Chapter 5

The B-model: complex geometry

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In this chapter we will introduce the language describing the so-called B-model of mirror symmetry. Recall that there is a conjecture due to Kontsevich, relating the derived category of coherent sheaves on a Calabi-Yau manifold (B-model) to the Fukaya category of its mirror (A-model). Hence, in the following we start by defining the basics of the B-model, an introduction to complex geometry, by considering varieties and coherent sheaves, and applying these basics to the examples of $\mathbb{A}^1$ and $\mathbb{P}^1$. Along the way, we note that $\text{Coh}_Y \cong \text{fgMod}_{\mathbb{A}(Y)}$ for an affine variety (such as $\mathbb{A}^1$), but that this does not hold for projective varieties (such as $\mathbb{P}^1$) or more general algebraic varieties. This will lead us to the definition of derived categories and eventually to the category of singularities.

5.1 Introduction to complex geometry

We will give a short discussion of the concepts of complex geometry that will be necessary to treat Mirror Symmetry. For the basic concepts, such as that of affine and algebraic varieties and sheaves, see [Algebraic Geometry by prof. Ben Moonen]. For more on coherent sheaves, see the eponymous article [Coherent algebraic sheaves by Serre].

5.1.1 Pre-sheaves, sheaves and coherent sheaves

So, what are coherent sheaves? To properly define these, we will first have to consider pre-sheaves. Let $S$ be a fixed topological space and denote by $\text{Open}_S$ the category of opens, with as objects the open subsets of $S$ (i.e. the topology of $S$) and as morphisms the embedding (if it exists). Finally, denote by $\text{Ab}$ the category of abelian groups with group homomorphisms.

**Definition 5.1.1.** A pre-sheaf $\mathcal{F}$ of abelian groups on $X$ is a contravariant functor $\mathcal{F} : \text{Open}_S \rightarrow \text{Ab}$. So if $i : U \hookrightarrow V$ is an embedding, then there is a group homomorphism $\mathcal{F}(i) : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, often called a restriction map and denoted by $\mathcal{F}(i)(\mathcal{F}(V)) = \mathcal{F}|_U(V)$.

Since the intersection of two open sets $U \cap V$ is open and we have canonical embeddings $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$, we obtain two subgroups $\mathcal{F}(i_U)(\mathcal{F}(U \cap V)) \subset \mathcal{F}(U)$ and $\mathcal{F}(i_V)(\mathcal{F}(U \cap V)) \subset \mathcal{F}(V)$. In a pre-sheaf, these need not be isomorphic.

**Definition 5.1.2.** A sheaf $\mathcal{F}$ is a pre-sheaf such that for each two open sets, the embedded subgroups of the intersection are isomorphic.

We say that the “abelian groups agree on the intersection”. This allows us to “glue” the groups together.

**Remark 5.1.3.** Note that we can similarly define sheaves of $k$-algebras for a field $k$ by taking $k$-algebras and their homomorphisms, as well as sheaves of $R$-modules for a given ring $R$.

Before defining coherent sheaves, we first recall the definition of algebraic varieties:
Definition 5.1.4. A (complex) algebraic variety \((X, \mathcal{O})\) is an irreducible topological space \(X\) with a subsheaf \(\mathcal{O}\) of the sheaf of complex-valued functions on \(X\) (as \(\mathbb{C}\)-algebra sheaves), with the following properties:
   
   (i) There exists a finite open cover \(\{U_1, \ldots, U_k\}\) of \(X\) such that \((U_i, \mathcal{O}|_{U_i})\) is an affine variety;
   
   (ii) The diagonal \(\Delta(X)\) is closed in the product space \(X \times X\).

The sheaf \(\mathcal{O}\) is called the structure sheaf and its elements are called regular functions.

Recall that an affine variety \(Y\) is a closed irreducible subset of \(\mathbb{C}^n\), or equivalently the zero set of a prime ideal \(I\) of \(\mathbb{C}[x_1, \ldots, x_n]\). We call the ring \(\mathbb{C}[x_1, \ldots, x_n]/I\) the coordinate ring of \(Y\) and denote it by \(A(Y)\) or even \(\mathbb{C}[Y]\). The structure sheaf \(\mathcal{O}_Y\) of \(Y\) is the sheaf that assigns to an open \(U \subset Y\) the \(\mathbb{C}\)-algebra \(\mathcal{O}_Y(U)\) of regular functions of \(U\): functions \(f : Y \to \mathbb{C}\) such that \(f|_U : U \to \mathbb{C}\) can be written as \(f|_U = g/h\) for \(g, h \in A(Y)\). Note that \(\mathcal{O}_Y(U)\) therefore carries a natural action of \(A(Y)\), making it into an \(A(Y)\)-module, so that we can consider \(\mathcal{O}_Y\) also as a sheaf of \(A(Y)\)-modules, which we will use later on.

Finally, we have all the necessary ingredients to define coherent sheaves.

Definition 5.1.5. Let \((X, \mathcal{O})\) be an algebraic variety and \(\mathcal{F}\) a sheaf of abelian groups on \(X\), then \(\mathcal{F}\) is a coherent sheaf if for every open \(U \subset X\):
   
   (i) \(\mathcal{F}(U)\) is a finitely generated \(\mathcal{O}(U)\)-module, we say that \(\mathcal{F}\) is a sheaf of \(\mathcal{O}\)-modules;
   
   (ii) for every \(n \in \mathbb{N}\), every morphism \((\mathcal{O}(U))^n \to \mathcal{F}(U)\) has a finitely generated kernel.

In fact, we can think of coherent sheaves as generalizations of vector bundles in which the dimensions of the fibre are not necessarily the same. In the case of a smooth affine variety \(Y\), examples of coherent sheaves are trivial bundles, vector bundles, skyscraper sheaves and sheaves supported in a finite number of points.

There is a nice correspondence between coherent sheaves over an affine variety and (finitely generated) modules over its coordinate ring: there exists a functor \(\mathcal{O}\) from modules to coherent sheaves and a functor \(\mathcal{M}\) back such that the compositions are naturally isomorphic to the identity. We will briefly discuss the construction of both functors, but we will not discuss the proof of the isomorphisms.

Let \(Y\) be an affine variety with coordinate ring \(A = A(Y)\), and consider a given finitely generated \(A\)-module \(M\). We note that we can define the constant sheaves \(\mathcal{C}_M\) and \(\mathcal{C}_A\) on \(Y\) by \(\mathcal{C}_M(U) = M\) and \(\mathcal{C}_A(U) = A\) for all open \(U \subset Y\). Since \(M\) is an \(A\)-module, \(\mathcal{C}_M\) is a sheaf of \(A\)-modules. On the other hand, we have mentioned before that we can consider the structure sheaf \(\mathcal{O}_Y\) as a sheaf of \(A\)-modules, so we can take the tensor product of the two sheaves: \(\mathcal{O}_Y \otimes_{\mathcal{C}_A} \mathcal{C}_M\): this tensor product is defined as the sheafification of the pre-sheaf given by

\[
U \mapsto \mathcal{O}_Y(U) \otimes_{\mathcal{C}_A(U)} \mathcal{C}_M(U) = \mathcal{O}_Y(U) \otimes_A M,
\]

where the last tensor product is the ordinary tensor product of \(A\)-modules. It turns out that this sheaf is coherent if \(M\) is finitely generated, so we define \(\mathcal{C}(M) = \mathcal{O}_Y \otimes_{\mathcal{C}_A} \mathcal{C}_M\), and turn it into a (covariant) functor by mapping a module-homomorphism \(\phi : M \to M'\) to the homorphism \(\mathcal{C}(\phi) = 1 \otimes \phi : \mathcal{O}_Y \otimes_{\mathcal{C}_A} \mathcal{C}_M \to \mathcal{O}_Y \otimes_{\mathcal{C}_A} \mathcal{C}_{M'}\).

For the inverse functor, consider a coherent sheaf \(\mathcal{F}\). Consider the so-called global sections \(\mathcal{F}(Y)\) of \(\mathcal{F}\) (this is often denoted \(\Gamma(Y, \mathcal{F})\)). By definition, \(\mathcal{F}\) is a sheaf of \(\mathcal{O}_Y\)-modules, so in particular \(\mathcal{F}(Y)\) is an \(\mathcal{O}_Y(Y)\)-module, so a fortiori an \(A\)-module. It turns out that for coherent sheaves, \(\mathcal{F}(Y)\) is finitely generated, so we set \(\mathcal{M}(\mathcal{F}) = \mathcal{F}(Y)\) and turn it into a (covariant) functor with the assignment \(\mathcal{M}(\phi) = \phi(Y) : \mathcal{F}(Y) \to \mathcal{F}(Y)\) for a homomorphism of sheaves \(\phi\).

The functors \(\mathcal{C}_* : \text{Coh}_Y \to \text{fgMod}_{A(Y)}\) and \(\mathcal{M}_* : \text{fgMod}_{A(Y)} \to \text{Coh}_Y\) are “inverse” to each other:

Theorem 5.1.6. For an affine variety \(Y\), the categories \(\text{Coh}_Y\) and \(\text{fgMod}_{A(Y)}\) are equivalent through the functors \(\mathcal{C}_*\) and \(\mathcal{M}_*\): there are canonical isomorphisms between \(\mathcal{M}(\mathcal{C}(M))\) and \(M\) and between \(\mathcal{C}(\mathcal{M}(\mathcal{F}))\) and \(\mathcal{F}\).

We will not prove this theorem here, see [Coherent algebraic sheaves by Serre] for the proof.

If we reconsider the examples of coherent sheaves mentioned above, denoting the coordinate field of \(Y\) by \(A = A(Y) = \mathbb{C}[x_1, \ldots, x_n]/I\), the theorem shows that the trivial bundles correspond

\footnote{Occasionally, \(\mathcal{F}\) is called an \(\mathcal{O}\)-module, understanding that \(\mathcal{O}\) and thus \(\mathcal{F}\) is a sheaf.}
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to $A^k$ as $A$-modules, that vector bundles correspond to projective $A$-modules and that skyscraper sheaves correspond to modules of the form $A/(x_1 - a_1, \ldots, x_n - a_n)^k$.

We will now work out this correspondence in the case of $A^1 = \mathbb{C}$. For more general varieties, Theorem 5.1.6 does not hold, and we will find an equivalent for the theorem in the subsequent sections. Finally, we will drop $fg$ in $fg\text{Mod}_{A(Y)}$ and implicitly assume finitely-generated modules.

5.1.2 Example: the affine line $A^1 = \mathbb{C}$

Let us work out the concepts outlined in the previous section in the specific and simple case of the affine variety $A^1 = \mathbb{C}$. We will distinguish between the variety and other copies of the complex plane by denoting the former by $A^1$ and the latter by $\mathbb{C}$. The coordinate ring of the affine line is $A = \mathbb{C}[x]$ and the structure sheaf assigns to an open subset $U \subset A^1$ the $C$-algebra of functions $f : \mathbb{C} \to \mathbb{C}$ whose restriction to $U$ can be written as $g/h$ for $g, h \in \mathbb{C}[x]$ with $h$ non-vanishing on $U$. We will consider the examples we referred to.

Trivial bundles. In the case of $\mathbb{A}^1$, a trivial bundle is of the form $\mathbb{A}^1 \times \mathbb{C}^k$ for some $k \in \mathbb{N}$. We can cast this into a sheaf by assigning to an open subset $U \subset \mathbb{A}^1$ the (continuous) sections $s : U \to \mathbb{A}^1 \times \mathbb{C}^k$ such that they compose with the projection $\pi : \mathbb{A}^1 \times \mathbb{C}^k \to \mathbb{A}^1$ to the identity, $\pi \circ s = \text{Id}_{\mathbb{A}^1}$. Actually, since the image of these sections has to be contained in $U \times \mathbb{C}^k$, we can consider them as the graph $s(x) = (x, f(x))$ of a regular function $f : U \to \mathbb{C}^k$. Noting that a regular function $f : U \to \mathbb{C}^k$ is in fact a tuple $f = (f_1, \ldots, f_k)$ of regular functions $f_i : U \to \mathbb{C} = \mathbb{A}^1$, i.e. $f_i \in \mathcal{O}(U)$, the (pre-) sheaf can be written as

$$U \mapsto \left( \mathcal{O}(U) \right)^k,$$

(the glueing condition is now clearly inherited from $\mathcal{O}$). Thus, the action of $\mathcal{O}(U)$ becomes evident: $(g, f) \mapsto g f$, where $(g f)(x) = g(x) f(x)$ has the vector $f(x) \in \mathbb{C}^k$ scaled by the factor $g(x) \in \mathbb{A}^1 = \mathbb{C}$. It is clear that the module is finitely generated, and any morphism from $(\mathcal{O}(U))^n$ to these regular functions clearly has a finite kernel, so the sheaf is coherent.

To which module does this coherent sheaf correspond? We take the global sections and find that the module is $(\mathcal{O}(\mathbb{A}^1))^k$, but any $f \in \mathcal{O}(\mathbb{A}^1)$ can be written as $g/h$ for $g, h \in A = \mathbb{C}[x]$ with $h$ non-vanishing on $\mathbb{A}^1$, so $h$ must be a constant and we find $\mathcal{O}(\mathbb{A}^1) = A$: the trivial bundle corresponds to $A^k$, where $k$ is the rank of the bundle, as we stated in the remarks after Theorem 5.1.6.

Vector bundles. Due to the simple nature of $\mathbb{A}^1$, all vector bundles can be trivialized, so these will not give new examples of coherent sheaves.

Skyscraper sheaves. Choose a fixed point $a \in \mathbb{A}^1$. We define a homomorphism of sheaves between the structure sheaf and itself given by

$$\cdot (x - a)^k : \mathcal{O}(U) \to \mathcal{O}(U) \quad f \mapsto f \cdot (x - a)^k.$$

The cokernel of this map (at $U$) is obviously given by $\mathcal{O}(U)/(x - a)^k \mathcal{O}(U)$. Now, if $a \notin U$, then $(x - a)^k$ is non-vanishing on $U$, so $g/(x - a)^k \in \mathcal{O}(U)$ for an arbitrary $g \in \mathcal{O}(U)$: we conclude that the cokernel is $\{0\}$ if $a \notin U$. The cokernel is thus the pre-sheaf given by

$$U \mapsto \begin{cases} 0, & a \notin U, \\ \mathcal{O}(U)/(x - a)^k, & a \in U, \end{cases}$$

at which point the $\mathcal{O}(U)$-action is evident and we see that the cokernel is a coherent sheaf. This sheaf is called a skyscraper sheaf.

The module corresponding to the skyscraper sheaf is $\mathcal{O}(\mathbb{A}^1)/(x - a)^k = A/(x - a)^k$, as we already remarked, using the identification we found when we were considering the trivial bundles.

This example is derived from the entry “Locally free sheaves and vector bundles” of the blog Rigorous Trivialities by Charles Siegel, where you can find the complete argument.
Sheaves supported in a finite number of points. These sheaves can be written as direct sums of skyscraper sheaves supported in a point. The corresponding module will therefore simply be the direct sum $\bigoplus_i A/(x - a_i)^{b_i}$.

In fact, there is the following theorem on $\mathbb{C}[x]$-modules, which can be proven using an identification of finite-dimensional $\mathbb{C}[x]$-modules with the Jordan normal form of a matrix:

**Theorem 5.1.7.** Every finitely generated $\mathbb{C}[x]$-module is a direct sum of copies of the modules $\mathbb{C}[x]$ and $\mathbb{C}[x]/(x - a)^k$ for an $a \in \mathbb{C}$ and $k \in \mathbb{N}$.

Thus, the modules corresponding to trivial bundles and skyscraper sheaves already give all the irreducible, finitely generated $A$-modules. Due to the equivalence of the categories $\text{Coh}\, A^1$ and $\text{Mod}_A$ proven in Theorem 5.1.6, we see that the trivial bundles and skyscraper sheaves generate all coherent sheaves of $A^1$.

We have seen that the correspondence $\text{Coh}\, Y \cong \text{Mod}_{A(Y)}$ makes working with coherent sheaves more controllable. However, this correspondence does not hold if the variety is not affine, so we will need to find a different way of relating coherent sheaves to modules. It turns out, for example for the projective variety $\mathbb{P}^1$, that it suffices to consider the derived categories: there is an algebra $A$ such that $D^b\, \text{Coh}\, \mathbb{P}^1 \cong D^b\, \text{Mod}_A$.

### 5.2 Derived categories

In words, derived categories are a way of considering complexes of modules up to quasi-isomorphisms. We could construct derived categories for a general category $\mathcal{C}$, but we will specify to the case of $R$-modules: $\mathcal{C} = \text{Mod}_R$.

**Definition 5.2.1.** Let $\text{Mod}_R$ be an arbitrary category. We denote the category of chain complexes in $\text{Mod}_R$ by $K^b\, \text{Mod}_R$ and let the objects and morphisms be given as follows.

- $\text{Obj}(K^b\, \text{Mod}_R) = \{ A = \bigoplus_{i \in \mathbb{Z}} (A_i, d^i) \}$, with $d_i : A_i \to A_{i+1}$ such that $d^2 = 0$ and $A_i \in \text{Obj}(\mathcal{C})$,
- $\text{Morph}(K^b\, \text{Mod}_R)(A, B) = \{ \phi = \bigoplus_{i \in \mathbb{Z}} \phi_i : A_i \to B_i \}$, such that $d^i \circ \phi_i = \phi_{i+1} \circ d^i$.

It is well known that a morphism $\phi$ of chain complexes induces a map $\phi_*$ on the cohomology level. We call $\phi$ a *quasi-isomorphism* if it induces an isomorphism $\phi_*$. The derived category $D^b\, \text{Mod}_R$ can now informally be seen as equal to $K^b\, \text{Mod}_R$ but with the quasi-isomorphisms promoted to isomorphisms. For the very precise reader, we will now make this more rigorous by the following procedure.

First let us give $K^b\, \text{Mod}_R$ the structure of a *homotopical category* by declaring the weak equivalences to be the quasi-isomorphisms. In such a homotopical category the weak equivalences satisfy the 2-out-of-6 property, i.e. if morphisms $g \circ f$ and $h \circ g$ are weak equivalences, then so are $f$, $g$, $h$ and $h \circ g \circ f$.

**Definition 5.2.2.** We define the derived category as the homotopy category of $K^b\, \text{Mod}_R$. The homotopy category is defined by the universal property that there exists a functor

$$Q : K^b\, \text{Mod}_R \to \text{Ho}(K^b\, \text{Mod}_R) =: D^b\, \text{Mod}_R,$$

such that $Q$ sends all weak equivalences to isomorphisms.

In order to gain some intuition for the concept of a derived category and connect it to something we have seen before, we will now consider the following theorem.

**Theorem 5.2.3.** Let $R$ be a smooth $A_\infty$-algebra. Then $D^b\, \text{Mod}_R = \text{Tw} \, R$

**Proof.** We will only give a sketch of the proof here. Recall from the definition of the twisted completion of $R$ that its objects are complexes of shifts and direct sums of $R$. These objects are thus always complexes of free $R$-modules and therefore $\text{Tw} \, R \subset D^b\, \text{Mod}_R$. When $R$ is smooth, we can use that every $R$-module $M$ has a finite free resolution. In the derived category this free resolution is isomorphic to $M$ itself. By the aforementioned, we can see this free resolution as an object of $\text{Tw} \, R$. This gives us equality and the proof is complete. \qed
5.2. Example: the projective line $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$

Let us turn to a slightly more challenging example, the projective line $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$. The problem of finding all coherent sheaves is a little bit more difficult for $\mathbb{P}^1$ than for $\mathbb{A}^1$, as we no longer have Theorem 5.1.6 available. The problem is still quite tractable however, since we know that $\mathbb{P}^1$ is the union of two affine open pieces isomorphic to $\mathbb{C}$; the coherent sheaves on these $\mathbb{C}$’s, subject to gluing conditions, will then provide us with the coherent sheaves of $\mathbb{P}^1$. This is what we will do: take $U_1 := \mathbb{C}$, with coordinate ring $\mathbb{C}[z]$, $U_2 := \mathbb{C} \setminus \{0\} \cup \{\infty\} \cong \mathbb{C}$, with coordinate ring $\mathbb{C}[w]$, and we will interpret $w$ as $z^{-1}$. This interpretation is natural given that the affine subvariety $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ is isomorphic to $\mathbb{P}^1(zw - 1) \subset \mathbb{C}^2$, with coordinate ring $\mathbb{C}[z, w]/(zw - 1) = \mathbb{C}[z, z^{-1}]$. The isomorphism is given by $z \mapsto (z, z^{-1})$, with inverse $(t, w) \mapsto t$.

Now, to define a coherent sheaf on $\mathbb{P}^1$ we need coherent sheaves $\mathcal{F}_1$ on $U_1$ and $\mathcal{F}_2$ on $U_2$, which (recall the discussion on the coherent sheaves on $\mathbb{C}$), correspond to finitely generated modules $M_1$ over $\mathbb{C}[z]$ and $M_2$ over $\mathbb{C}[w]$. We require that these sheaves satisfy the gluing condition: there exists an isomorphism of sheaves

$$\varphi_{12} : \mathcal{F}_1|_{U_1 \cap U_2} \to \mathcal{F}_2|_{U_1 \cap U_2}.$$ 

In terms of the coordinate ring of $U_1 \cap U_2$ this translates to the condition that we have an isomorphism of $\mathbb{C}[z, z^{-1}]$-modules

$$\varphi : M_1 \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] \to M_2 \otimes_{\mathbb{C}[w]} \mathbb{C}[z, z^{-1}].$$

Let us turn to a specific example, one that is crucially important and will in fact (almost) completely characterize all coherent sheaves on $\mathbb{P}^1$. Take $M_1 = \mathbb{C}[z]$, and $M_2 = \mathbb{C}[w]$, which of course corresponds to choosing on both $U_1$ and $U_2$ the trivial line bundle as the sheaves. For the gluing map, consider the following isomorphism of $\mathbb{C}[z, z^{-1}]$-modules

$$\varphi_i : \mathbb{C}[z, z^{-1}] \xrightarrow{\cong} \mathbb{C}[z, z^{-1}], \quad 1 \mapsto z^i, \quad i \in \mathbb{Z}.$$ 

Because we have an inverse for $z$ in $\mathbb{C}[z, z^{-1}]$ it is trivial to check that $\varphi_i$ is indeed an isomorphism for each $i \in \mathbb{Z}$. The sheaves corresponding to these modules are line bundles on the affine opens, so obviously they glue together to give a line bundle on $\mathbb{P}^1$. For each $i \in \mathbb{Z}$ we have such a line bundle, where $i$ intuitively measures the level of “twisting” of the bundle. This signals a crucial difference between coherent sheaves on $\mathbb{A}^1$ and $\mathbb{P}^1$: because the only $\mathbb{C}[z]$-module isomorphism $\mathbb{C}[z] \to \mathbb{C}[z]$ is the identity, there is only one line bundle on $\mathbb{C}$, the trivial one. On $\mathbb{P}^1$ however, we have just shown that for every $i \in \mathbb{Z}$ we have a line bundle, which we will denote by $\mathcal{O}(i)$, and this characterizes all the line bundles (the so-called locally free sheaves). We will also write $\mathcal{O}$ for the trivial bundle $\mathcal{O}(0)$.

To generate all of $\text{Coh}\mathbb{P}^1$ we also need the skyscraper sheaves. Since these are defined locally, there are no global twisting subtleties: the skyscrapers on $\mathbb{P}^1$ are described completely analogously to the skyscrapers on $\mathbb{C}$, save that on $\mathbb{P}^1$ we have the extra point at infinity to put our skyscraper on. Thus, a skyscraper at a point $p \in U_1$ corresponding to the point $\lambda \in \mathbb{C}$ is given by a the module $\mathbb{C}[z]/(z - \lambda)$, and if $p \neq 0$, $p$ is also in $U_2$ with corresponding point $\lambda^{-1} \in \mathbb{C} \setminus \{0\} \cup \{\infty\}$ and module $\mathbb{C}[w]/(w - \lambda^{-1})$.

The skyscrapers alone will not quite do however. To see this, consider the module $\mathbb{C}[z]/(z - \lambda)^n$, $n \in \mathbb{N}$ for instance. The basis $(z - \lambda)^k$, $0 \leq k \leq n$ defines the Jordan normal form for the multiplication of $z$, that is, in this basis the matrix of multiplication by $z$ in $\mathbb{C}[z]/(z - \lambda)^n$ is

$$\begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & \cdots & 0 & \lambda
\end{pmatrix}$$

which immediately implies the indecomposability of $\mathbb{C}[z]/(z - \lambda)^n$ as a $\mathbb{C}[z]$-module. Clearly, we cannot hope to recover all the coherent sheaves with just the skyscrapers; we must also include these generalized skyscraper sheaves given by the modules $\mathbb{C}[z]/(z - \lambda)^n$, $n \in \mathbb{N}$. Now the following theorem asserts that the coherent sheaves on $\mathbb{P}^1$ we have discussed so far are all that is needed.
Theorem 5.2.4. Every coherent sheaf on $\mathbb{P}^1$ is a direct sum of line bundles $O(i)$ and generalized skyscraper sheaves.

Proof. A coherent sheaf on $\mathbb{P}^1$ restricts to coherent sheaves on $U_1$ and $U_2$. On these affine opens we already know all coherent sheaves, since $\text{Mod } C[z] \cong \text{Coh } C$, and we know by the structure theorem for finitely generated modules over a PID that these are all of the form $C[z]/(z-\lambda)^n$, $n \in \mathbb{Z}_{>0}$.

We only need to consider nontrivial gluing maps when we want to construct line bundles, of the form $C[z,z^{-1}]$, so the fact that the only possible gluing maps $C[z,z^{-1}] \xrightarrow{\phi} C[z,z^{-1}]$ over $C[z,z^{-1}]$ are the isomorphisms $\phi_i : 1 \mapsto z^i$ implies the theorem.

We can ask whether there exists an algebra $A$ such that $\text{Mod } A \cong \text{Coh } \mathbb{P}^1$, but this is too much to hope for. Later on, we will indeed obtain a description of the coherent sheaves of $\mathbb{P}^1$ along these lines, but first we must delve a little bit deeper into the categorical structure of $\text{Coh } \mathbb{P}^1$; specifically, we will study its derived category.

5.3 Concrete description of derived categories

We will here study the derived categories of coherent sheaves. We will introduce these techniques by studying $\text{Coh } \mathbb{P}^1$ and compare this to $\text{Coh } \mathbb{A}^1$.

Let us first look at $\mathbb{A}^1$: we can identify all its elements with the maximal ideals of $C[x]$: $\mathbb{A}^1 = \text{Spec}(C[x]) = \{ (x-a) | a \in C \}$. The category of coherent sheaves $\text{Coh } \mathbb{A}_1$ will then consist of the trivial line bundle and $n$-dimensional skyscraper sheaves at every point. By the correspondence between the coherent sheaves of $\mathbb{A}_1$ and the finitely generated modules of its coordinate ring $C[x]$, we can identify the trivial line bundle with $C[x] =: \mathcal{L}$ itself and the skyscraper sheaves with the modules $C[x]/(x-a)^n =: S_{a,n}$. We can then denote the generating elements of $\text{D}^b \text{Mod } A_1$ by $S_{a,n}[i]$ and $\mathcal{L}[i]$, where these are complexes with respectively $S_{a,n}$ and $\mathcal{L}$ at degree $i$ and zeros elsewhere. Note however that this description of the objects is not unique. We can make two isomorphic complexes by

$$
\cdots \to 0 \to \cdots \to S_{a,n} \to 0 \to \cdots
$$

$$
\cdots \to 0 \to \mathcal{L} \to \mathcal{L} \to 0 \to \cdots
$$

These two complexes have the same homology and are thus isomorphic in the derived category.

5.3.1 A concrete description of $\text{D}^b \text{Coh } \mathbb{P}^1$

Now let’s do the same on $\mathbb{P}^1$. We can view it as the projective space of $C[X,Y]$ as follows

$$
P_1 = \text{Proj}(C[X,Y]) \cong \{ \text{homogeneous max. ideals, not containing } (X,Y) \} = \{ (aX - bY) \}. \quad (5.2)
$$

The ideals containing $(X,Y)$ are left out, because these would correspond to functions being zero on $(0,0)$. This point is not in $\mathbb{P}^1$ however, because we always demand that one of the two homogeneous coordinates is nonzero. From theorem 5.2.4 we know that the indecomposable coherent sheaves on $\mathbb{P}^1$ are the line bundles $O(i)$ and the skyscrapers $S_{a,n}$. Analogous to the usual correspondence, the category of these coherent sheaves is equivalent to the category of graded modules on $C[X,Y]$ modulo the finite dimensional graded modules

$$
P_1 \cong \text{Coh } \mathbb{P}_1 \cong \text{grMod } C[X,Y]/\text{fdgrMod } C[X,Y] =: \text{Coh } C[X,Y], \quad (5.3)
$$

where $\text{fdgrMod} = \{ M \in \text{grMod } C[X,Y] \mid M \text{ finite dimensional} \}$. The reason that the finite dimensional modules are divided out is because in these modules $X$ and $Y$ must be nilpotent. One can see this by acting over and over with $X$ and $Y$ on an element of the module. This must stop because of the finite dimensionality, making $X$ and $Y$ nilpotent. Modules in which $X$ and $Y$ are nilpotent correspond to the maximal ideals containing $(X,Y)$ and these ideals were excluded in equation (5.2). Note that the name ‘category of tails’ is indeed a conveniently chosen name, since two objects are only different when they differ in their infinitely long tails.
5.3. CONCRETE DESCRIPTION OF DERIVED CATEGORIES

A small remark is needed about this new category, because what exactly is a quotient of categories? Let $N \subset C$ be two categories. We can form the quotient $C/N$ and its projection functor $Q : C \to C/N$ as the category with the universal property that given a functor $F : C \to D$, there exists an $\tilde{F}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
N & \xrightarrow{f} & C \\
\text{Q} & \downarrow & \text{f} \\
& & \text{C/N}
\end{array}
$$

That said, we are now ready to describe $\text{Tails } C[X,Y]$. It has two kinds of indecomposable objects:

$$
\begin{align*}
C[X,Y]/(ax-by)^n &= (M_i = C) \oplus (M_{i+1} \cong C) \oplus (M_{i+2} \cong C) \oplus \ldots, \\
C[X,Y](i) &= C \oplus (CX + CY) \oplus (CX^2 + CXY + CY^2) \oplus .
\end{align*}
$$

(5.4) (5.5)

The objects $C[X,Y]/(ax-by)^n$ are the (generalized) skyscrapers localized at the point $(a:b)$, while the $C[X,Y](i)$ correspond to the line bundles $O(i)$. The morphisms in this category are the degree preserving maps. The $i$ in the objects of (5.4) determines the degree of $X$ and $Y$ in the graded module. If $i = 0$ we have $\text{deg}(X) = \text{deg}(Y) = 1$. These shifts will all give different objects. Try, for example, to write down an isomorphism between $C[X,Y](0)$ and $C[X,Y](1)$. Because the morphisms are degree-preserving we will always have a one-dimensional kernel, because the higher-degree parts are higher-dimensional as well. With the more geometrical picture from the previous section in mind, this is exactly the statement that the skyscrapers localized at the point $(a:b)$ are equal modulo a finite dimensional graded module.

Recall the definition of a quiver. A quiver $\Gamma$ is a directed graph given by the following data:

(i) The set $V$ of vertices of $\Gamma$

(ii) The set $E$ of edges of $\Gamma$

(iii) Two maps $s : E \to V$ giving the start of the edge and $t : E \to V$ giving the target of the edge.

A path in the quiver is a sequence of edges $a_n a_{n-1} \ldots a_1$ such that $s(a_i) = t(a_{i-1})$, $1 < i \leq n$. We may associate to a quiver $\Gamma$ an associative algebra, called the path algebra of $\Gamma$, as in the following definition.
Definition 5.3.1. Let $\Gamma$ be a quiver. The path algebra $PT$ of $\Gamma$ is the algebra (over $C$) generated by all paths of length $\geq 0$ in $\Gamma$. Note that for every vertex $i$ we also have the trivial path of zero length $e_i$. Multiplication of paths $a$ and $b$ is the concatenation $ab$, if $b$ ends at the vertex where $a$ starts. If $a$ and $b$ cannot be concatenated in this way, the product is $0$.

With this definition $PT$ is an associative algebra over $C$. Note that if $V$ is a finite set, $PT$ is unital: 

$$1 = e_1 + \ldots + e_n,$$

where $n = |V|$. As an easy example consider the following quiver

$$\begin{array}{ccc}
& X & \\
e_1 & \rightarrow & \rightarrow \\
& Y & & e_0
\end{array}$$

Observe that $PT = C[X]$, so this quiver provides us with the path algebra for the coordinate ring of $A^1$, and $\text{Coh} A^1 \cong \text{Mod} PT$. We have already remarked that such a construction is not possible for the projective line. However it is possible to find an algebra $A$ whose modules correspond to the coherent sheaves of $P^1$ if we pass to the derived category: $D^b \text{Coh} P^1 \cong D^b \text{Mod} A$, and we shall see that $A$ is the path algebra of a specific quiver.

The indecomposables of $D^b \text{Coh} P^1$ are given by (5.6). Consider the object $E := O \oplus O(1) = C[x,y] \oplus C[x,y](1)$, the direct sum of the trivial line bundle with the degree 1 line bundle. This object is specifically crafted so that it intuitively “touches everything in $D^b \text{Coh} P^1$” in a sense we will make more precise in a moment. Set $A := \text{End}(E)$. The requirement that all maps are degree preserving is quite restrictive: A map $C[x,y] \rightarrow C[x,y](1)$ must necessarily send $1 \mapsto 0$ because $1$ has degree 0 in $C[x,y]$ but there are no nontrivial degree 0 elements $C[x,y](1)$. By similar reasoning, a map $C[x,y](1) \rightarrow C[x,y]$ can only send 1 to (a scalar multiple) of $x$ or (a scalar multiple) of $y$, and must send $x$ and $y$ to 0. Furthermore, the only self maps $C[x,y](i) \rightarrow C[x,y](i)$ are scalar multiples of the identity, so $A$ is spanned by

1. $e_0$, the identity map on $C[x,y]$ and the zero map on $C[x,y](1)$,
2. $e_1$, the zero map on $C[x,y]$ and the identity map on $C[x,y](1)$,
3. $X$, the map $C[x,y](1) \rightarrow C[x,y], 1 \mapsto X$,
4. $Y$, the map $C[x,y](1) \rightarrow C[x,y], 1 \mapsto Y$.

Now observe that the relations these four generators satisfy make $A$ exactly into the path algebra of the quiver $\Gamma$:

$$\begin{array}{ccc}
& X & \\
e_1 & \rightarrow & \rightarrow \\
& Y & & e_0
\end{array}$$

This quiver is known as the Kronecker quiver, and $A = PT$.

Now how does $A$ relate to $D^b \text{Coh} P^1$? Take $S \in D^b \text{Coh} P^1$, and substitute $S$ by a resolution of $S$ that is equivalent to $S$ in the derived category. Consider the Hom-space of $S$ paired with $E$, $\text{Hom}(E,S)$ (the homomorphisms are over $C[x,y]$). The homology $H_*\text{Hom}(E,S)$ carries an obvious $A$-module structure, so we have a functor

$$F : D^b \text{Coh} P^1 \rightarrow D^b \text{Mod} A, \quad S \mapsto H_*\text{Hom}(E,S).$$

Claim 5.3.2. The functor $F$ defines an equivalence of categories.

We will not attempt to prove this claim. The “special” object $E$ is known as a tilting object in this situation.

The approach outlined here using quivers is easily generalized to higher $P^n$. Consider the Beilinson quiver $\Gamma^n$

$$\begin{array}{ccc}
& X_0^n & \\
e_n & \rightarrow & \rightarrow \\
& X_{n-1}^n & & e_{n-1}
\end{array}$$

which has $n + 1$ vertices with $n + 1$ arrows between two consecutive vertices, then $D^b \text{Coh} P^n \cong D^b \text{Mod} PT^n$. 
5.4. CATEGORY OF SINGULARITIES

Remark 5.3.3. The method will usually fail when we leave projective space, but in many special cases something similar can be done. For instance, when the variety is an elliptic curve one can still choose an appropriate tilting object \( E \) for which the associated \( A = \text{End}(E) \) is a path algebra. The equivalence \( D^b \text{Coh} \mathbb{P} \cong D^b \text{Mod} A \) will no longer be true, but if one replaces \( D^b \text{Mod} A \) with the larger twisted completion \( \text{Tw}_A \), one does obtain an equivalence. Note that the twisted completion is only larger when \( A \) is not smooth, according to theorem 5.2.3.

5.4 Category of singularities

In this last section about the B-model we will have a short look at the category of singularities. The general idea is to take a general variety and then determine its smooth and singular points. We will then kill the smooth part, such that only the interesting part remains. We will then include a short teaser about Orlov’s theorem but will not proceed in this direction.

5.4.1 Smooth and singular points

Let \( V \) be a general variety for which we again want to look at the derived category of coherent sheaves. The question we now ask ourselves is if there are criteria for \( V \) being smooth. A theorem of Serre tells us the following.

Theorem 5.4.1 (Serre). A variety \( V \) is smooth if and only if every sheaf has a finite resolution of locally free projectives.

We will not provide a proof here, but we can illustrate the theorem with two examples.

- Let’s look at the affine variety \( V = \mathbb{A}^2 = \mathbb{C}^2 \), with coordinate ring \( R = \mathbb{C}[X, Y] \) and consider a skyscraper sheaf at the origin. Because the origin in \( \mathbb{C} \) is obviously smooth, we should, according to Serre, now be able to find a finite resolution for \( \mathbb{C}[X, Y]/(X, Y) \cong \mathbb{C} \), corresponding to the skyscraper sheaf at \((0, 0)\). Recall that a resolution of a \( R \)-module \( M \) should be an exact sequence of \( R \)-modules starting with \( 0 \leftarrow M \leftarrow \ldots \). We can try

\[
0 \leftarrow C \leftarrow \mathbb{C}[X,Y] \xleftarrow{X} \mathbb{C}[X,Y] \oplus \mathbb{C}[X,Y] \xleftarrow{(Y,X)} \mathbb{C}[X,Y] \leftarrow 0.
\]

Clearly the modules \( \mathbb{C}[X,Y] \) and \( \mathbb{C}[X,Y] \oplus \mathbb{C}[X,Y] \) are free over \( R = \mathbb{C}[X,Y] \), so we have found a finite free resolution.

- Let’s then look at \( V \) being the two intersecting lines \( X = 0 \) and \( Y = 0 \), with coordinate ring \( R = \mathbb{C}[X,Y]/XY \). The origin is now not a smooth point anymore. If we put a skyscraper sheaf at the origin, Serre predicts that we will not find a finite resolution. The skyscraper sheaf corresponds to the module \( R/(X,Y) \cong \mathbb{C} \). Let’s try the following resolution

\[
0 \leftarrow C \leftarrow R \xleftarrow{X} R \oplus R \xleftarrow{(Y,X)} R \xleftarrow{Y} R \oplus R \leftarrow \ldots.
\]

Note the difference between this resolution and the previous one. We now do not have zero kernel in the map from \( R \) to \( R \oplus R \), so we need to add an extra module in the resolution. We then see that this becomes a repeating sequence and the resolution will thus be infinite.

5.4.2 Orlov’s theorem

Now define the derived category of perfect complexes on \( R \). This is a subcategory of \( D^b \text{Coh} R \), containing all objects with a finite resolution of locally free projectives. According to Serre’s theorem we see that this subcategory coincides with the full category in the case that \( R \) is smooth. For singular varieties we can form the quotient and call it the category of singularities

\[
D^b \text{Sing} R := D^b \text{Coh} R/(\text{perfect objects}).
\]

Now consider an \( R \) of the form \( \mathbb{C}[X_1, \ldots, X_n]/(f) \) where \( f \) is a function of degree \( n \). Orlov’s theorem relates the derived category of such \( R \) to its category of singularities. This hand-wavingly
means that to study the huge category of coherent sheaves of this type of variety, it is enough to look at the smaller category of singularities. In other words, the singularities still contain the interesting part of $R$. 