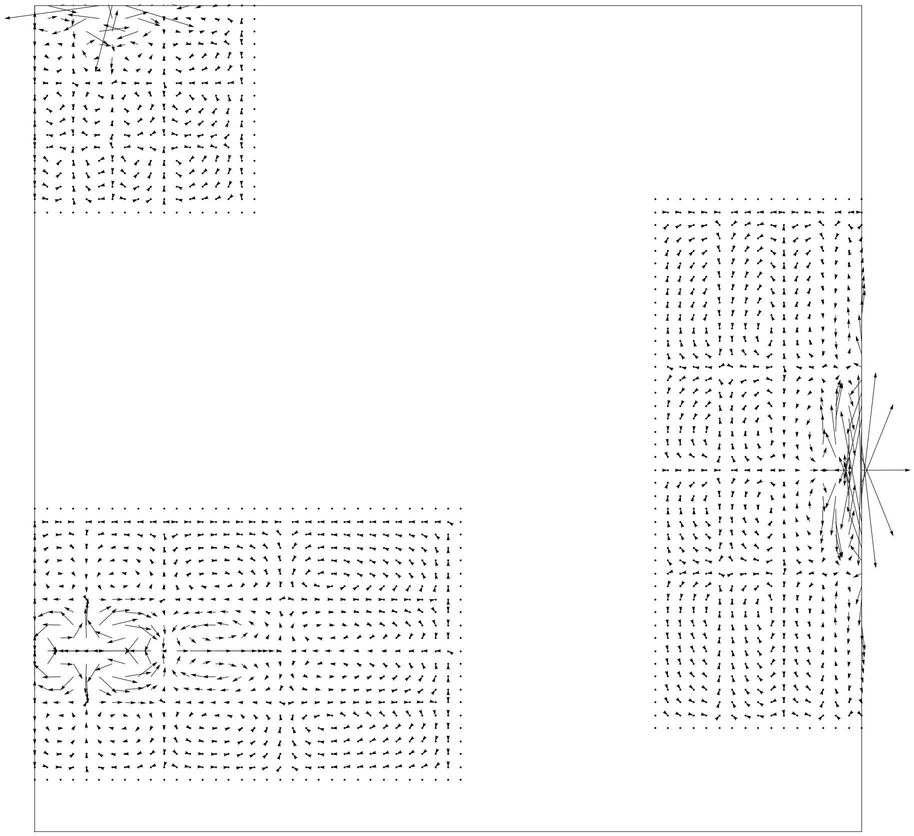

FIGURE 4. Some divergence free wavelets on $\square$ with $n = 2$, constructed using the univariate wavelets Ψ and Ψ^+ from Example 4.4

and $\langle \tilde{\mathbf{\Psi}}^{\mathrm{gr}}, \tilde{\mathbf{\Psi}}^{\mathrm{gr}} \rangle_{L_2(\square)^n}^{-\frac{1}{2}} \tilde{\mathbf{\Psi}}^{\mathrm{gr}}$, the Helmholtz decomposition of $\mathbf{u}$ is given by

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n} \langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n}^{-1} \mathbf{\Psi}^{\mathrm{df}} + \langle \mathbf{u}, \tilde{\mathbf{\Psi}}^{\mathrm{gr}} \rangle_{L_2(\square)^n} \langle \tilde{\mathbf{\Psi}}^{\mathrm{gr}}, \tilde{\mathbf{\Psi}}^{\mathrm{gr}} \rangle_{L_2(\square)^n}^{-1} \tilde{\mathbf{\Psi}}^{\mathrm{gr}}.$$

In order to have the Helmholtz decomposition in terms of primal basis functions only, simply use that the second term is equal to $\mathbf{u}$ minus the first term, i.e., that it is equal to

$$\langle \mathbf{u}, \tilde{\mathbf{\Psi}} \rangle_{L_2(\square)^n} \mathbf{\Psi} - \langle \mathbf{u}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n} \langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n}^{-1} \mathbf{\Psi}^{\mathrm{df}}.$$

An alternative way to arrive at the above formulas is by realizing that in the Helmholtz decomposition $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1$ is the best approximation to $\mathbf{u}$ from $\mathbf{H}_{0,\Gamma}(\mathrm{div}0; \square)$ in the $L_2(\square)^n$-norm , i.e., that $\langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{u} - \mathbf{u}_1 \rangle_{L_2(\square)^n} = 0$. Writing $\mathbf{u}_1 = \vec{\mathbf{u}}_1^\top \mathbf{\Psi}^{\mathrm{df}}$, i.e., $\vec{\mathbf{u}}_1$ is the coefficient vector of $\mathbf{u}_1$ with respect to the basis $\mathbf{\Psi}^{\mathrm{df}}$, one arrives at

$$\langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n} \vec{\mathbf{u}}_1 = \langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{u} \rangle_{L_2(\square)^n}. \tag{6.1}$$

Since $\langle \mathbf{\Psi}^{\mathrm{df}}, \mathbf{\Psi}^{\mathrm{df}} \rangle_{L_2(\square)^n}$ is symmetric positive definite (and boundedly invertible), this system can be iteratively solved with e.g. conjugate gradients. Of course in practical computations, the infinite vectors have to be truncated. This can be done by computing a Galerkin approximation from the span of a predefined finite subset of $\mathbf{\Psi}^{\mathrm{df}}$, or by running an adaptive wavelet scheme on (6.1).

Remark 6.2. An alternative scheme for computing the Helmholtz decomposition in wavelet coordinates (in $\mathbb{R}^n$), i.e., for solving (6.1), was proposed in [DP09]. This scheme was shown to be convergent for some (globally supported) wavelets, but turned out to be divergent for some other wavelet collections.

Finally in this section, we give expressions of div and **curl** operators in terms of wavelet coordinates, as well as corresponding norm equivalences. Apart from applications in solving equations involving **grad**-div or **curl**-**curl** operators (cf. [Urb02, Ch. 3]), these results allow to verify whether functions are div- or **curl**-free by computing wavelet coefficients. Our expressions improve upon those from [Urb02] in the sense that for $\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathrm{div};\square)$, without computing $\mathrm{div}\,\mathbf{u}$, they give an expression for $\mathrm{div}\,\mathbf{u}$ in terms of a *basis* for $L_2(\Omega)$ (instead of in terms of an overcomplete system). The same remark applied to the **curl** operator.

Proposition 6.3. (a). *On* $\mathbf{H}_{0,\Gamma}(\mathrm{div};\square)$*, we have*

$$\mathrm{div}\,\mathbf{u} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \langle \mathbf{u}, \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}\rangle_{L_2(\square)^n}\,\mathrm{div}\,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(n)},$$

and

$$\|\mathrm{div}\,\mathbf{u}\|_{L_2(\square)}^2 \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)|\langle \mathbf{u}, \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}\rangle_{L_2(\square)^n}|^2.$$

(b). *In* two *dimensions, on* $\mathbf{H}_{0,\tilde{\Gamma}}(\mathrm{rot};\square) := \{\mathbf{u} \in \mathbf{H}(\mathrm{rot};\square) : \mathbf{u}\times\mathbf{n} = 0 \text{ on } \tilde{\Gamma}\}$*, we have*

$$\mathrm{rot}\,\mathbf{u} = -\sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(1)}\rangle_{L_2(\square)^2}\mathrm{rot}\,\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(1)}$$

and

$$\|\mathrm{rot}\,\mathbf{u}\|_{L_2(\square)^2}^2 \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} (4^{|\lambda_1|} + 4^{|\lambda_2|})|\langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(1)}\rangle_{L_2(\square)^2}|^2.$$

(c). *In* three *dimensions, on* $\mathbf{H}(\mathbf{curl};\square)$*, we have*

$$\mathbf{curl}\,\mathbf{u} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{2} \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}\rangle_{L_2(\square)^3}\mathbf{curl}\,\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(k)}$$

and

$$\|\mathbf{curl}\,\mathbf{u}\|_{L_2(\square)^3}^2 \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} (4^{|\lambda_1|} + 4^{|\lambda_2|} + 4^{|\lambda_3|}) \sum_{k=1}^{2} |\langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)}\rangle_{L_2(\square)^3}|^2.$$

Proof. (a). Since $\Psi\otimes\cdots\otimes\Psi$, $\tilde{\Psi}\otimes\cdots\otimes\tilde{\Psi}$ are biorthogonal Riesz bases for $L_2(\square)$, Proposition 5.4(b) shows that for $\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathrm{div};\square)$,

$$\begin{aligned}\mathrm{div}\,\mathbf{u} &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \langle \mathrm{div}\,\mathbf{u}, \tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\rangle_{L_2(\square)}\psi_{\lambda_1}\otimes\cdots\otimes\psi_{\lambda_n}\\ &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \langle \mathbf{u}, \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(n)}\rangle_{L_2(\square)^n}\,\mathrm{div}\,\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(n)}\end{aligned}$$

and

$$\|\operatorname{div}\mathbf{u}\|^2_{L_2(\square)} \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} |\langle \operatorname{div}\mathbf{u}, \tilde{\psi}_{\lambda_1}\otimes\cdots\otimes\tilde{\psi}_{\lambda_n}\rangle_{L_2(\square)}|^2$$
$$= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \Big(\sum_{m=1}^{n} 4^{|\lambda_m|}\Big)|\langle \mathbf{u}, \tilde{\boldsymbol{\psi}}^{(n)}_{\boldsymbol{\lambda}}\rangle_{L_2(\square)^n}|^2.$$

(b). Since $\Psi^+\otimes\Psi^+$ and $\tilde{\Psi}^-\otimes\tilde{\Psi}^-$ are biorthogonal Riesz bases for $L_2(\square)$, we have $\operatorname{rot}\mathbf{u} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}}\langle \operatorname{rot}\mathbf{u}, \psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2}\rangle_{L_2(\square)}\tilde{\psi}^-_{\lambda_1}\otimes\tilde{\psi}^-_{\lambda_2}$ as well as the norm equivalence $\|\operatorname{rot}\mathbf{u}\|^2_{L_2(\square)} \eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} |\langle \operatorname{rot}\mathbf{u}, \psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2}\rangle_{L_2(\square)}|^2$.

Using that $(\mathbf{u}\times\mathbf{n})(\psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2})$ vanishes at $\partial\square$, integration by parts shows that $\langle \operatorname{rot}\mathbf{u}, \psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2}\rangle_{L_2(\square)} = -\langle \mathbf{u}, \begin{bmatrix}-\partial_2\\ \partial_1\end{bmatrix}\psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2}\rangle_{L_2(\square)^2}$. Straightforward calculations show that for any $\mathbf{A}^{\boldsymbol{\lambda}}\in\mathbb{R}^{2\times 2}$ that satisfies (5.2), $\begin{bmatrix}-\partial_2\\ \partial_1\end{bmatrix}\psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2} = \pm\sqrt{4^{|\lambda_1|}+4^{|\lambda_1|}}\boldsymbol{\psi}^{(1)}_{\boldsymbol{\lambda}}$ and $\pm\sqrt{4^{|\lambda_1|}+4^{|\lambda_1|}}\,\tilde{\psi}^-_{\lambda_1}\otimes\tilde{\psi}^-_{\lambda_2} = \operatorname{rot}\tilde{\boldsymbol{\psi}}^{(1)}_{\boldsymbol{\lambda}}$, with which the proof is easily completed.

(c). We define the biorthogonal Riesz bases $\underline{\boldsymbol{\Sigma}} = \{\underline{\boldsymbol{\sigma}}^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla},\, k\in\{1,2,3\}\}$, $\underline{\tilde{\boldsymbol{\Sigma}}} = \{\underline{\tilde{\boldsymbol{\sigma}}}^{(k)}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda}\in\boldsymbol{\nabla},\, k\in\{1,2,3\}\}$ for $L_2(\square)^3$ by

$$\underline{\boldsymbol{\sigma}}^{(k)}_{\boldsymbol{\lambda}} := \begin{cases}\psi_{\lambda_1}\otimes\psi^+_{\lambda_2}\otimes\psi^+_{\lambda_3}\mathbf{e}_1 & k=1,\\ \psi^+_{\lambda_1}\otimes\psi_{\lambda_2}\otimes\psi^+_{\lambda_3}\mathbf{e}_2 & k=2,\\ \psi^+_{\lambda_1}\otimes\psi^+_{\lambda_2}\otimes\psi_{\lambda_3}\mathbf{e}_3 & k=3,\end{cases} \quad \underline{\tilde{\boldsymbol{\sigma}}}^{(k)}_{\boldsymbol{\lambda}} := \begin{cases}\tilde{\psi}_{\lambda_1}\otimes\tilde{\psi}^-_{\lambda_2}\otimes\tilde{\psi}^-_{\lambda_3}\mathbf{e}_1 & k=1,\\ \tilde{\psi}^-_{\lambda_1}\otimes\tilde{\psi}_{\lambda_2}\otimes\tilde{\psi}^-_{\lambda_3}\mathbf{e}_2 & k=2,\\ \tilde{\psi}^-_{\lambda_1}\otimes\tilde{\psi}^-_{\lambda_2}\otimes\tilde{\psi}_{\lambda_3}\mathbf{e}_3 & k=3,\end{cases}$$

Then with $\mathbf{Z}^{\boldsymbol{\lambda}} := \begin{bmatrix}0 & -2^{|\lambda_3|} & 2^{|\lambda_2|}\\ 2^{|\lambda_3|} & 0 & -2^{|\lambda_1|}\\ -2^{|\lambda_2|} & 2^{|\lambda_1|} & 0\end{bmatrix} = -(\mathbf{Z}^{\boldsymbol{\lambda}})^\top$, an easy calculation shows that

$$\mathbf{curl}\begin{bmatrix}\underline{\tilde{\boldsymbol{\psi}}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\psi}}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\psi}}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} = \mathbf{Z}^{\boldsymbol{\lambda}}\begin{bmatrix}\underline{\tilde{\boldsymbol{\sigma}}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\sigma}}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\sigma}}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}, \quad \mathbf{curl}\begin{bmatrix}\underline{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\sigma}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\sigma}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} = -\mathbf{Z}^{\boldsymbol{\lambda}}\begin{bmatrix}\underline{\boldsymbol{\psi}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\psi}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\psi}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}.$$

For $\boldsymbol{\lambda}\in\boldsymbol{\nabla}$, we set

$$\begin{bmatrix}\boldsymbol{\sigma}^{(1)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\sigma}^{(2)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\sigma}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} := \mathbf{A}^{\boldsymbol{\lambda}}\begin{bmatrix}\underline{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\sigma}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\boldsymbol{\sigma}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}, \qquad \begin{bmatrix}\tilde{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\sigma}}^{(2)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\sigma}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} := \mathbf{A}^{\boldsymbol{\lambda}}\begin{bmatrix}\underline{\tilde{\boldsymbol{\sigma}}}^{(1)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\sigma}}}^{(2)}_{\boldsymbol{\lambda}}\\ \underline{\tilde{\boldsymbol{\sigma}}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}.$$

Then

$$\mathbf{curl}\begin{bmatrix}\tilde{\boldsymbol{\psi}}^{(1)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\psi}}^{(2)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\psi}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} = \mathbf{A}^{\boldsymbol{\lambda}}\mathbf{Z}^{\boldsymbol{\lambda}}(\mathbf{A}^{\boldsymbol{\lambda}})^\top\begin{bmatrix}\tilde{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\sigma}}^{(2)}_{\boldsymbol{\lambda}}\\ \tilde{\boldsymbol{\sigma}}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}, \quad \mathbf{curl}\begin{bmatrix}\boldsymbol{\sigma}^{(1)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\sigma}^{(2)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\sigma}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix} = -\mathbf{A}^{\boldsymbol{\lambda}}\mathbf{Z}^{\boldsymbol{\lambda}}(\mathbf{A}^{\boldsymbol{\lambda}})^\top\begin{bmatrix}\boldsymbol{\psi}^{(1)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\psi}^{(2)}_{\boldsymbol{\lambda}}\\ \boldsymbol{\psi}^{(3)}_{\boldsymbol{\lambda}}\end{bmatrix}.$$

Any orthogonal $\mathbf{A}^{\boldsymbol{\lambda}}\in\mathbb{R}^{3\times 3}$ that satisfies (5.2) is of the form $\mathbf{A}^{\boldsymbol{\lambda}} = \begin{bmatrix}\multicolumn{2}{c}{\mathbf{Q}} & \begin{matrix}0\\0\end{matrix}\\ 0\ 0 & 1\end{bmatrix}\mathbf{A}^{\boldsymbol{\lambda}}_p$, where $\mathbf{A}^{\boldsymbol{\lambda}}_p\in\mathbb{R}^{3\times 3}$ is some orthogonal matrix that satisfies (5.2), and $\mathbf{Q}\in\mathbb{R}^{2\times 2}$ is orthogonal. Taking $\mathbf{A}_p^{\boldsymbol{\lambda}}$ to be the Householder transformation from (5.3) for $n = 3$, a direct calculation shows that

$$-\mathbf{A}^{\boldsymbol{\lambda}}\mathbf{Z}^{\boldsymbol{\lambda}}(\mathbf{A}^{\boldsymbol{\lambda}})^\top = \sqrt{4^{|\lambda_1|} + 4^{|\lambda_2|} + 4^{|\lambda_3|}} \begin{bmatrix} \hat{\mathbf{Q}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix},$$

where $\hat{\mathbf{Q}} \in \mathbb{R}^{2\times 2}$ is some orthogonal matrix, and thus

$$\begin{bmatrix} \langle \mathbf{u}, \mathbf{curl}\, \boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(1)} \rangle_{L_2(\square)^3} \\ \langle \mathbf{u}, \mathbf{curl}\, \boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(2)} \rangle_{L_2(\square)^3} \\ \langle \mathbf{u}, \mathbf{curl}\, \boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(3)} \rangle_{L_2(\square)^3} \end{bmatrix} = \sqrt{4^{|\lambda_1|} + 4^{|\lambda_2|} + 4^{|\lambda_3|}} \begin{bmatrix} \hat{\mathbf{Q}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} \begin{bmatrix} \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(1)} \rangle_{L_2(\square)^3} \\ \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(2)} \rangle_{L_2(\square)^3} \\ \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(3)} \rangle_{L_2(\square)^3} \end{bmatrix}.$$

From these calculations and by integration by parts we conclude that for $\mathbf{u} \in \mathbf{H}(\mathbf{curl}; \square)$,

$$\begin{aligned} \mathbf{curl}\,\mathbf{u} &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} \langle \mathbf{curl}\,\mathbf{u}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3} \tilde{\underline{\boldsymbol{\sigma}}}_{\boldsymbol{\lambda}}^{(k)} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} \langle \mathbf{curl}\,\mathbf{u}, \boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3} \tilde{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}^{(k)} \\ &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} \langle \mathbf{u}, \mathbf{curl}\,\boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3} \tilde{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}^{(k)} = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{2} \langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3} \mathbf{curl}\, \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(k)}, \end{aligned}$$

as well as

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{u}\|_{L_2(\square)^3}^2 &\eqsim \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} |\langle \mathbf{curl}\,\mathbf{u}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3}|^2 \\ &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} |\langle \mathbf{curl}\,\mathbf{u}, \boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3}|^2 = \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} \sum_{k=1}^{3} |\langle \mathbf{u}, \mathbf{curl}\,\boldsymbol{\sigma}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3}|^2 \\ &= \sum_{\boldsymbol{\lambda}\in\boldsymbol{\nabla}} (4^{|\lambda_1|} + 4^{|\lambda_2|} + 4^{|\lambda_3|}) \sum_{k=1}^{2} |\langle \mathbf{u}, \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(k)} \rangle_{L_2(\square)^3}|^2. \qquad \square \end{aligned}$$

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