$A_\infty$-algebras, $A_\infty$-categories and the twisted completion

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1 $A_\infty$-algebras and $A_\infty$-categories

In this chapter we lay the groundwork for the Fukaya-category $\text{Fuk}(X)$, the A-part of mirror symmetry. We motivate and expand upon the notion of $A_\infty$-algebras and $A_\infty$-categories. Here $A$ stands for ‘associative’ and $\infty$ for the relaxation thereof up to higher homotopies, without bound on the degree of the homotopies. In the next chapter, we will see that $\text{Fuk}(X)$ is a Calabi-Yau $A_\infty$-category.

1.1 Motivation from Koszul duality

1.1.1 Algebra structure on $\text{Ext}^*_A$

We start off by (re)introducing some notions and definitions:

- A projective module is a module $P$ that has the following lifting property (with $f, g$ homomorphisms):

$$
\begin{array}{c}
P \\ \xrightarrow{f} \\
\downarrow \\
M
\end{array}
\xrightarrow{ \exists! \  g} 
\begin{array}{c}
N \\
\xrightarrow{f} \\
\downarrow \\
M
\end{array}
$$

- A projective resolution $P^\bullet$ is an exact sequence of projective modules $P$;
- The space $\text{Ext}^*_A$ is defined as $H(\text{Hom}(P^\bullet, N))$, with $P^\bullet$ a projective resolution of $M$.

Now let $M$ be an $A$-module with projective resolution $P^\bullet$ and let $f \in Z(\text{Hom}(P^k, M))$ (that is, $df = f \circ \pi_{k+1} = 0$):

$$
P_k \xleftarrow{\pi_{k+1}} P_{k+1} \xleftarrow{\pi_{k+2}} P_{k+2} \cdots \xleftarrow{f_{k+2}} P_{k+2} \cdots
$$

$$
M \xrightarrow{f} P_0 \xleftarrow{\pi_0} P_1 \xleftarrow{\pi_1} P_2 \cdots
$$
The projectivity of $P^*$ gives the lift $f_0$ of $f$. Then, the surjectivity of $\pi_1$ on the image of $f_0 \circ \pi_{k+1}$ ($\text{im} \pi_1 = \ker f_0 \supset \text{im} f_0 \circ \pi_{k+1}$) gives us a second lifting $f_1$ of $f_0 \circ \pi_{k+1}$. Continuing this process, we obtain from $f$ a chain map $f_k : P_{k+i} \to P_i$.

On $\text{Hom}(P^*,P^*)$, the obvious multiplication is given by the composition of maps; it has a grading $\text{Hom}(P^*,P^*)^n = \bigoplus_k \text{Hom}(P^{k+n},P^k)$ and a differential $\delta$ given by $\delta f = [d,f]$. This differential is a derivation, turning $\text{Hom}(P^*,P^*)$ into a differentially graded algebra.

Note that $\ker \delta$ are precisely the chain maps, the lifting of $f \in Z(\text{Hom}(P^k,M))$ is unique up to chain equivalence, and $f = dg$ on $Z(\text{Hom}(P^k,M))$ implies $f = \delta h$ (where $h$ is the lifting of $g$, with all odd-numbered maps set to zero). Hence, we obtain an isomorphism $H(\text{Hom}(P^k,M)) \simeq H(\text{Hom}(P^*,P^*))$, that is, a DGA structure on $\text{Ext}_A^*$.

1.1.2 Example of Köszul duality: $\mathbb{C}$ as $\mathbb{C}[x,y]$-module

Let $A = \mathbb{C}[x,y]$. A projective resolution of the module $\mathbb{C} \simeq A/(x,y)$ is given by

$$
\begin{array}{c}
\mathbb{C} & \xleftarrow{\pi} & A & \xleftarrow{(x,y)} & A \oplus A & \xleftarrow{(-y,x)} & A & \xleftarrow{0} & 0
\end{array}
$$

Consider a module map $f \in Z(\text{Hom}(A \oplus A, \mathbb{C}))$. As it must vanish on $A(-y,x)$ and $x, y$ act as the zero map on $\mathbb{C}$, $f$ must vanish away from the identity in either copy of $A$: let us write $\pi$ for the projection onto $\mathbb{C}$, and define $a\xi + b\eta = (a\cdot \pi, b \cdot \pi) : A \oplus A \mapsto \mathbb{C}$, for $a, b \in \mathbb{C}$.

We see that $(a\xi + b\eta) \circ (c\xi + d\eta) = (bc - ad)\zeta$, where $\zeta = \pi$, but with the highest $A$ in the projective resolution as domain. Hence, $\text{Ext}_A^*(\mathbb{C}, \mathbb{C}) \simeq \lambda(\xi, \eta)$, the exterior algebra in two variables.

In fact, $\text{Ext}_A^*(\mathbb{C}, \mathbb{C})$, as an algebra, has a representation on $\mathbb{C} = \text{Ext}_A^*(\mathbb{C}, \mathbb{C})/(\xi, \eta)$ which is zero except in the degree 0 part (i.e. $a\pi \cdot \lambda = a\lambda$, $a, \lambda \in \mathbb{C}$, $\xi\lambda = \eta\lambda = 0$), and $\text{Ext}^*_{\text{Ext}_A^*(\mathbb{C}, \mathbb{C})}(\mathbb{C}, \mathbb{C}) \simeq A$. This phenomenon is called Köszul duality, and $A$ is called a Köszul algebra.

1.1.3 Example failing Köszul duality: $\mathbb{C}$ as $\mathbb{C}[x]/\langle x^n \rangle$-module

Let $A = \mathbb{C}[x]/\langle x^n \rangle$. A projective resolution of the module $\mathbb{C} = A/(x)$ is given by

$$
\begin{array}{c}
\mathbb{C} & \xleftarrow{\pi} & A & \xleftarrow{x} & A & \xleftarrow{x^{-1}} & A & \xleftarrow{x^{-2}} & A \cdots
\end{array}
$$
This resolution never stops, corresponding to the fact that $A$ is ‘not smooth’. Write $t_k$ for the trivial module map $\pi : A \to \mathbb{C}$, applied in degree $k$, and write $t = t_1$.

From the above diagram, we learn that $t \circ t = x^{n-2}$ acting on $A$ in degree 2. For $n = 2$ we get $t \circ t = t_2$, $\text{Ext}_A(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}[t]$. For $n > 2$, $\pi \circ x^{n-2} = 0$ so $t \circ t = 0$, $\text{Ext}_A(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}[t_1, \ldots]/(t_1 \circ t_1 = 0)$; note that this does not depend on $n$ anymore. Thus, we conclude that the algebra structure of $\text{Ext}$ only remembers the quadratic part of the relations on $A$.

### 1.2 Generalisation of associativity

In this section we discuss a few different ways to see associativity, and motivate a natural generalisation.

Let $A$ be a graded vector space over a field $k$ equipped with a set of maps $\mu_n : A^\otimes n \to A$, $n \geq 0$. We denote multinary (or $n$-ary) operators $\mu_k : A^\otimes n \to A$, i.e. it acts on $n$ arguments. Note that the degree of such a map is

$$\deg \mu_n = 1 + \dim A - \dim A^\otimes n = 2 - n.$$ 

Let us consider a few perspectives on associativity. First, we represent the binary operator $\mu_2$ by ‘$ullet \odot \bullet$’. Then we write for the associativity condition as

$$(x_1 \odot x_2) \odot x_3 = x_1 \odot (x_2 \odot x_3).$$

As a function, the same rule is written

$$\mu_2(\mu_2(x_1, x_2), x_3) - \mu_2(x_1, \mu_2(x_2, x_3)) = 0.$$ 

And as an operator we have

$$\mu_2 \circ (\mu_2 \otimes 1) - \mu_2 \circ (1 \otimes \mu_2) = 0.$$ 

Another way to write this rule is by means of a diagram. We write arguments $x_1, \ldots, x_n$ as a series of arcs. Each $\mu_k$ contracts $k$ such arcs. The associativity rule is then written like

$$\mu_2 \circ (\mu_2 \otimes 1) - \mu_2 \circ (1 \otimes \mu_2) = 0.$$ 

Of course, we mathematicians like to be minimalistic, so we usually do not write any of the labels. As an example, $\mu_4(x_1, \mu_2(x_2, x_3), x_4, x_5)$ is simply written as
Let us see what is happening here, so that we can cook up similar rules for higher operators $\mu_{k \geq 3}$. Note that the recipe for generalised associativity should contain the following ingredients:

(i) Applying pairs of operators $\mu_i, \mu_j$, with $i + j = k - 1$ to the arguments $x_1, \ldots, x_k$;
(ii) Doing so without changing the order of the arguments;
(iii) Alternate-summing over all configurations via some rule;
(iv) Having the result equal 0.

With $R_k$ the $k$-th order associativity rule, it should look like

$$R_k := \sum_{1 \leq i, j \leq k-1} \sum_{i+j=k-1} \pm \mu_i \left( x_1, \ldots, x_{a-1}, \mu_j(x_a, \ldots, x_{a+j}), x_{a+j+1}, \ldots, x_k \right) = 0,$$

where $\pm$ is the signature of each term. Now apply the power of the diagrams: write the first few $R_k$ as defined in (2):

$$R_1 = \quad \text{(3)}$$

Note that this is nothing but the familiar rule $d^2 = 0$ for derivations.

$$R_2 = \quad \text{(4)}$$

Here we have the Leibnitz rule, writing $\mu_1 \equiv d$ and $\mu_2$ as $\bullet \circ \bullet$ we regain the familiar form of the distributive law of derivations $d(x_1 \circ x_2) = (dx_1) \circ x_2 + x_1 \circ (dx_2)$.

$$R_3 = \quad \text{(5)}$$

When setting $dx_i = 0$ for all $i$, that is, when acting on closed forms and $\mu_3(x_1, x_2, x_3) = 0$, we retrieve the classical associative law.

As a remark: we have tacitly been assuming $\mu_0 : k \to A$ to be zero. In some cases, as we will see later in the course, we do need $\mu_0$. We then must also consider diagrams in which new arcs are added, rather than contracted, e.g.,

1.3 $A_\infty$ algebras

The above motivates the formal definition, where we follow [1]:

1. Since we are generalising associativity rules we do not count terms like $\mu_2(\mu_2(x_1, x_3), x_2)$.
Definition 1. Let \( k \) be a field. An \( A_\infty \)-algebra (also known as strongly homotopy associative algebra, or sha algebra), is a \( \mathbb{Z} \)-graded vector space

\[
A = \bigoplus_{p \in \mathbb{Z}} A^p
\]

endowed with graded maps \( \mu_n : A^{\otimes n} \to A \) (\( n \geq 1 \)) of degree \( 2 - n \) satisfying

\[
\sum_{r+s+t = n} (-1)^{r+s+t} \mu_{r+1+t}(1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t})
\] (6)

For the purposes of this course, we will not be too precise about the signature, usually simply writing \( \pm \), rather than the full exponent.\(^2\) For our purposes, therefore, we can replace rule 6, by the pictorial

\[
\sum_{\text{arcs}} \pm x_1 \ldots x_n = 0,
\] (7)

where the summation runs over all possible arcs for \( \mu_s \). Note that equations (3,4,5) are in accordance with this definition.

From this, the following is immediate:

Corollary 1. For \( A_\infty \) algebra \( A \), we have special (familiar) cases:

- If \( \mu_{n \geq 2} = 0 \), \( A \) is a complex.
- If \( \mu_2 \neq 0 \), and \( \mu_{n \neq 2} = 0 \), \( A \) is an ordinary associative algebra.
- If \( \mu_{n \geq 3} = 0 \), \( A \) is a differential \( \mathbb{Z} \)-graded (dg) algebra.

We are interested in relations between \( A_\infty \) algebras. This motivates the definition of morphisms between different such algebras (still following \[1\]):

Definition 2. An \( A_\infty \)-morphism \( F : A \to B \) is a family

\[
F_n : A^{\otimes n} \to B
\]

of graded degree \( 1 - n \) maps such that

\[
\sum_{r+s+t = n} (-1)^{r+s+t} F_{r+1+t}(1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = \sum_{1 \leq i_1 \leq n; i_1 + \ldots + i_s = n} (-1)^{\sigma} \mu_{i_1}(F_{i_1} \otimes \ldots \otimes F_{i_s}),
\] (8)

\(^2\) Note that, in fact, additional signs appear, when applied to elements, due to the Koszul sign rule

\( (f \otimes g)(x \otimes y) = (-1)^{\deg f \deg x} f(x) \otimes g(y) \),

for graded maps \( f, g \), and homogeneous elements \( x, y \).
where the signature $\sigma = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \ldots + (i_{r-1} - 1)$ depends on the permutation of the $i_j$-indices. In pictures:

$$\sum_{\text{arcs}} \pm x_1^{x_2} \cdots x_{n-1}^{x_n} F_{n-s+1}^{\mu_s} = \sum_{r=1}^{n} \sum_{\text{arcs}} \pm x_1^{x_2} \cdots F_{i_2}^{F_{i_1} \cdots F_{i_{u}}^{\mu_r}}, \quad (9)$$

where we sum over all possible arcs for $\mu_s$ on the left, and all possible arcs for $F_i$, through $F_{i_u}$, for every $r$ on the right.

The morphism $F$ is a (quasi-)isomorphism if $F_1$ is a quasi-isomorphism, $F$ is strict if $F_1 \neq 1 = 0$, and the strict morphism, such that $F_1 = 1_A$ is the identity morphism. Loosely speaking, we can see $F$ as a map where $F_1$ is the core part, and $F_i \geq 2$ are (small) high order corrections, much like a Taylor series near its expansion point.

The composition of morphisms $G : A \rightarrow B, F : B \rightarrow C$ takes the form

$$(F \circ G)_n = \sum (-1)^{\sigma} F_{r} \circ (G_{i_1} \otimes \ldots \otimes G_{i_{u}}),$$

where the summation and $\sigma$ are just as in the right hand side of (8).

In this course, we are mainly interested in complexes, up to quasi-isomorphism. We can pull a quasi-isomorphism through the algebra, inducing a quasi-isomorphism between complexes:

$$A \xrightarrow{\cong} B; \quad \mu_1 \xrightarrow{\cong} \bar{\mu}_1$$

The higher order operations are defined inductively; having defined $\bar{\mu}_n$, we define $\bar{\mu}_{n+1}$ in accordance with the generalised associativity rules (6).

## 2 Minimal Model

Let $(A, \mu_1)$ be a dg-algebra. Our goal is to construct an $A_\infty$ structure on the homology of $A$ that is quasi-isomorphic to $A$ itself. We follow a special case of Kadeishvili’s construction. In this section, we write $A^i$ for a module over $A$.

We start by noting that the degree-one map $\mu_1 : A \rightarrow A$ acts a differential on the $A$-modules $A^i$. For familiarity, in this section we will write $d \equiv \mu_1$. For differentials we know that $d^2 = 0$, meaning $B^i \equiv \text{im } d^{i-1} \subseteq \ker d^i \equiv Z^i$. Therefore, there are subspaces $H^i$ such that $Z^i = B^i \oplus H^i$, and $L^i$ such that

$$A^i = Z^i \oplus L^i = B^i \oplus H^i \oplus L^i,$$

where we have

- $B^i$ the coboundaries in $A^i$;
• $H^i$ a complement of the coboundaries $B^i$ within the cocycles $Z^i$;
• $L^i$ a complement of the cocycles $B^i$ in $A^i$.

Note that the $H^i$ and $L^i$ are non-unique! The cochain complex is schematically depicted below.

For clarity, the $0 \in B^i$ has been added to the diagram. In this diagram, mappings by $d^*$ cannot cross the drawn arrows.

We identify the homology $H(A) := \bigoplus H^n$, that is: the hatched part of the above diagram. Let $\iota : H(A) \hookrightarrow A$ be the inclusion map, and $\pi \equiv \pi_H : A \to H(A)$ be the projection map. Consider a homotopy $h : A \to A$ from $\mathbb{1}_A$ to $\pi$, and such that it satisfies, for every $n$

- $h^n|_{B^n} = 0$;
- $h^n|_{H^n} = \left(d^n|_{H^n}\right)^{-1}$;
- $h^n|_{L^n} = 0$.

Since $h$ is a homotopy, it follows that $\pi = \mathbb{1}_A - (hd + dh)$; and from the restriction conditions above, we have that $h^{n+1}d^n = \pi_{L^n}$, and $d^{n-1}h^n = \pi_{B^n}$. It follows that $hdh = h$ (much like the Hodge star codifferential definition $\delta = \pm \star \ast^{-1} d\ast$).

The above definition of $h$ can be summarized by the diagram obtained by simply reversing the arrows and writing 0 on top.

Using $\pi$ and $\iota$, we can go back and forth between $A$ and $H(A)$. This will allow us to transfer the operators $\mu$ on $A$ to $H(A)$.

Starting in $H(A)^{\otimes k}$, we do so by taking the sum of all possible ways to repeatedly apply $\mu$ operators from the original algebra $A$, such that we end up in $H(A)$. This
Figure 1: An example tree such as to be summed over (here $k = 7$). The formal expression of this specific chain of operations is $\pi(\mu_2(h\mu_3(h\mu_3(\iota(a_1),\ldots),\ldots),\iota a_7)$.

gives rise to operator $\mu_{H(A)} : H(A)^{\otimes k} \to H(A)$:

$$\mu_{H(A)}(a_1, \ldots, a_k) = \sum_{\text{trees}} \pm a_1 \ldots a_k. \quad (10)$$

By ‘trees’ we mean the application of $\mu$’s in $A$ mentioned above. Incoming external lines are identified with $i$ so as to be able to work in $A$. Nodes are identified with $\mu_i$, where $i$ is the number of incoming lines. The (sole) outgoing external line is identified with $\pi$, so as to end back up in $H(A)$. Note that the degree of $\mu_i$ is 1, so that internal lines should be identified with $h$. Additionally, the property $h dh = h$ makes trivial nodes with $\mu_1$ obsolete, saving us from infinite trees. An example tree is given in (fig.1).

These trees are a typical example of a nice expression that you would never want to explicitly calculate. Luckily, since we work in homology, we are saved by quasi-isomorphism arguments. In fact, a different choice of $h$ generally leads to a different, but quasi-isomorphic, expression for $M_{H(A)}$. The general information is therefore contained in the structure, rather than in the explicit expression of the trees.

3 Alternative construction: Bar formalism

$A_\infty$ structures can also be defined via the Bar coalgebra. However, we will need a few definitions before we can do that, please be patient:

The reduced tensor product $\bar{T}V$ over some vector space $V$ is the direct sum $\bar{T}V := \bigoplus_{i=1}^{\infty} V^{\otimes i}$ that starts at 1 instead of 0. You can view this as a coalgebra: a coalgebra has a splitting operation $\Delta : A \to A \otimes A$ that is similar to the product of an algebra:

- algebra: $\mu : A \otimes A \to A$, $\mu(\mu \otimes 1) - \mu(1 \otimes \mu) = 0$
- coalgebra: $\Delta : A \to A \otimes A$, $(1 \otimes \Delta)\Delta - (\Delta \otimes 1)\Delta = 0$
In our case, $\tilde{T}V$ is a coalgebra by the following:

$$
\Delta(v_1 \otimes \cdots \otimes v_k) = \sum_{k=1}^{i-1} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_i)
$$

and $\Delta(v_i) = 0$

On co-algebras we can co-have co-derivations: a coderivation is a map (perhaps regrettably also denoted $d$) $\tilde{T}V \rightarrow \tilde{T}V$ for which $\Delta d = (d \otimes 1)\Delta + (1 \otimes d)\Delta$.

Now let $SA$ be the suspension of $A$. That is: $(SA)^k := A^{k+1}$, or $A$ shifted by one degree. Then, finally, an $A_\infty$-structure on $A$ can be seen as a coderivation $d$ on $\tilde{T}(SA)$ such that $d^2 = 0$. ($\tilde{T}(SA)$ is also called the Bar coalgebra $B(A)$.)

How would this map $d : \tilde{T}(SA) \rightarrow \tilde{T}(SA)$ look? Consider the restriction to a single degree $d_i : (SA)^\otimes i \rightarrow \tilde{T}(SA)$; we can split this into a sum $\oplus d_i^j$, where $d_i^j$ takes out $j$ degrees. It can be shown that the definition of a coderivation determines all $d_i^j$ uniquely by $d_i^1$. In fact, from $d^2 = 0$ it follows that we can identify $d_i^1$ with $\mu_i$ by $\mu_i = S^{\otimes 1} \circ d_i^1 \circ S^{-1}$ (with $S$ is the suspension isomorphism), thus giving the $A_\infty$ structure we recognise.

For an ordinary algebra, the Bar resolution is the resolution of the algebra as a bimodule over itself:

$$
\tilde{T}(SA) : A \leftarrow A \otimes A \xleftarrow{\mu_2} A \otimes A \xleftarrow{\mu_2 \otimes 1 - 1 \otimes \mu_2} A \otimes A \leftarrow \cdots
$$

Clearly, the algebra structure coincides with the differential on the Bar resolution of $A$.

4 $A_\infty$-categories

Any $k$-algebra can be seen as $k$-linear category $C$ with one object, $\bullet$. In that view, the actual algebra corresponds to the morphisms $\text{Hom}_{\mathcal{C}}(\bullet, \bullet)$, where the composition of two morphisms corresponds to the algebra product.

In much the same way, an $A_\infty$ algebra can be seen as $A_\infty$ category with one object. Furthermore, it is possible to define an $A_\infty$ functor as a generalisation of $A_\infty$ morphisms. A more detailed discussion of $A_\infty$ categories can be found on http://ncatlab.org/nlab/show/A-infinity-category, but is outside the scope of this chapter.

4.1 Representations of $A_\infty$-categories and twisted completion

When we view an algebra as a category $C$ with one object, we can view the morphisms of the category as the elements of its representation. The representation itself is from that point of view a functor $C \rightarrow \text{Vect}(\mathbb{C})$. This leads to the following category:
Definition 3. $\text{A-mod}$ is the category that has representations (or modules) of $A$ as objects and morphisms of modules as morphisms.

It is possible that different algebras have equivalent module categories; this is the subject of morita theory. For example, $\mathbb{C}$ and $\text{Mat}_n(\mathbb{C})$ are morita equivalent, since in both cases $A$-$\text{mod}$ is the category of complex vector spaces.

Switching to $A_\infty$ algebras, a representation of $A$ becomes an $A_\infty$ functor $A \to \text{dgVect}$, where $\text{dgVect}$ is the category of differentially graded vector spaces. Here, too, different algebras can have equivalent representations. In particular, the twisted completion $\text{Tw}A$ of $A$ has the same representation.

The twisted completion of an $A_\infty$ category $A$ has as objects the pairs $(M, \delta)$, where $M$ is the formal direct sum $\bigoplus e_i[j_i]$ with $e_i$ objects in $A$ and $j_i$ a shift; $\delta$ is an (upper triangular) matrix of elements in $\text{Hom}_A(e_i, e_j)$ shifted by $j_j - j_i$. ($\delta$ must be upper triangular because $\mu_1(\delta) + \mu_2(\delta, \delta) + \cdots = 0$.)

References