Chapter 1

Representations

1.1 Representations of Groups and Algebras

A *representation of a group* G on a finite-dimensional complex vector space V is a group homomorphism $\rho : \mathsf{G} \to \mathsf{GL}(V)$. If G has some additional structure like topological space, complex variety or a real manifold, we ask that ρ is a corresponding morphism: a continuous map, a polynomial map or a smooth map.

A *representation of a finitely generated complex algebra* A on a finite dimensional complex vector space v is an algebra morphism $\rho : A \to \mathsf{End}(V)$. We say that such a map gives V the structure of an A-module. When there is little ambiguity about the map ρ, we sometimes call V itself a representation of A. For any element $x \in A, v \in V$ we will shorten $\rho(x)v$ to $x \cdot v$ or xv.

A *morphism* ϕ between two representations ρ_V and ρ_W is a vector space map $\phi : V \to W$ such that the following diagram is commutative

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho_V(x)"'] & W \arrow[d, "\rho_W(x)"] \\ V \arrow[r, "\phi"] & W. \end{tikzcd}$$

In short we can also write $\phi(xv) = x\phi(v)$. morphism is also sometimes called an A-linear map. The set of A-linear maps is denoted by $\mathsf{Hom}_A(V, W)$

A *subrepresentation* of V is a subspace W such that $x \cdot W \subset W$ forall $x \in A$. Note that a morphism maps subrepresentations to subrepresentations so in particular for any morphism ϕ the spaces $\mathsf{Ker}\phi$ and $\mathsf{Im}\phi$ are subrepresentations.

A representation V is called *simple* if its only subrepresentations are 0 and V. This is equivalent to saying that ρ_V is a surjective algebra morphism. If V and W are representations we can construct new representations from them: the direct sum $V \oplus W = \{(v,w)|v \in V, w \in W\}$ has a componentwise action $x(v,w) = (xv, xw)$. A representation that is not isomorphic to the direct sum of two non-trivial representations is called *indecomposable*. If a representation is a direct sum of simple representations it is called *semisimple*. The decomposition of a semisimple into simple components is unique up to a permutation of the factors.

Homomorphisms between (semi)simple representations can be described easily using Schur's Lemma

Lemma 1.1 (Schur)**.** *Let S and T be simple representations then*

$$\mathsf{Hom}_{\mathsf{G}}(S,T) = \begin{cases} 0 & \textit{if } S \neq T \\ \mathbb{C} & \textit{if } S = T \end{cases}$$

Corollary 1.2. *If $V \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ and $W \cong V \cong S_1^{\oplus f_1} \oplus \cdots \oplus S_k^{\oplus f_k}$ where some of the e and $f's$ can be zero. then*

$$\mathsf{Hom}_{\mathsf{G}}(V,W) = \mathsf{Mat}_{f_1 \times e_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{f_k \times e_k}(\mathbb{C}).$$

To every representation ρ_V we can associate its character. This is the composition of ρ_V with the trace map $\mathsf{Tr} : \mathsf{End}(V) \to \mathbb{C}$.

$$\chi_V : \mathsf{Tr} \circ \rho_V.$$

χ_V is an element of A^* and it is invariant under isomorphism: $\rho_V \cong \rho_W \Rightarrow \chi_V = \chi_W$. The main theorem of representation theory of finitely generated algebras now states that the opposite is also true for semisimple representations.

Theorem 1.3.
1. *If V and W are semisimple representations then $V \cong W$ if and only if $\chi_V = \chi_W$.*
2. *V is not semisimple then there is a unique semisimple representation V^S such that $\chi_V = \chi_{V^S}$.*

The proof of this theorem is quite lengthy and in can be found in the course notes on representation theory.

If G is a finite group then one can construct its group algebra. This is the complex algebra with as basis the elements of the group and as multiplication the linear extension of the multiplication of the group. We will now have a closer look at the representations of these group algebras.

If we're considering only group representations we can construct even more new representations:

- The tensor product $V \otimes W = \mathsf{Span}(v_i \otimes w_j | (v_i),\ (w_j)$ are bases for V, W) has as action $x(v \otimes w) = gv \otimes gw$.
- The dual space $V^* = \{f : V \to \mathbb{C} | f \text{ is linear}\}$ has a contragradient action: $(g \cdot f)v = f(g^{-1} \cdot v)$.
- The space of linear maps $\mathsf{Hom}(V, W)$ can be identified with $V^* \otimes W$ and hence the action is $(g \cdot f)v = g \cdot (f(g^{-1} \cdot v))$. Note that this means that the elements of $\mathsf{Hom}_A(V, W)$ are in fact the maps that are invariant under the action of A.

The characters of the representations are elements of $\mathbb{C}\mathsf{G}^*$. This is a finite dimensional vector space with dual basis $g^*, g \in \mathsf{G}$. On this space we can put an hermitian product such that $\langle g^*, h^* \rangle = \delta_{gh}/|\mathsf{G}|$.

The representation theory of finite groups can be summarized as

Theorem 1.4. *Let* G *be a finite group then we have that*

1. *Every representation is semisimple.*
2. *A representation* V *is simple if and only if* $\langle \chi_V, \chi_V \rangle = 1$
3. *The number of isomorphism classes simple representations is the same as the number of conjugacy classes in* G.
4. *The characters of simple representations form an orthonormal basis for the subspace of class functions is* $\mathbb{C}G^*$.
5. *If* S *is simple then* $\langle \chi_V, S \rangle$ *is the multiplicity of* S *inside* V.
6. V *is completely determined by its character* χ_V.
7. $\mathbb{C}\mathsf{G} \cong \mathsf{End} S_1 \oplus \cdots \mathsf{End} S_k \cong \mathsf{Mat}_{\dim S_1} \oplus \cdots \oplus \mathsf{Mat}_{\dim S_k}$.

Note a group (not necessarily finite) for which (1) holds is called a reductive group. Other examples of reductive groups are GL_n, SL_n, SO_n and finite products of these groups.

1.2 Finite Subgroups of SL_2

In the rest of this chapter we will apply the theory to the case where V is a two-dimensional representation of a finite group. First of all we do not have to consider all finite groups. The quotient only depends on the image of $\rho_V : \mathsf{G} \to \mathsf{GL}(V)$. So we only have to consider finite subgroups of GL_2. We can do even better as we only need to determine these subgroups up to conjugation, we can bring them into a standard form: every finite subgroup of GL_2 can be conjugated to a subgroup of U_2. To prove this we can define a hermitian form on V as follows:

$$\langle v, w\rangle := \sum_{g\in\mathsf{G}} (g\cdot v)(g\cdot w)^\dagger$$

The action of G keeps this form invariant $\langle hv, hw\rangle := \sum_{g\in\mathsf{G}}(gh\cdot v)(gh\cdot w)^\dagger = \langle v, w\rangle$, so if we choose an orthonormal basis for this form G will act as unitary matrices according to this basis. Because $U_2 = U_1 \times SU_2$, G can also be written as the product of as subgroup of U_1 and a subgroup of SU_2. The subgroups of U_1 are the cyclic groups $\mathbb{Z}_n$, so we will now look at finite subgroups of SU_2 which are also the finite subgroups of SL_2.

Theorem 1.5. *Every finite subgroup of SU_2 can be conjugated to one of the following groups:*

- $\mathcal{C}_n$ *a cyclic group of order n generated by* $\begin{bmatrix} e^{2\pi/n} & 0 \\ 0 & e^{-2\pi/n}\end{bmatrix}$
- $\mathcal{D}_n$ *The binary dihedral group of order $4n$ generated by* $\begin{bmatrix} e^{2\pi/n} & 0 \\ 0 & e^{-2\pi/n}\end{bmatrix}$ *and* $\left[\begin{smallmatrix} 0 & i \\ i & o\end{smallmatrix}\right]$.
- $\mathcal{T}$ *The binary tetrahedral group.*
- $\mathcal{O}$ *The binary octahedral group.*
- $\mathcal{I}$ *The binary icosahedral group.*

Proof. The group SU_2 can be mapped onto $SO_3(\mathbb{R})$. Embed $\mathbb{R}^3$ in $\mathsf{Mat}_2(\mathbb{C})$ as the subspace of traceless antihermitian matrices $\mathbb{H}$

$$\mathbb{R}\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right] + \mathbb{R}\left[\begin{smallmatrix} 0 & i \\ i & 0\end{smallmatrix}\right] + \mathbb{R}\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right].$$

On this subspace we can put a scalar product $\langle A, B\rangle := Tr(AB^\dagger)$ SU_2 acts on this subspace by conjugation and the conjugation respects the scalar product: $\langle U\cdot A, U\cdot B\rangle = Tr(UAU^{-1}(UBU^{-1})^\dagger) = Tr(UAU^{-1}(UBU^{-1})^\dagger) = Tr(UAB^\dagger U^{-1}) = Tr(AB^\dagger)$. Therefore the action of SU_2 on $\mathbb{H}$ factors through the orthogonal group of $\langle,\rangle$. As SU_2 is connected the image of SU_2 will be contained in SO_3.

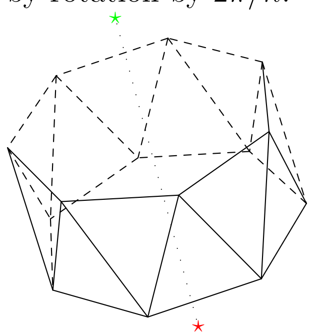
One can check that the kernel of this map is $\{1, -1\} \subset SU_2$ and as the real dimension of SU_2 and $SO_3(\mathbb{R})$ are both 3 the map will be surjective. SU_2 is called the double cover of SO_3.

Now we will show that any finite subgroup of SO_3 is either a cyclic group C_n a dihedral group D_n or one of the groups of a Platonic solid.

Let G now be a finite subgroup of SO_3 with order n. The elements of $G - \{1\}$ are rotations so we can associate to each element its poles i.e. the intersection points of the rotation axis with the unit sphere. Let P be the set of poles of elements of G. Every pole is mapped to a pole under the action of G. So we can partition P into orbits of G. To every pole p we can associate m_p, the number of rotations with this pole. Note that poles in the same orbit have the same m_p. It is also the order of the subgroup of G that fixes p.

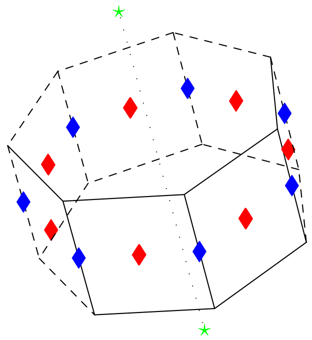
The $n-1$ non-trivial rotations in G consist of $m-1$ rotations for each pair of poles. That is $\frac{1}{2}(m-1)\frac{n}{/}m$ for each orbit. Hence $n-1 = \frac{1}{2}n(\sum \frac{(m-1)}{m})$ where the summation is over the orbits Since $m \geq 2$ we have $(m-1)/m > 1/2$ and so we can only have 2 of 3 orbits if G is non-trivial.

1. The case of two orbits. Suppose these have n/m_1 and n/m_2 elements. Then $2/n = 1/m_1 + 1/m_2$ implies $n/m_1 = n/m_2 = 1$ and we have two orbits with one pole in each. This is the case when G is a cyclic group C_n generated by rotation by $2\pi/n$.

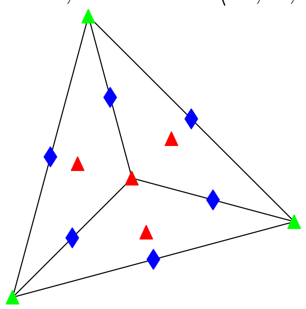


2. The case of three orbits. Then $1 + 2/n = 1/m_1 + 1/m_2 + 1/m_3$ so one of the $m_i = 2$. Take $m_3 = 2$ so $1/m_1 + 1/m_2 = 1/2 + 2/n$. There are only a few possibilities:

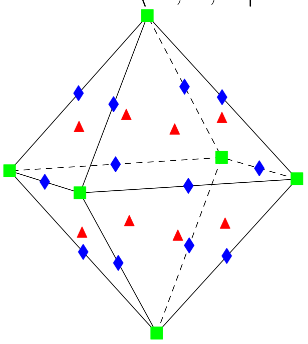
   - $m_1 = 2, m_2 = m, n = 2m$ (This is the dihedral case $G = D_{2n} = \langle X, Y, Z | X^2 = Y^m = Z^2 = XYZ = 1\rangle$)



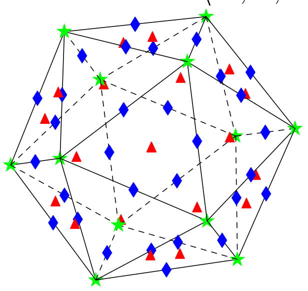
- $m_1 = 3, m_2 = 3, n = 12$ (This is the symmetry group of the tetrahedron, $G = T = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = 1 \rangle$)



- $m_1 = 3, m_2 = 4, n = 24$ (This is the symmetry group of the cube, $G = O = \langle X, Y, Z | X^3 = Y^4 = Z^2 = XYZ = 1 \rangle$)



- $m_1 = 3, m_2 = 5, n = 60$ (This is the symmetry group of the dodecahedron $G = I = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = 1 \rangle$)



Now let $\tilde{G}$ be a subgroup of SU_2. If $\tilde{G}$ has an even number of elements then it contains -1, because this is the only element in SU_2 of order 2. This means that $\tilde{G}$ is the inverse image of a finite subgroup of SO_3, these are called the binary dihedral, tetrahedral, etc. groups, note that binary cyclic is again cyclic. These groups can be expressed in generators and relations by introducing a new generator T that commutes with all others and $T^2 = 1$, the relations of the original group are then put equal to T instead of one.

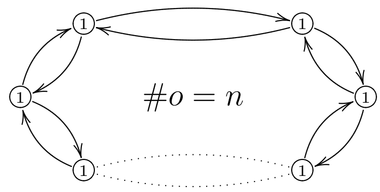
- the binary dihedral case $G = \tilde{D}_{4n} = \langle X, Y, Z | X^2 = Y^m = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary tetrahedral case $G = \tilde{T} = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary octahedral case $G = \tilde{O} = \langle X, Y, Z | X^3 = Y^4 = Z^2 = XYZ = T, T^2 = 1\rangle$
- the binary dodecahedral case $G = I = \langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$

If $\tilde{G}$ has an odd number of elements then it is isomorphic to its image which must be cyclic. $\square$

1.3 Character tables and McKay Quivers

We will now determine the character tables of these subgroups of SL_2. We will need these to determine the rings of invariants $\mathbb{C}[V]^G$. In order to do this more easily we will associate also to each group a combinatorial object: The McKay quiver. This is a directed graph of which the vertices correspond to the simple representations of G and the number of arrows from S_i to S_J is the multiplicity of S_j inside $V \otimes S_i$. Inside the vertices we will put the dimension of the representations.

- $\mathcal{C}_n$ is a cyclic group generated by g. It has n simple one-dimensional representations corresponding to the n roots of 1. $\chi_{S_k}(g) = e^{2k\pi i/n}$. $V = S_1 \oplus S_{n-1}$ and $S_i \otimes S_j = S_{i+j}$ where the sum is modulo n. The McKay quiver looks like:



The other groups are products of $\mathcal{C}_2$ and subgroups of SO_3 so their characters are also products of characters of those groups and characters χ_{S_0}, χ_{S_1}.

- The binary dihedral group can be rewritten as $\langle g, s | g^{2n} = 1, s^2 = g^n, gs = sg^{-1}\rangle$. The one-dimensional representations must map $g \mapsto \epsilon$ and $s \mapsto \pm\sqrt{\epsilon^n}$ with $\epsilon = \pm 1$, therefore there are 4 one-dimensional representations. We will denote these representations by $\sigma_{\pm}^{\pm}$ where the lower index is the sign of g and the upper the sign of s.

  For every $0 \leq k \leq n$ one can define the two-dimensional representation

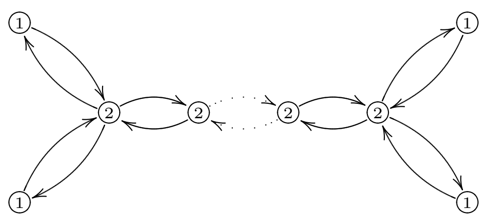
  $$\rho_i : g \mapsto \begin{pmatrix} e^{\pi k/n} & 0 \\ 0 & e^{-\pi k/n} \end{pmatrix}, s \mapsto \begin{pmatrix} 0 & i^k \\ i^k & 0 \end{pmatrix}.$$

  Using the characters one can check that $\rho_0 = \sigma_+^+ \oplus \sigma_+^-$ and $\rho_n = \sigma_-^+ \oplus \sigma_+^-$, all the other representations are simple and non-isomorphic. As $4n = 4 \cdot 1^2 + (n-1) \cdot 2^2$ we now know all simple representations.

  The tensor products of these representations are given by

  $$\begin{aligned} \rho_1 \otimes \rho_j &= \rho_{i-1} \oplus \rho_{i+1} \\ \rho_1 \otimes \sigma_+^{\pm} &= \rho_1 \\ \rho_1 \otimes \sigma_-^{\pm} &= \rho_{n-1} \end{aligned}$$

  This makes our McKay quiver look like



- The binary tetrahedral group $\langle X, Y, Z | X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$ has 3 one-dimensional representations mapping Z, T to 1 and X to $e^{2\pi k/3}$, denote these by σ_k. The simple two-dimensional representations look like $\rho_i := \sigma_i \otimes \rho_V$ where ρ_V is the standard representation. Finally the 3-dimensional representation τ coming from the symmetries of the tetrahedron is also simple. There are no more simple representations because

  $$24 = 3 \cdot 1^2 + 3 \cdot 2^2 + 3^3.$$

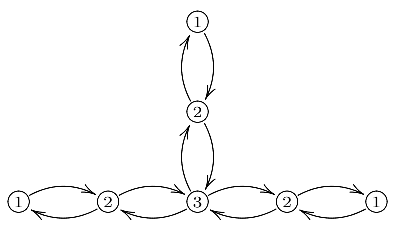
  the formulas for the tensor products are

  $$\begin{aligned} \rho_i \otimes \rho_V &= \sigma_i \oplus \tau \\ \tau \otimes \rho_V &= \rho_V \oplus \rho_1 \oplus \rho_2. \end{aligned}$$

So the character table is

σ_0	1	1	1	1	1	1	1
σ_1	1	$e^{2\pi/3}$	$e^{-2\pi/3}$	$e^{2\pi/3}$	1	$e^{-2\pi/3}$	1
σ_2	1	$e^{-2\pi/3}$	$e^{2\pi/3}$	$e^{-2\pi/3}$	1	$e^{2\pi/3}$	1
ρ_0	2	-1	-1	1	-2	1	0
ρ_1	2	$-e^{-2\pi/3}$	$-e^{2\pi/3}$	$e^{-2\pi/3}$	-2	$e^{2\pi/3}$	0
ρ_2	2	$-e^{2\pi/3}$	$-e^{-2\pi/3}$	$e^{2\pi/3}$	-2	$e^{-2\pi/3}$	0
τ	3	0	0	0	3	0	-1

and the McKay quiver looks like



- The binary octahedral group $\langle X, Y, Z | X^3 = Y^4 = Z^2 = XYZ = T, T^2 = 1\rangle$ has 2 one-dimensional representations mapping X, T to 1 and Y to $(-1)^k$, denote these by σ_k. The standard representation ρ_V and its tensor product with σ_1 give two simple 2-dimensional representations. The same holds for the 3-dimensional representation τ coming from the symmetries of the octahedron. We denote these four representations by $\rho_i = \rho_V \otimes \sigma_i, \tau_i = \tau \otimes \sigma_i$.

  The tensor product $\tau \otimes \rho_V$ decomposes as a direct sum of ρ_V and a four-dimensional simple representation ν. Finally we can construct another 2-dimensional representation ϱ as a summand of $\nu \otimes \rho_V = \varrho \oplus \tau_0 \oplus \tau_1$. There are no more simple representations because

$$48 = 2 \cdot 1^2 + 3 \cdot 2^2 + 2 \cdot 3^3 + 4^2.$$

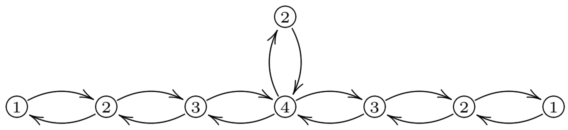
  the formulas for the tensor products are

$$\begin{aligned} \rho_i \otimes \rho_V &= \tau_i \oplus \sigma_i \\ \tau_i \otimes \rho_V &= \nu \oplus \rho_i \\ \varrho \otimes \rho_V &= \nu \end{aligned}$$

Giving a character table

σ_0	1	1	1	1	1	1	1	1
σ_1	1	1	1	1	-1	-1	-1	1
ϱ	2	-1	-1	2	0	0	0	2
ρ_0	2	-1	1	-2	0	$\sqrt{2}$	$-\sqrt{2}$	0
ρ_1	2	-1	1	-2	0	$-\sqrt{2}$	$\sqrt{2}$	0
τ_0	3	0	0	3	1	-1	-1	-1
τ_1	3	0	0	3	-1	1	1	-1
ν	4	1	-1	-4	0	0	0	0

and a McKay quiver like



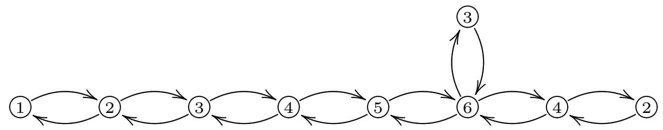
- The binary icosahedral group $\tilde{I} = \langle X, Y, Z | X^3 = Y^5 = Z^2 = XYZ = T, T^2 = 1\rangle$. Up to conjugation $\tilde{I}/\{1, T\}$ can identified with A_5 in 2 different ways. A_5 has only one one-dimensional representation: the trivial. It has also 2 3-dimensional representations τ_1, τ_2 as symmetry groups of the icosahedron identifying Y with a rotation over either $\pi/5$ or $2\pi/5$. These 2 can be pulled back two two-dimensional representations σ_1, σ_2 of $\tilde{I}$ using the map $SU_2 \to SO_3$. Out of these we can construct 2 four-dimensional ν_1, ν_2, one 5-dimensional, μ, and one 6-dimensional, ζ, representations by the equations.

$$\begin{aligned} \nu_1 \oplus \mu &= \tau_1 \otimes \tau_2 \\ \nu_2 \oplus \rho_1 &= \rho_1 \otimes \tau_1 \\ \zeta &= \rho_1 \otimes \tau_2 \end{aligned}$$

This gives us the following Character table

σ	1	1	1	1	1	1	1	1	1
ρ_1	2	-1	1	-2	0	$\frac{1+\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
ρ_2	2	-1	1	-2	0	$\frac{1-\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
τ_1	3	0	0	3	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
τ_2	3	0	0	3	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
ν_1	4	1	1	4	0	-1	-1	-1	-1
ν_2	4	1	-1	-4	0	1	-1	-1	1
μ	5	-1	-1	5	1	0	0	0	0
ζ	6	0	0	-6	0	-1	1	1	-1

and McKay Quiver



1.4 GAP instruction sheet

1. GAP is case-sensitive and instructions end with a semicolon.
2. Assignments are done by :=, comparisons by =.
3. A list is given by [`a`, `b`, `c`], a list of natural numbers from one to 10 is written as [`1..10`]. The i^{th} item of a list l is `l`[`i`].
4. A matrix is given by a list of its rows: [[`a11`, `a12`], [`a21`, `a22`]].
5. The loops and conditional clauses are written as
   (a) `for` variable`in` list`do` instructions`od;`
   (b) `while` condition`do` instructions`od;`
   (c) `if` condition`then` instructions`else` instructions`od;`
6. a function is defined as:
   name `:=` `function`(variables)
   `local` variables;
   instructions;
   `return` expression;
   `end;`
7. Gap can work over different fields, the standard one is the field of rational numbers. To define other numbers one can use the expressions `E`(`n`) and `ER`(`n`) which stand for the expressions $e^{2\pi/n} = \cos 2\pi/n + i \sin 2\pi/n$ and $\sqrt{n}$. These elements live in the field `CF`(`n`). To find square roots of other elements one can use `RootsOfUPol`(`F`, `p`) (this gives a list of roots of the polynomial p in the field F), provided you take your field big enough.

   To take roots of other numbers, one must use `x := indeterminate(Rationals,"x");`
   `pol := x^ 2 - x -1;`
   `phi := RootsOfUPol(CF(5),p);`
   If one must take roots of elements that are not in the rationals one must take instead of `Rationals` the field over which the coefficients of the minimal polynomial are defined.
8. To define a group generated by matrices one can use the command `G := Group`(matrix1,matrix2,$\cdots$); The character table can be constructed by the command `t := CharacterTable(G);`. It can be displayed by `Display(t);` and the characters of the simples can be put in a list by `Irr`(`G`).

Chapter 2

Affine Quotients

2.1 Review of Algebraic Geometry

In algebraic geometry one studies the connections between algebraic varieties, which are sets of solutions of polynomial equations, and complex algebras.

An affine variety is a subset $X \subset \mathbb{C}^n$ that is defined by a finite set of polynomial equations.

$$X := \{x \in \mathbb{C}^n | f_1(x) = 0, \ldots, f_k(x) = 0\}$$

A morphism between two varieties $X \in \mathbb{C}^n$ an $Y \in \mathbb{C}^m$ is a map $\phi : X \to Y$ such that there exist a polynomial map $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ such that $\phi = \Phi|_X$. Such a morphism is an isomorphism if ϕ is invertible and ϕ^{-1} is also a morphism.

The affine varieties with their morphisms form a category which we will denote by AffV.

We can consider $\mathbb{C}$ as a variety, so it makes sense to look at the morphisms from a variety X to $\mathbb{C}$, these maps are also called the regular functions on X. They are closed under point wise addition and multiplication so they form a commutative $\mathbb{C}$-algebra: $\mathbb{C}[X]$.

This algebra can be described with generators and relations. To every variety $X \in \mathbb{C}^n$ the set of polynomial functions that are zero on X form an ideal in $\mathbb{C}[x_1, \cdots, x_n]$. If we divide out this ideal we get the ring of polynomial functions on X.

$$\mathbb{C}[X] := \mathbb{C}[x_1, \ldots, x_n]/(f|\forall x \in X : f(x) = 0)$$

This algebra is finitely generated by the x_i and it also has no nilpotent elements because $f(x)^n = 0 \Rightarrow f(x) = 0$.

A morphism between varieties, $\phi : X \to Y$, will also give an algebra morphism between the corresponding rings but the arrow will go in the opposite direction:

$$\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X] : g \mapsto g \circ \phi.$$

On the other hand if R is a finitely generated commutative $\mathbb{C}$-algebra without nilpotent elements, by definition we will call this an affine algebra. The category of affine algebras together with algebra morphisms will be denoted by AffA.

Every $R \in \mathsf{AffA}$ can be written as a quotient of a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ with an ideal $\mathfrak{i}$. Because polynomial rings are Noetherian, $\mathfrak{i}$ is finitely generated by f.i. $f_1, \dots, f_k$. Therefore we can associate to R the variety $V(R)$ in $\mathbb{C}^n$ defined by the f_i. Although this variety depends on the choice of generators of R and $\mathfrak{i}$, one can prove that different choices will give isomorphic varieties. One can do even more if $\varphi : R \to S$ is an algebra morphism and $R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{i}$ and $S = \mathbb{C}[y_1, \cdots, y_m]/\mathfrak{j}$ one can find polynomials $g_1, \dots, g_n \in \mathbb{C}[y_1, \cdots, y_m]$ such that $g_i + \mathfrak{j} = \varphi(x_i) + \mathfrak{j}$. These functions define a map $G : \mathbb{C}^m \to \mathbb{C}^n$ such that $G(V(S)) = V(R)$ so $G|_V(S) : V(S) \to V(R)$ is a morphism of varieties. So an algebra morphism gives rise to a morphism of varieties in the opposite direction.

The main theorem of algebraic geometry now states that the operations $V(-)$ and $\mathbb{C}[-]$ are each other's inverses:

Theorem 2.1. *The category* AffV *and the category* AffA *are anti-equivalent. So working with affine varieties is actually the same as working with affine algebras but all maps are reversed. The anti-equivalence is given by the contravariant functors* $V(-)$ *and* $\mathbb{C}[-]$*, so*

$$\mathbb{C}[V(R)] \cong R \text{ and } V(\mathbb{C}[X]) \cong X$$

One can also give a more intrinsic description of $V(R)$. For every point $p \in V(R)$ on can look at the embedding $p \to V(R)$. From the algebraic point of view this will give a map from $R \to \mathbb{C}[p] = \mathbb{C}$, so points correspond to maps from R to $\mathbb{C}$ which are determined by their kernels. As $\mathbb{C}$ is an algebraicly closed field these kernels correspond to the maximal ideals of R. So we can also define $V(R)$ as the set of all maximal ideals of R.

This last definition only describes $V(R)$ as a set. We want to give $V(R)$ some more structure. This can be done by introducing the Zariski Topology. This topology can be defined by its closed sets: $C \subset V(R)$ is closed if there is an ideal $\mathfrak{c} \lhd R$ such that $C = \{\mathfrak{m} \in V(R) | \mathfrak{c} \subset \mathfrak{m}\}$. Now if $R = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{i}$ we can see $\mathfrak{c}$ as generated by polynomials (c_i) so the points in $V(R)$ that lie on C are exactly those for which the c_i are zero. So closed sets are subset that correspond to zeros of polynomial functions. Every closed set C will give us a morphism

$R \to \mathbb{C}[C] \cong R/\mathfrak{c}$ which is a surjection. Conversely every surjection $R \to S$ will give us an embedding of a closed subset $V(S)$ in $V(R)$.

Open functions on the other hand are unions of subsets for which certain polynomials are nonzero. Contrarily to closed subsets, open subsets can not always be considered as affine varieties. F.i. in C^2 the complement of the origin is an open subset but it isomorphic to an affine variety. Basic open sets which are set on which one polynomial f does not vanish can be considered as the variety corresponding to the ring $R[1/f]$. This construction is called a localization

The Zariski topology is not the same as the ordinary complex topology on $V(R) \subset \mathbb{C}^n$. The ordinary topology has lots more closed (open) sets. For instance a closed ball with finite radius around a point is closed in the ordinary topology of $\mathbb{C}^n$, but not in the Zariski topology because the zeros of a polynomial form a hypersurface in C^n and hypersurfaces never contain closed balls.

The translation table

Geometry	Algebra
Affine Variety	Affine Algebra
Morphism	Algebra Morphism
Point	Maximal Ideal
Closed Set	Semiprime Ideal (i.e. $f^n \in \mathfrak{i} \Rightarrow f \in \mathfrak{i}$)
Intersection closed Sets	Sum of ideals
Union of closed sets	Intersection of ideals
Basic Open Set	Localization of a function
Embedding of a closed subvariety	Surjection
The image is dense	Injection
Irreducible (open sets always intersect)	No zero divisors
Connected	does not contain idempotents
Dimension	longest ascending chain of prime ideals
Tangent space in p	$(\mathfrak{m}/\mathfrak{m}^2)^*$

2.2 Quotients and rings of invariants

Suppose now we have a reductive group G and let Ω be the set of its simple representations up to isomorphism. Let V be a finite dimensional representation with dimension k. V can also be considered as a variety. for every point $v \in V$ we can define the orbit $\mathsf{G} \cdot v := \{g \cdot v | g \in \mathsf{G}\}$. Orbits never intersect so we can partition V into its orbits. We will denote the set of all orbits by V/G.

A natural question one can ask if whether this set can also be given the structure of an affine variety. In the case of finite groups it will be possible, but for general reductive groups there will be extra complications.

We can take a closer look at the problem by looking at the algebraic side of the story. The ring of polynomial functions over V is $R = \mathbb{C}[V] \cong \mathbb{C}[X_1, \dots, X_k]$ is a graded polynomial ring if we give the X_i degree 1.

On R we have an action of G:

$$\mathsf{G} \times \mathbb{C}[V] \to \mathbb{C}[V] : (g, f) \mapsto g \cdot f := f \circ \rho_V(g^{-1}).$$

This action is linear and compatible with the algebra structure: $g \cdot f_1 f_2 = (g \cdot f_1)(g \cdot f_2)$. As $g \cdot X_i : \sum_j \rho_V(g^{-1})_{ij} X_j$ is homogeneous of degree 1 the G-action maps homogeneous elements of to homogeneous elements with the same degree. This means that all homogeneous components R_κ are finite dimensional rational representations of G.

We can decompose every R_κ as a direct sum of simple representations

$$R_\kappa = \bigoplus_{\omega \in \Omega} W_\kappa^\omega \text{ with } W_\kappa^\omega \cong \omega^{\oplus e_{\kappa\omega}}.$$

If we define then the *isotopic components* of R as $R^\omega = \oplus_\kappa W_\kappa^\omega$. Now we can regroup the terms in the direct sum of our ring to obtain

$$R \cong \bigoplus_{\kappa=0, \omega \in \Omega}^{\infty} W_\kappa^\omega = \bigoplus_{\omega \in \Omega} R^\omega.$$

In words, the ring R is the direct sum of its isotopic components. Note also that if α is a endomorphism of R as a G-representation then α will map isotopic components inside themselves because of Schur's lemma (try to prove this as an exercise).

One isotopic component interests us specially: the isotopic component of the trivial representation $\mathbb{1}$. This component consists of all functions that are invariant under the G-action. It is not only a vector space but it is also a graded ring: the ring of invariants $S = R^{\mathsf{G}} = \{f \in R | g \cdot f = f\} = \oplus_{\kappa \in \mathbb{N}} W_\kappa^{\mathbb{1}}$.

Consider a function $f \in S$. The map $\mu_f : R \to R : x \to fx$ is an endomorphism of R as a representation: $\mu_f(g \cdot x) = f(g \cdot x) = (g \cdot f)(g \cdot x) = g \cdot fx = g \cdot \mu_f x$. So $fR^\omega \subset R^\omega$ for every ω. Put in another way we can say that all isotopic components are S-modules.

We are now ready to prove the main theorem:

Theorem 2.2. *If G is a reductive group and V a finite dimensional representation then the ring of invariants $S = \mathbb{C}[V]^{\mathsf{G}}$ is finitely generated.*

Proof. To prove that S is finitely generated we first prove that this ring is noetherian. Suppose that

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$$

is an ascending chain of ideals in S. Multiplying with R we obtain a chain of ideals in R:

$$\mathfrak{a}_1 R \subset \mathfrak{a}_2 R \subset \mathfrak{a}_3 R \subset \cdots .$$

This chain is stationary because R is a polynomial ring and hence noetherian. Finally we show that $\mathfrak{a}_i R \cap S = \mathfrak{a}_i$. Multiplication with $\mathfrak{a}_i$ maps the isotopic components into themselves so

$$(\mathfrak{a}_i R) \cap S = \left(\mathfrak{a}_i \bigoplus_{\omega \in \Omega} R^\omega\right) \cap S = \left(\bigoplus_{\omega \in \Omega} \mathfrak{a}_i R^\omega\right) \cap S = \mathfrak{a}_i R^1 = \mathfrak{a}_i S = \mathfrak{a}_i.$$

Now let $S_+ = \oplus_{\kappa \geq 1} W_\kappa^1$ denote the ideal of S generated by all homogeneous elements of nonzero degree. Because S is Noetherian, S_+ is generated by a finite number of homogeneous elements: $S_+ = f_1 S + \cdots + f_r S$. We will show that these f_i also generate S as a ring.

Now $S = \mathbb{C} + S_+$ so $S_+ = \mathbb{C}f_1 + \cdots + \mathbb{C}f_r + S_+^2$, $S_+^2 = \sum_{i,j} \mathbb{C}f_i f_j + S_+^3$ and by induction

$$S_+^t = \sum_{i_1 \ldots i_t} \mathbb{C}f_{i_1} \cdots f_{i_t} + S_+^{t+1}.$$

So $\mathbb{C}[f_1, \ldots, f_r]$ is a graded subalgebra of S and $S = \mathbb{C} + S_+ = \mathbb{C}[f_1, \ldots, f_r] + S_+^t$ for every t. If we look at the degree d-part of this equation we see that

$$S_d = \mathbb{C}[f_1, \ldots, f_r]_d + (S_+^t)_d.$$

Because S_+^t only contains elements of degree at least t, $(S_+^t)_d = 0$ if $t > d$. As the equation holds for every t we can conclude that

$$S_d = \mathbb{C}[f_1, \ldots, f_r]_d \text{ and thus } S = \mathbb{C}[f_1, \ldots, f_r]$$

□

Now because S is finitely generated and does not have nilpotent elements, it corresponds to a variety $V(S)$ and the embedding $S \subset R$ gives a

$$\pi : V(R) \to V(S) : \mathfrak{m} \mapsto \mathfrak{m} \cap S$$

The map π is a projection because if $\mathfrak{s}$ is a maximal ideal in S then $\mathfrak{s}R$ is not equal to R because $\mathfrak{s}R \cap S = \mathfrak{s} \neq S$. Therefore $\mathfrak{s}R$ will be contained in a maximal ideal $\mathfrak{m} \lhd R$ (there may be more) so $\pi(\mathfrak{m}) = \mathfrak{s}$.

Furthermore if $\mathfrak{m} \lhd R$ then $\mathfrak{m} \cap S = g \cdot \mathfrak{m} \cap S$ so points of $V(R)$ in the same orbit are mapped to the same point in $V(S)$.

The reverse implication is however not true, points that are mapped to the same point do not need to lie in the same orbit. Because π is a continuous map $\pi^{-1}(x)$ must be a closed subset, so if there exists an orbit $\mathcal{O}$ that is not closed and $w = \pi(\mathcal{O})$ we know that $\pi^{-1}(w)$ must contain points outside $\mathcal{O}$. Note that if G is finite then this problem does not occur because all orbits contain only a finite number of points and are hence closed.

The projection map π has a special property. Suppose $\psi : V(R) \to X$ is a map to another variety that is constant on orbits ($\psi(gv) = \psi(v)$) then the corresponding $\psi^* : \mathbb{C}[X] \to R$ maps $\mathbb{C}[X]$ inside S. This means that we have a map τ from $V(S)$ to X such that $\psi = \tau \circ \pi$.

$$\begin{tikzcd} V(R) \arrow[r, "\psi"] \arrow[d, "\pi"'] & X \\ V(S) \arrow[ur, "\exists\tau"'] & \end{tikzcd}$$

This means that π is the closest we can get to a quotient in the category of varieties, therefore we will call π a categorical quotient and we will denote $V(S)$ by $V /\!\!/ \mathsf{G}$.

Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are two closed disjoint orbits. These correspond to two ideals $\mathfrak{o}_1, \mathfrak{o}_2 \lhd R$ the fact that they are disjoint translates to $\mathfrak{o}_1 + \mathfrak{o}_2 = R$ and the fact that they are orbits to $g \cdot \mathfrak{o}_i = \mathfrak{o}_i$. Their images under the projection must also be disjoint because $\mathfrak{o}_1 \cap S + \mathfrak{o}_2 \cap S = (\mathfrak{o}_1 + \mathfrak{o}_2) \cap S = S$. The inverse image of a point in $V /\!\!/ \mathsf{G}$ contains at most one closed orbit. It also contains exactly one closed orbit: let $\mathcal{O}$ be an orbit in $\pi^{-1}(p)$ with minimal dimension then this must be closed because otherwise $\bar{\mathcal{O}} \setminus \mathcal{O}$ would consist of orbits of smaller dimension.

We can summarize all this in a theorem

Theorem 2.3. *If V is a finite dimensional representation of a reductive group, then there exists a unique variety $V /\!\!/ \mathsf{G} = V(\mathbb{C}[V]^{\mathsf{G}})$ such that*

1. *The points are in one-to-one correspondence with the closed orbits in V.*

2. *The projection $V \to V /\!\!/ \mathsf{G}$ is a categorical quotient.*

3. *If G is finite then as a set $V /\!\!/ \mathsf{G} = V/\mathsf{G}$.*

2.3 Kleinian singularities

In this section we will determine generators and relations for the rings of invariants $\mathbb{C}[V]^G$.

Theorem 2.4. *Every Kleinian singularity is generated by three invariants.*

Proof. In order to prove this we define a map $\varrho : \mathbb{C}[V] \to \mathbb{C}[V]^G$, the Reynolds operator,

$$\varrho(f) = \frac{1}{|G|} \sum_{g \in G} f^g.$$

This map is a projection $\varrho^2 = \varrho$ and it is the identity operation on $\mathbb{C}[V]^G$. So to get a basis for the ring of invariants we can look at the set of images of all the monomials in $\mathbb{C}[V]$.

$$\varrho X^i Y^j.$$

We will now consider the different types

A_n If g is the generator of the cyclic group then $g \cdot X = \zeta X, g \cdot Y = \zeta^{-1} Y$ with $\zeta = e^{2\pi/k}$. Therefore

$$\begin{aligned} \varrho X^i Y^j &= \sum_{k=1}^{n} g^k \cdot X^i Y^j \\ &= \sum_{k=1}^{n} \zeta^{k(i-j)} X^i Y^j \\ &= \begin{cases} 0 & i \neq j \mod n \\ n X^i Y^j & i = j \mod n \end{cases}. \end{aligned}$$

From this one can deduce that all invariants are generated by X^n, Y^n and XY.

D_n The elements of this group can be written as

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^l s^l g^k = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}^k \quad 1 \leq k \leq 2n, 1 \leq l \leq 4$$

Therefore

$$\begin{aligned}
\varrho X^i Y^j &= \sum_{k=1}^{2} ng^k \cdot X^i Y^j + sg^k \cdot X^i Y^j \\
&= \sum_{k=1}^{n} \zeta^{k(i-j)} (X^i Y^j + i^{i+j} X^j Y^i) \\
&= \begin{cases} 0 & i \neq j \mod 2n \text{ or} \\ X^i Y^j - Y^j X^i & i = j \mod 2n \text{ and } i \text{ is odd} \\ X^i Y^j + Y^j X^i & i = j \mod 2n \text{ and } i \text{ is even} \end{cases}.
\end{aligned}$$

From this one can deduce that all invariants are generated by X^2Y^2, $XY(X^{2n} - Y^{2n})$ and $X^{2n} + Y^{2n}$.

E_6 The group is generated by two matrices of order 6:

$$\begin{pmatrix} \frac{i+1}{2} & -\frac{i+1}{2} \\ \frac{-i+1}{2} & \frac{-i+1}{2} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{i+1}{2} & \frac{i+1}{2} \\ -\frac{-i+1}{2} & \frac{-i+1}{2} \end{pmatrix}$$

Using calculations in GAP we can find 3 generators: $XY(X^4 - Y^4)$, $(X^4 - Y^4)^2 + 16X^4Y^4$, $(X^4 + Y^4)^3 - 36X^4Y^4(X^4 + Y^4)$.

E_7 The group is generated by $E6$ and a matrix of order 8:

$$\begin{pmatrix} \frac{\sqrt{2}+\sqrt{2}i}{2} & 0 \\ 0 & \frac{\sqrt{2}-\sqrt{2}i}{2} \end{pmatrix}$$

Therefore we can express the invariants of $E7$ in terms of those of $E6$: $(XY(X^4 - Y^4))^2$, $(X^4 - Y^4)^2 + 16X^4Y^4$, $X^{17}Y - 34X^{13}Y^5 + 34X^5Y^{13} - XY^{17}$.

E_8 The group is generated by $E7$ and a matrix of order 10:

$$\begin{pmatrix} \frac{\sqrt{2}+\sqrt{2}i}{2} & 0 \\ 0 & \frac{\sqrt{2}-\sqrt{2}i}{2} \end{pmatrix}$$

□

Theorem 2.5. *The ideal of relations between the 3 generators of the ring of invariants is generated by one element. These relations are*

- A_n $XY + Z^n$,
- D_n $X^{n+1} - XY^2 + Z^2$,

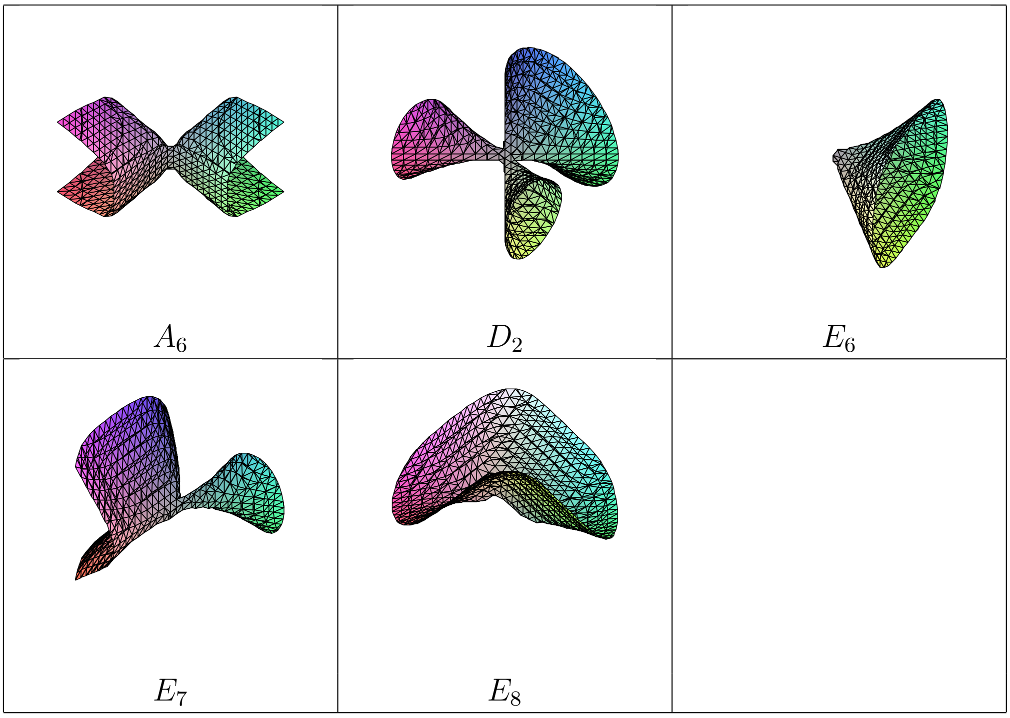
E_6 $X^4 + Y^3 + Z^2$,

E_7 $X^3Y + Y^3 + Z^2$,

E_8 $X^5 + Y^3 + Z^2$.

Proof. The dimension of the quotient space must be two because the map $\mathbb{C}^2 \to V/G$ has finite fibers. If the ideal would be generated by more than one generator $\mathbb{C}[X, Y, Z]/\mathfrak{p}$, its corresponding variety would not be two-dimensional. The exact relation can be easily deduced from the generators above. $\square$

To give you a flavor of what these quotient varieties look like, we include some graphs of the real parts of the equations.



Chapter 3

Smash Products and Preprojective Algebras

In this chapter we will associate several noncommutative algebras with these Kleinian singularities

3.1 Cayley-Hamilton Algebras

From linear algebra we recall that every $n \times n$-matrix A satisfies its characteristic polynomial

$$\chi_A(X) := \det(A - X1)$$

the coefficients of this polynomial can be expressed as symmetric functions of the eigenvalues of A, (λ_i)

$$\chi_A(X) = X^n - \sum_i \lambda_i X^{n-1} + \cdots + (-1)^k \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n.$$

We can rewrite these coefficients in function of the traces of the powers of A using the expressions $\mathsf{Tr} A^k = \sum_i \lambda_i^k$.

$$\chi_A(X) = X^n - \mathsf{Tr} A X^{n-1} + \frac{(\mathsf{Tr} X)^2 - \mathsf{Tr} X^2}{2} + \cdots \quad (*)$$

In this way we have found an expression of the characteristic polynomial using only traces.

Now we will expand this notion of characteristic polynomials to a broader class of algebras. If $\mathcal{A}$ is a finitely generated algebra with finitely generated center $\mathcal{Z}(A)$ then a trace function is a linear map $\mathcal{A} \to \mathcal{Z}(A)$ satisfying the conditions

- $\forall a, b \in \mathcal{A} : \mathsf{Tr}(ab) = \mathsf{Tr}(ba)$,
- $\forall a, b \in \mathcal{A} : \mathsf{Tr}(\mathsf{Tr}(a)b) = \mathsf{Tr}(a)\mathsf{Tr}(b)$.

For these algebras it makes sense to define a characteristic polynomial of order n $\chi_a^n(X) \in \mathcal{Z}(A)[X]$ using the expression $(*)$. Now we will call $\mathcal{A}$ a Cayley-Hamilton Algebra of order n if and only if $\mathsf{Tr}1 = n$ and every element satisfies its own characteristic polynomial of order n: $\chi_a^n(a) = 0$.

A morphism between n^{th} Cayley-Hamilton Algebras will be an algebra morphism $\phi : \mathcal{A} \to \mathcal{B}$ that commutes with the trace: $\mathsf{Tr}_B \circ \phi = \phi \circ \mathsf{Tr}_A$. We can use these morphisms to turn the class of n^{th} Cayley-Hamilton Algebras into a category: CH_n.

The importance of these Cayley-Hamilton Algebras is that the allow a nice generalization of the algebra-geometry correspondence to non-commutative algebras.

CH_n has an object of special importance: the matrix-algebra $M_n = \mathsf{Mat}_{n\times n}(\mathbb{C})$. We will call a morphism of $\mathcal{A}$ to M_n a trace preserving representation.

Let A be a CH-algebra defined by a finite number of generators and relations. This means that we can write A as a quotient of a free algebra:

$$A \cong \mathbb{C}\langle Y_1, \dots, Y_k\rangle/\mathcal{R} \text{ with } \mathcal{R} = (r_1, \dots, r_l).$$

We will write the generators of A as $y_i := Y_i \mod \mathcal{R}$. We can also If $\mathcal{A}$ is generated by $y_1, \dots, y_k$ then we define we can associate to every trace preserving representation ρ a point in $\mathbb{C}^{n^2k} = (M_n)^k$: $(\rho(y_1), \dots, \rho(y_k))$. The set of all points corresponding to a trace preserving representation will be denoted by $\mathsf{trep}_n\mathcal{A}$.

On the other hand if we have a k-tuple of matrices $(A_1 \dots, A_k)$ that satisfies the relations $r_1, \dots, r_k$ and the trace relations $\mathsf{Tr}w(A_1 \dots, A_k) = [\mathsf{Tr}w(y_i)](A_1 \dots, A_k)$ we can construct a morphism

$$\rho : A \to \mathsf{Mat}_{n\times n}(\mathbb{C}) : y_i \mapsto A_i$$

This implies that we can consider $\mathsf{trep}_n\mathcal{A}$ as the closed subset of $\mathbb{C}^{n^2k}$ where the functions

$$\begin{aligned} f_{mij} &: \mathbb{C}^{n^2k} \to \mathbb{C} : (A_1 \dots, A_k) \mapsto [r_m(A_1 \dots, A_k)]_{ij} \\ t_{wij} &: \mathbb{C}^{n^2k} \to \mathbb{C} : (A_1 \dots, A_k) \mapsto [\mathsf{Tr}w(y_i)](A_1 \dots, A_k)_{ij} - \mathsf{Tr}w(A_1 \dots, A_k)\delta_{ij} \end{aligned}$$

are zero ($[r_m(A_1 \dots, A_k)]_{ij}$ is the coefficient on the i^{th} and the j^{th} column of the matrix $r_m(A_1 \dots, A_k)$). However, the ideal $\mathfrak{n}_{\mathcal{A}}$ generated by the f_{mij}, t_{wij} is not necessarily semiprime, so it is not always that $\mathbb{C}[\mathsf{trep}_n A] \cong \mathbb{C}[M_n^k]/\mathfrak{n}_\mathcal{A}$ but we have that

$$\mathbb{C}[\mathsf{trep}_n\mathcal{A}] \cong \mathbb{C}[M_n^k]/\sqrt{\mathfrak{n}_\mathcal{A}} \text{ with } \sqrt{\mathfrak{n}_\mathcal{A}} = \{a|\exists\ell : a^\ell \in \mathfrak{n}_\mathcal{A}\}.$$

So $\mathsf{trep}_n A$ can be considered as a variety and as such it is independent of the choice of generators of $\mathcal{A}$. We will call it the representation variety of $\mathcal{A}$. $\mathcal{A}$ will be called reduced if $\mathfrak{n}_\mathcal{A} = \sqrt{\mathfrak{n}_\mathcal{A}}$.

On the other hand we can construct a Cayley-Hamilton Algebra for every closed subset of $X \subset (M_n)^k$ closed under the GL_n-action: Denote by $\mathbb{M}[M_n^k]$ the subspace of $M_n \otimes \mathbb{C}[M_n^k]$ all polynomial maps

$$\phi : M_n^k \to M_n \text{ such that } \phi(g \cdot (M_1, \dots, M_k)) = g\phi(M_1, \dots, M_k)g^{-1}.$$

This space is an algebra using the pointwise matrix multiplication and it has a trace function $(\mathsf{Tr} f)(M_1, \cdots, M_k) = \mathsf{Tr} f(M_1, \cdots, M_k)$. It is easy to check that it is in fact a Cayley-Hamilton Algebra of order n.

For X we can define the subset $\mathfrak{n}_X = \{\phi \in \mathbb{M}[M_n^k] | \forall x \in X : f(x) = 0\}$ This subset is in fact an ideal and $\mathsf{Tr}\mathfrak{n}_X \subset \mathfrak{n}_X$ so

$$\mathbb{M}[X] := \mathbb{M}[M_n^k]/\mathfrak{n}_X$$

is also a Cayley-Hamilton Algebra of order n.

The expected generalization of the commutative setting would be that $X \mapsto \mathbb{M}[X]$ and $\mathcal{A} \mapsto \mathsf{trep}_n\mathcal{A}$ are inverses of each other giving an equivalence of categories between CH_n and $\mathsf{GL}_n - \mathsf{AffV}$. This is only partly true

Theorem 3.1. *For every reduced Cayley-Hamilton Algebra $\mathcal{A}$, we have that*

$$\mathbb{M}[\mathsf{trep}_n\mathcal{A}] \cong \mathcal{A}.$$

This means that reduced Cayley-Hamilton Algebras are completely described by their representation variety.

The converse is not true: there are non-isomorphic GL_n-varieties giving the same CH-algebra. If $\mathcal{A}$ is not reduced the theorem still holds if we redefine $\mathbb{M}[\mathsf{trep}_n\mathcal{A}]$ as $\mathbb{M}[M_n^k]/(M_n(\mathfrak{n}_\mathcal{A}) \cap \mathbb{M}[M_n^k])$.

For the GL_n-action on $(M_n)^k$ we can construct the quotient $\pi : (M_n)^k \to (M_n)^k /\!\!/ \mathsf{GL}_n$. Note that if $(M_1, \dots, M_k) \in \mathsf{trep}_n\mathcal{A}$ then also $g \in (M_1, \dots, M_k) \in \mathsf{trep}_n\mathcal{A}$ and one can check that two representations are in the same if and only if they are isomorphic. It makes sense to look at $\pi(\mathsf{trep}_n\mathcal{A})$ we will call this the quotient variety of $\mathsf{trep}_n\mathcal{A}$ and we will denote it $\mathsf{tiss}_n\mathcal{A}$. $\mathbb{C}[\mathsf{tiss}_n\mathcal{A}]$ consist of all GL_n-invariant polynomial functions from $\mathsf{trep}_n\mathcal{A}$ to $\mathbb{C}$ and it is easy to see that it classifies the closed GL_n-orbits in $\mathsf{trep}_n\mathcal{A}$.

We have a nice description of this quotient

Theorem 3.2.

1. $\mathbb{C}[(M_n)^k]^{\mathsf{GL}_n}$ *is generated by functions of the form*
$$(M_1, \cdot, M_k) \mapsto \mathsf{Tr} M_{i_1} \cdots M_{i_l}.$$
where some of the i_j *may be the same and* $0 > l > n^2$*.*

2. $\mathbb{C}[\mathsf{trep}_n\mathcal{A}]^{\mathsf{GL}_n} = \mathsf{Tr}(\mathcal{A})$ *or equivalently* $\mathsf{tiss}_n\mathcal{A} = V(\mathsf{Tr}(\mathcal{A}))$*.*

3. *The points of* $\mathsf{tiss}_n\mathcal{A}$ *are in one to one correspondence with the semisimple trace preserving representations of* $\mathcal{A}$*.*

We will omit the proofs of this theorem.

3.2 Smash Products

Now we will associate to every Kleinian singularity (V, G) an CH-algebra whose quotient variety is isomorphic to $V/\!\!/G$.

As we know from the second lesson, the action of G on V gives rise to an action on the polynomial ring $C[V] \cong \mathbb{C}[X, Y]$. Construct the vector space $\mathbb{C}[V]^{|}G|$ we can identify the standard basis elements with the elements in the group, such that every element of this space can be written uniquely as a sum of $f(X, Y)g$ where $f(X, Y)$ is a polynomial function and g is an element of G. We can now define a product on this vector space

$$f_i(X, Y)g_i \times f_j(X, Y)g_j = (f_i g_i \cdot f_j)g_i g_j,$$

in this expression the $\cdot$ denotes the action of G on $\mathbb{C}[V]$. One can easily check that this product is associative and by linearly extending it to the whole vector space one obtains an algebra: the smash product of $\mathbb{C}[V]$ and G. In symbols we write $\mathbb{C}[V]\#G$. The center of this algebra can be easily determined: if $z = \sum_g f_g g \in \mathcal{Z}$ then

$$\forall f \in \mathbb{C}[V] : [z, f] = f_g(f - g \cdot f)g = 0 \text{ and } \forall h \in g : [z, h] = (h \cdot f_g - f_g)gh$$

The first equation implies that $f_g = 0$ if $g \neq 1$ and the second implies that f_1 must be a G-invariant function so we can conclude that

$$\mathcal{Z}(\mathbb{C}[V]\#G) \cong \mathbb{C}[V]^G.$$

We can also embed this algebra into $M_{|G|}(\mathbb{C}[V])$: if we consider $\mathbb{C}[V]^{|}G|$ as a space of row vectors then multiplication on the right with g can be modeled with a permutation matrix and multiplication with a function f with the diagonal matrix

$$\begin{pmatrix} g_1 \cdot f & & \\ & \ddots & \\ & & g_{|G|} \cdot f \end{pmatrix}.$$

The standard trace on $M_{|G|}(\mathbb{C}[V])$ gives a trace on $\mathbb{C}[V]\#G$ making it a CH-algebra of order $|G|$. For this trace we have that

$$\mathsf{Tr}(\sum f_g g) := \sum \pi(f_g)\mathsf{Tr}g = \sum_{g\in G} g \cdot f_1 \in \mathcal{Z}(\mathbb{C}[V]\#G)$$

The fact that the trace comes from an ordinary matrix trace implies that the CH-identity is satisfied. Note also that the trace maps surjectively onto $\mathbb{C}[V]^G$.

3.3 Quivers

A *quiver* $Q = (Q_0, Q_1, h, t)$ consists of a set of vertices Q_0, a set of arrows Q_1 between those vertices and maps $h, t : A \to V$ which assign to each arrow its head and tail vertex. We also denote this as

$$\textcircled{\scriptsize h(a)} \xleftarrow{a} \textcircled{\scriptsize t(a)}.$$

The *Euler form* of Q is the bilinear form $\chi_Q : \mathbb{Z}^{\#V} \times \mathbb{Z}^{\#V} \to \mathbb{Z}$ defined by the matrix

$$m_{ij} = \delta_{ij} - \#\{a | \textcircled{\scriptsize i} \xleftarrow{a} \textcircled{\scriptsize j}, \}$$

where δ is the Kronecker delta.

A sequence of arrows $a_1 \dots a_p$ in a quiver Q is called a *path of length* p if $t(a_i) = h(a_{i+1})$, this path is called a cycle if $t(a_p) = h(a_1)$. A path of length zero will be defined as a vertex. A quiver is strongly connected if for every couple of vertices (v_1, v_2) there exists a path p such that $s(p) = v_1$ and $t(p) = v_2$.

A *dimension vector* of a quiver is a map $\alpha : Q_0 \to \mathbb{N}$, the size of a dimension vector is defined as $|\alpha| := \sum_{v\in Q_0} \alpha_v$. A couple (Q, α) consisting of a quiver and a dimension vector is called a *quiver setting* and for every vertex $v \in Q_0$, α_v is referred to as the dimension of v. A setting is called *sincere* if none of the vertices has dimension 0. For every vertex $v \in Q_0$ we also define the dimension vector

$$\epsilon_v : V \to \mathbb{N} : w \mapsto \begin{cases} 0 & v \neq w, \\ 1 & v = w. \end{cases}$$

An *α-dimensional complex representation* W of Q assigns to each vertex v a linear space $\mathbb{C}^{\alpha_v}$ and to each arrow a a matrix

$$W_a \in \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

The space of all α-dimensional representations is denoted by $\mathsf{Rep}(Q, \alpha)$.

$$\mathsf{Rep}(Q, \alpha) := \bigoplus_{a \in A} \mathsf{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C})$$

To the dimension vector α we can also assign a reductive group

$$\mathsf{GL}_\alpha := \bigoplus_{v \in V} \mathsf{GL}_{\alpha_\mathsf{v}}(\mathbb{C}).$$

An element of this group, g, has a natural action on $\mathsf{Rep}(Q, \alpha)$:

$$W := (W_a)_{a \in A},\ W^g := (g_{t(a)} W_a g_{s(a)}^{-1})_{a \in A}$$

The quotient of this action will be denoted as

$$\mathsf{iss}(Q, \alpha) := \mathsf{Rep}(Q, \alpha) /\!\!/ \mathsf{GL}_\alpha.$$

Quiver representations can be seen as representations of an algebra, called the path algebra. If we take all the paths, including the one with zero length, as a basis we can form a complex vector space $\mathbb{C}Q$. On this space we can put a noncommutative product, by concatenating paths. By the concatenation of two paths $a_1 \dots a_p$ and $b_1 \dots b_q$ we mean

$$a_1 \dots a_p \cdot b_1 \dots b_q := \begin{cases} a_1 \dots a_p b_1 \dots b_q & s(a_p) = t(b_1) \\ 0 & t(a_p) \neq h(b_1) \end{cases}$$

For a vertex v and a path p we define vp as p if p ends in v and zero else. On the other hand pv is p if this path starts in v and zero else.

The vector space $\mathbb{C}Q$ equipped with this product is called the *path algebra*. The set of vertices $Q_0 = \{v_1, \dots, v_k\}$ forms a set of mutually orthogonal idempotents for this algebra. The subalgebra generated by these vertices is isomorphic to $\mathbb{C}Q_0 = \mathbb{C}^{\oplus k}$ and this is also the degree zero part if we give $\mathbb{C}Q$ a gradation using the length of the paths.

Suppose we have an n-dimensional representation ρ of the path algebra, then we can decompose the vector space $\mathbb{C}^n$ into a direct sum

$$\mathbb{C}_n := \rho(v_1)\mathbb{C}^n \oplus \dots \oplus \rho(v_k)\mathbb{C}^n.$$

Note that $\rho(v_i)$ acts like the identity on $\rho(v_i)\mathbb{C}^n$. Choosing bases in $\rho(v_i)\mathbb{C}^n$ we can associate with every arrow a of Q a matrix W_a corresponding to the map

$$\rho(a)|_{\rho(t(a))\mathbb{C}^n} : \rho(t(a))\mathbb{C}^n \to \rho(h(a))\mathbb{C}^n$$

These matrices give us a quiver representation W with dimension

$$\alpha : Q_0 \mapsto \mathbb{N} : v_i \mapsto \mathsf{Dim}\rho(v_i)\mathbb{C}^n.$$

The identification of ρ with a quiver representation depends on the choice of the base and hence there is still an action of GL_α working on this representation. Making the quotient we can say that every equivalence class of n-dimensional $\mathbb{C}Q$-representations defines uniquely an α-dimensional representation class of Q, for a certain α of size n. This implies that if we take take the quotient in $\mathsf{Rep}_n\mathbb{C}Q$ or in $\mathsf{Rep}(Q,\alpha)$ we get the same. In symbols we have

$$\mathsf{iss}_n\mathbb{C}Q \cong \bigsqcup_{|\alpha|=n} \mathsf{iss}(Q,\alpha).$$

Viewing quiver representations of a quiver as representations of a algebra we easily translate the concepts simple and semisimple representation to the quiver language.

A representation W is called *simple* if the only collections of subspaces $(\mathsf{V}_v)_{v\in V}$, $\mathsf{V}_v \subseteq \mathbb{C}^{\alpha_v}$ having the property

$$\forall a \in A : W_a \mathsf{V}_{s(a)} \subset \mathsf{V}_{t(a)}$$

are the trivial ones (i.e. the collection of zero-dimensional subspaces and $(\mathbb{C}^{\alpha_v})_{v\in V}$).

The direct sum $W \oplus W'$ of two representations W, W' has as dimension vector the sum of the two dimension vectors and as matrices $(W \oplus W')_a := W_a \oplus W'_a$. A representation equivalent to a direct sum of simple representations is called *semisimple*.

From this point of view an orbit of a quiver representation is closed if and only if this representation is semisimple. So one can also consider $\mathsf{iss}(Q,\alpha)$ as the space classifying all semisimple α-dimensional representation classes.

Path algebras of quiver perform a similar function as free algebras (in fact free algebras are path algebras of quivers with one vertex). Every algebra over $\mathbb{C}Q_0 = \mathbb{C}^{\oplus k}$ can be seen as a quotient of a path algebra of a quiver with k vertices. If we want to study the representations of $A = \mathbb{C}Q/\mathcal{I}$, we can see them as representations of $\mathbb{C}Q$ satisfying some relations. Therefore it makes sense to consider the variety of α-dimensional representations of A, this is the closed subset of $\mathsf{Rep}(Q,\alpha)$ satisfying the relations in $\mathcal{I}$. We will denote this variety as $\mathsf{Rep}(A,\alpha)$.

Because this is a closed subset of $\mathsf{Rep}(Q, \alpha)$, the quotient $\mathsf{Rep}(A, \alpha)/\!\!/\mathsf{GL}_\alpha$ can be seen as the image of $\mathsf{Rep}(A, \alpha)$ under the quotient map $\mathsf{Rep}(Q, \alpha) \to \mathsf{iss}(Q, \alpha)$. Again it classifies the semisimple α-dimensional representations up to isomorphism and hence we denote it by $\mathsf{iss}(A, \alpha)$.

3.4 Getting Quiver representations from Smash Products

To go from the smash product to quivers, we recall from the first chapter that

$$\mathbb{C}G = \begin{pmatrix} \mathsf{Mat}_{n_1\times n_1} & & \\ & \ddots & \\ & & \mathsf{Mat}_{n_k\times n_k} \end{pmatrix}$$

Where k is the number of simple representations of G and $n_1, \dots, n_k$ are the dimensions of these representations. Now let e denote the element that corresponds to the matrix having a 1 in the upper left corner for each representation and zeros everwhere else.

$$\begin{pmatrix} 1 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \in \mathbb{C}G.$$

This element is the sum of k idempotent elements with a unique one, we denote these by $e_1, \dots, e_k$. Each of these corresponds to a unique simple representation of G. These representations can be seen as $S_i \cong \mathbb{C}Ge_i$ The e_i also have the property that for a $\mathbb{C}G$-representation W the dimension of the subspace e_iW is the same as the multiplicity of S_i inside W. Another important property is that the ideal generated by e is the full group algebra, $\mathbb{C}Ge\mathbb{C}G = \mathbb{C}G$, this is because matrix algebras have no proper ideals and e has a nonzero value in every matrix component of $\mathbb{C}G$.

Given the algebra $A = \mathbb{C}[V]\#G$, we look at the subspace $\Pi := eAe$. This space is again an algebra but its unit element is now e instead of 1, it is called the *preprojective algebra* of G and V. The preprojective algebra is closely related to A: they have the same representation theory. If W is a representation of A then the subspace eW is a representation of Π. On the other hand if W is a representation of Pi we can turn it into a representation of A by taking the tensor product

$$W_A := A \otimes_\Pi W = Ae \otimes_{eAe} W$$

These two operations are each other's inverse:

$$e(Ae \otimes_{\Pi} W) = eAe \otimes_{eAe} W = W$$
$$Ae \otimes_{\Pi} eW = Ae \otimes_{eAe} eA \otimes_A W = AeA \otimes_A W = A \otimes_A W = W.$$

So instead of looking at the representations of A we can consider representations of Π. Π has the advantage that it is an algebra over $e\mathbb{C}Ge = \mathbb{C}^{\oplus k}$ so it is the quotient of a path algebra of a quiver.

Not all representations of A are of interest, we were only considering trace preserving representations of A. These corresponds to representations of Π with a special dimension vector.

If W is a trace preserving representation of A then it is also a trace preserving representation of $\mathbb{C}Q$. Because $\mathsf{Tr}_{\mathbb{C}G} g = |G|\delta_{g1}$ we know that W must be isomorphic to the regular representation of G. This means that $e_iW = e_ieW$ is n_i-dimensional where $n_i = \dim S_i$. So eW is an α-dimensional representation with $\alpha_i = n_i = \dim S_i$.

We can conclude that

$$\mathsf{trep} A /\!\!/ \mathsf{GL}_n \cong \mathsf{Rep}(\Pi, \alpha) /\!\!/ \mathsf{GL}_\alpha$$

Finally we need an explicit description of Π in terms of its quiver and its relations. The vertices of the quiver are already known these are the simple representations. The arrows will correspond to generators of A, these sit inside the degree 1 part: $A_1 = (\mathbb{C}X + \mathbb{C}Y)\mathbb{C}G$.

Now for every couple i, j we can choose a basis for the subspace $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j \subset \mathbb{C}[V]\#G$. The union of all these bases forms a basis $\{a_1, \dots, a_l\}$ for the space eA_1e. We can now construct a quiver Q_G with vertices the set $\{e_1, \dots, e_k\}$ and as arrows $\{a_1, \dots, a_l\}$. If the arrow a_ℓ sits in $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j$ then we let it run from e_i to e_j: $h(a_\ell) = e_j, t(a_\ell) = e_i$.

Theorem 3.3. *Q_G is the McKay Quiver of G.*

Proof. The dimension of $e_i(\mathbb{C}X + \mathbb{C}Y)\mathbb{C}Ge_j$ is the same as the multiplicity of e_i in $V \otimes \mathbb{C}G$ $\square$

To obtain the relations we show first write out $eXY - YXe$ in terms of these arrows. In the case of A_n this is easy to do, eXe will correspond to all clockwise arrows $(i \xrightarrow{a_i} i+1)$ and eYe to all anticlockwise arrows $(i-1 \xleftarrow{b_i} i)$. Therefore the relation will turn out to be

$$\sum_i a_{i-1}b_i - b_{i+1}a_i$$

Because every term above consists only of paths from i to i, all these terms must be zero separately. For the other cases one can do similar things but the calculations become more complicated. We will only state the general result without proof.

Theorem 3.4 (Crawley-Boevey)**.** *Choose an involution on the arrows $\cdot^* : Q_1 \to Q_1$ such that $h(a^*) = t(a)$ and $t(a^*) = h(a)$. Take a subset $A \subset Q_1$ such that every arrow can uniquely be written as a or a^* with $a \in A$. Now we can express the preprojective algebra as*

$$\Pi_G := \mathbb{C}Q_G/(\sum_{a \in A} aa^* - a^*a).$$

Chapter 4

Resolutions of Singularities

4.1 Non-affine varieties

Up until now we have only seen affine varieties, however one can construct far bigger class of varieties by gluing affine varieties together.

Let V be a topological space, such that it is the union $V_1 \cup \cdots \cup V_n$ of open subsets, and suppose that each V_i and each intersection $V_i \cap V_j$ have the structure of reduced affine varieties and that the natural embedding $V_i \cap V_j \to V_i$ can be seen as morphisms of affine varieties, then we will call V a (pre)variety.

Morphisms between varieties are maps $\phi : V \to W$ such that the $\phi|_{V_i \cap \phi^{-1}(W_j)}$ are morphisms of affine varieties.

The standard examples of non-affine varieties can be done using the Proj -construction. If A is a graded ring such it is generated by $X_1, \ldots, X_n$ with degree 0 and $Y_0, \cdots Y_m$ with degree 1 end let $r_1, \ldots r_p$ the homogeneous relations between the generators, then we can define a subset

$$\mathsf{Proj}\ A := \{(x_1, \ldots, x_n, y_0, \ldots, y_m) \in \mathbb{C}^n \times \mathbb{P}^m | r_i(x_1, \ldots, x_n, y_0, \ldots, y_m) = 0\}$$

This set is well defined because if r_i is of degree d then

$$r_i(x_1, \ldots, x_n, \lambda y_0, \ldots, \lambda y_m) = 0 \Leftrightarrow \lambda^d r_i(x_1, \ldots, x_n, \lambda y_0, \ldots, \lambda y_m) = 0.$$

Now we can cover by affine open subset V_i corresponding to the locus where y_i is nonzero:

$$V_i = \{(x_1, \ldots, x_n, \lambda \frac{y_0}{y_i}, \ldots, \frac{y_{i-1}}{y_i} \frac{y_{i+1}}{y_i}, \ldots, \frac{y_m}{y_i}) | (x_1, \ldots, x_n, y_0, \ldots, y_m) \in \mathsf{Proj}\ A\} \subset \mathbb{C}^{n+m}$$

Note that $\mathbb{C}[V_i] = A[y_i^{-1}]$ and $\mathbb{C}[V_i \cap V_j] = A[y_i^{-1}, y_j^{-1}]$ so $\mathsf{Proj}\ A$ is indeed a variety.

This variety can be mapped to $V(A_0)$ by the ordinary projection

$$(x_1, \dots, x_n, y_0, \dots, y_m) \mapsto (x_1, \dots, x_n).$$

With all this in mind we can introduce the concept of a resolution of a singularity:

Definition 4.1. If V is a singular reduced variety then the morphism of varieties $\pi : W \to V$ is a resolution of V if

- π is a surjection,
- there is an open parts $U_V \subset V$ and $U_W \subset W$ such that $\pi|_{U_W}$ is an isomorphism between U_W and U_V,
- W is smooth (if W is not smooth we call π a partial resolution). The locus of W for which π is not one-to-one is called the exceptional fiber.

4.2 Blow-ups

An interesting way of constructing resolutions is by using blow-ups. Suppose V is an affine variety and $\mathfrak{n} \lhd \mathbb{C}[V]$ is an ideal corresponding to the closed subset X. The blow-up of X is then defined as

$$\tilde{V} = \mathsf{Proj}\ \mathbb{C}[V] \oplus \mathfrak{n}t \oplus \mathfrak{n}^2 t^2 \oplus \cdots .$$

The standard projection $\pi : \tilde{V} \to V$ is at least a partial resolution because if $p \in V \setminus X$ and $y_0 t, \dots y_m t$ are the generators of $\mathfrak{n}t$ is there must be at least one y_i that is not zero on p, so the preimage of p will only contain the point

$$(x_1(p), \dots, x_n(p), y_0(p), \dots, y_m(p))$$

We will now calculate the blow ups corresponding to the kleinian singularities. The ring $\mathbb{C}[X, Y, Z]/(XY - Z^2)$ has a unique singularity in the point $(0, 0, 0)$ because

$$(\partial_X, \partial_Y, \partial_Z) r = (Y, X, 2Z) = 0 \Leftrightarrow (X, Y, Z) = (0, 0, 0).$$

The blow-up is (using the convention $x = Xt, y = Yt, z = Zt$)

$$\begin{aligned}
&\mathsf{Proj}\ \mathbb{C}[X, Y, Z]/(r) \oplus (X, Y, Z)t \oplus (X, Y, Z)^2 t^2 \oplus \cdots \\
&= \frac{\mathbb{C}[X, Y, Z, x, y, z]}{(XY - Z^2, Xy - xY, xZ - Xz, Yz - yZ, Xy - Zz, xy - z^2)} \\
&= \{(X, Y, Z, X, Y, Z) \in \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^2 | XY - Z^2\} \cup \{(0, 0, 0, x, y, z) \in \mathbb{P}^2 | xy - z^2 = 0\}
\end{aligned}$$

where the last bit is the exceptional fiber, it is a conic and hence as a variety it is isomorphic to $\mathbb{P}_1$. We can cover the blow-up variety by two parts corresponding to

$x \neq 0$ we can choose coordinates $X, \eta = y/x, \zeta = z/x$ the relation $XY - Z^2$ becomes $\cancel{X}^2(\eta - \zeta^2) = 0$. which is smooth.

$y \neq 0$ we can choose coordinates $\xi = x/y, Y, \zeta = z/y$ the relation $XY - Z^2$ becomes $\cancel{Y}^2(\xi - \zeta^2) = 0$. which is smooth.

$z \neq 0$ is not necessary because it implies that both $x, y \neq 0$.

The ring $\mathbb{C}[X,Y,Z]/(XY - Z^n), n \geq 3$ has a unique singularity in the point $(0,0,0)$ because

$$(\partial_X, \partial_Y, \partial_Z)r = (Y, X, 3Z^2) = 0 \Leftrightarrow (X,Y,Z) = (0,0,0).$$

The blow-up is

$$\frac{\mathbb{C}[X,Y,Z,x,y,z]}{(XY - Z^n, Xy - xY, \dots, Xy - Z^{n-1}z, xy - Z^{n-2}z^2)}$$
$$= \{(X,Y,Z,X,Y,Z) \in \mathbb{C}^3 \setminus \{0\} \times \mathbb{P}^2 | XY - Z^2\} \cup \{(0,0,0,x,y,z) \in \mathbb{P}^2 | xy = 0\}$$

where the last bit is the exceptional fiber, it is a union of 2 projective lines that intersect in the point $(0,0,0,0,0,1)$.

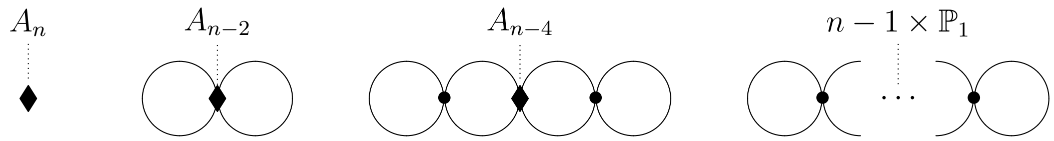
We can cover the blow-up variety by three parts corresponding to

$x \neq 0$ we can choose coordinates $X, \eta = y/x, \zeta = z/x$ the relation $XY - Z^n$ becomes $\eta - \zeta^n X^{n-2} = 0$. which is smooth.

$y \neq 0$ we can choose coordinates $\xi = x/y, Y, \zeta = z/y$ the relation $XY - Z^n$ becomes $\xi - \zeta^2 Y^{n-2} = 0$. which is smooth.

$z \neq 0$ we can choose coordinates $\xi = x/z, \eta = y/z, Z$ the relation $XY - Z^n$ becomes $\xi\eta - Z^{n-2} = 0$, which has a singularity if $n > 3$, but this singularity is 'smaller' so we can blow it up again.

Diagramatically we get the following



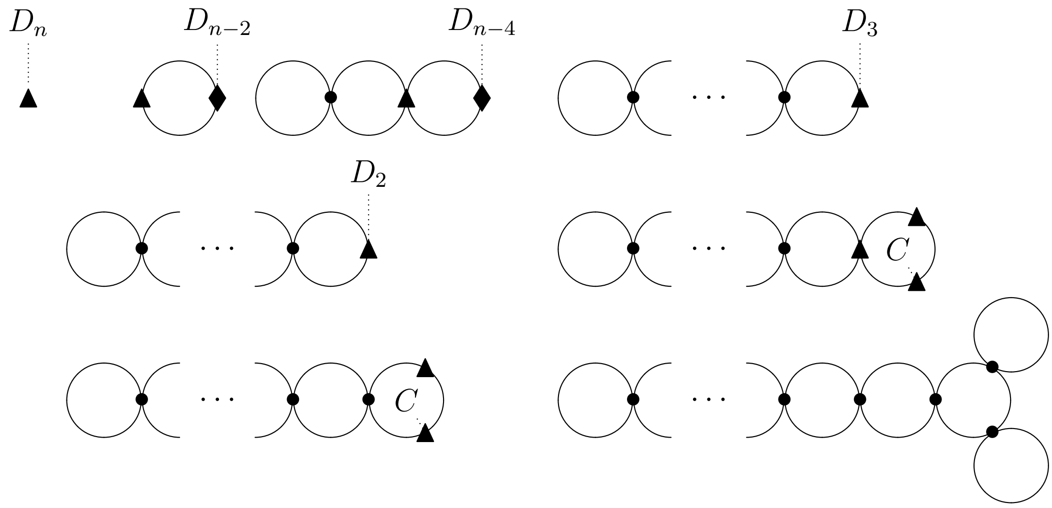
For D_2 we can do the following: The first blow-up has exceptional fiber $z = 0$ because the relation becomes $Xx^2 + Xy^2 + z^2 = 0$. If we look at the chart for $y \neq 0$ we get the relation

$$\xi^3 Y + \xi Y + \zeta^2$$

which has three singularities, for $\xi = 0, \pm i$ if we blow these 3 up (define $\xi_\pm = \xi \pm i$), we get 3 exceptional fibers of the form $\bar{\xi} y + \bar{\zeta}^2$ with $\bar{\zeta} = \zeta t$, $\bar{x}i = t(\xi + 0, \pm i)$ (depending on the point blown up). One can check easily that there are no further singularities.

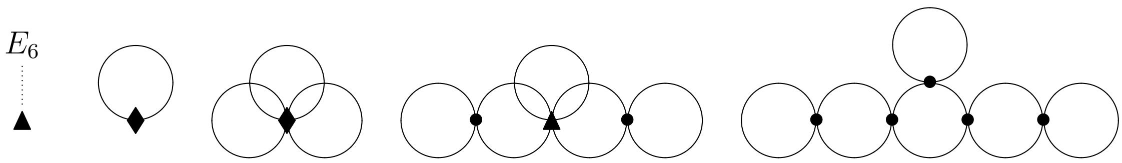
If $n > 2$ then the exceptional fiber is $z = 0$. There are two singularities in the blow-up, the one corresponding to $(1, 0, 0)$ which is of the type D_{n-2} and the one in $(0, 1, 0)$ which has local equation $\xi^{n+1} Y^{n-1} + \xi Y + \zeta^2$. The blow-up of this last singularity has as exceptional fiber a conic (look at the degree 2-part) and there are no further singularities.

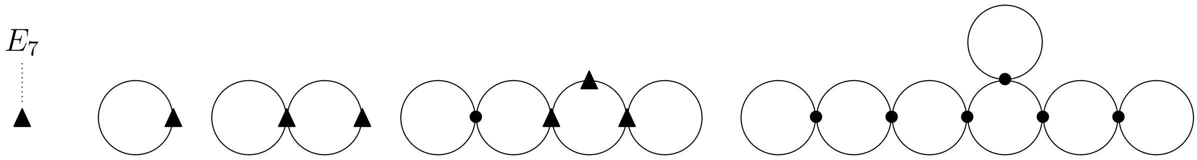
The diagram looks as

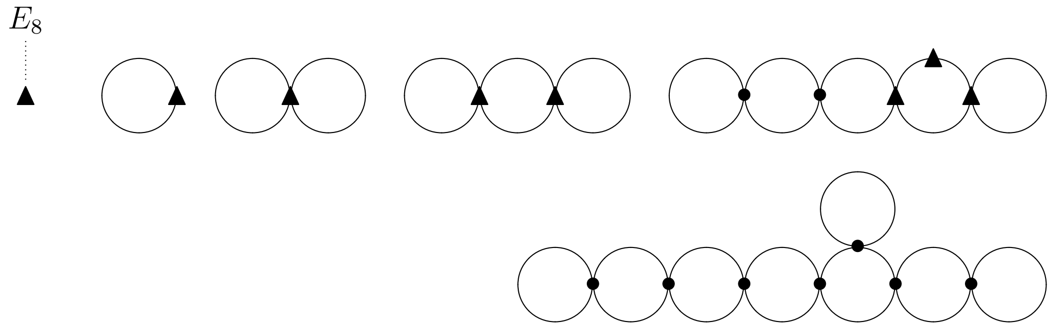


Below we show a picture of the resolution of D_2.

Finally we do the diagrams E_6, E_7, E_8.







Mystery: Construct a quiver where the vertices correspond to the $\mathbb{P}_1$'s in the exceptional fiber in the blow up of V/G and there is an arrow from v_i to v_j if their corresponding $\mathbb{P}_1$'s intersect. This quiver is isomorphic to the McKay quiver of G without the vertex corresponding to the trivial representation.

Chapter 5

Semi-invariants and Moduli spaces

5.1 Semi-invariants

As we have seen in the previous chapter it is possible to get a resolution of an affine variety by constructing the Proj of a graded ring of which the degree zero part is the ring of regular functions of the original variety. The method we used for this was blow-ups. In invariant theory it is also possible to do a different construction using semi-invariants.

If G is a reductive group then a *multiplicative character* of G is a group morphism $\theta : G \to \mathbb{C}^* : g \mapsto g^\theta$. We will write the action of θ exponentially because it will be very handy later on. The characters of G form an additive group if we define $g^{\theta_1+\theta_2} := g^{\theta_1} g^{\theta_2}$, we will also use the shorthand $n\theta = \theta + \cdots + \theta$.

If G acts on a variety V then a function $f \in \mathbb{C}[V]$ is called a *θ-semi-invariant* if

$$\forall g \in G : g \cdot f = g^\theta f.$$

The subspace of θ-semi-invariants will be denoted by $\mathbb{C}[V]_\theta$. This space does not form a ring, it is only a module over the ring of invariants $\mathbb{C}[V]^G$.

We can construct an $\mathbb{N}$-graded ring by taking the direct sum of all $n\theta$-semi-invariants with $n \in \mathbb{N}$:

$$\mathsf{SI}_\theta[V] = \bigoplus_{n \in \mathbb{N}} \mathbb{C}[V]_{n\theta}.$$

It is easy to extend the proof of theorem to show that $\mathsf{SI}_\theta[V]$ is also finitely generated as an algebra over $\mathsf{SI}_\theta[V]_0 = \mathbb{C}[V]^G$.

A point $p \in V$ is called θ-semi-stable if there is an $f \in \mathbb{C}[V]_{n\theta}$ such that $f(p) \neq 0$. The set of θ-semi-stable points will be denoted by V_θ^{ss}. Note that V_θ^{ss} itself is not necessarily an affine variety but if $f_1, \cdots, f_k$ forms a set of homogeneous generators of $\mathsf{SI}_\theta[V]$ over $\mathbb{C}[V]^G$ then V_θ^{ss} can be covered with affine varieties corresponding to the rings

$$R_i = \mathbb{C}[V][f_i^{-1}] \; V(R_i) = \{p \in V | f_i(p) \neq 0\}$$

These varieties and rings have G-actions on them coming from the G-action on V and one can take the categorical quotient of these varieties. Their ring are of the form

$$\mathsf{SI}_\theta[V][f_i^{-1}]_0$$

and hence one can cover $\mathsf{Proj}\ \mathsf{SI}_\theta[V]$ with these quotients varieties. Out of this one can conclude

Theorem 5.1. *The variety* $\mathsf{Proj}\ \mathsf{SI}_\theta[V]$ *classifies the closed orbits in* V_θ^{ss}. *If there exists a* θ*-semi-invariant that is non-zero in a point of* V *then* V_θ^{ss} *is open and dense in* V *and the image of the map* $V_\theta^{ss} /\!\!/ G \to V /\!\!/ G$ *is dense.*

5.2 semi-stable representations of quivers

If Q is a quiver and α a dimension vector then we can look at the θ-semi-invariants of the GL_α-action on $\mathsf{Rep}(Q, \alpha)$. We will denote this set by $\mathsf{Rep}_\theta^{ss}(Q, \alpha)$, the quotient of this set by the GL_α-action we be denoted by $\mathsf{M}_\theta^{ss}(Q, \alpha)$ and is called the moduli space of θ-semistable representations.

First of all we have to look at the multiplicative characters of GL_α. For the general linear group GL_n the characters are given by powers of the determinant, so the group of characters is isomorphic to $\mathbb{Z}$. As GL_α consists of $k = \#Q_0$ components each one isomorphic to a general linear group, the group of characters will be isomorphic to $\mathbb{Z}^k$:

$$\theta = (\theta_1, \ldots, \theta_k) : \mathsf{GL}_\alpha \to \mathbb{C}^* : (M_1, \ldots, M_k) \mapsto \det M_1^{\theta_1} \cdots \det M_k^{\theta_k}.$$

In the case of invariants we had a nice description using traces of cycles, for semi-invariants we can do a similar thing. A way to construct a θ-semi invariant is the following: let $i_1, \ldots, i_s$ be the vertices for which θ_{i_ℓ} is positive, while $j_1, \ldots, j_t$ be the ones with a negative θ_{j_ℓ}. Now chose for each i and j $|\theta_i\theta_j|$ elements in $j\mathbb{C}Qi$

and put all these in a $\sum_j \theta_j \times \sum_i \theta_i$-matrix D over $\mathbb{C}Q$.

$$D := \begin{bmatrix} j_1\leftarrow i_1 & \dots & j_1\leftarrow i_1 & j_1\leftarrow i_s & \dots & j_1\leftarrow i_s \\ \vdots & |\theta_{j_1}\theta_{i_1}|\times & \vdots & \vdots & |\theta_{j_1}\theta_{i_s}|\times & \vdots \\ j_1\leftarrow i_1 & \dots & j_1\leftarrow i_1 & j_1\leftarrow i_s & \dots & j_1\leftarrow i_s \\ \vdots & & & \ddots & & \vdots \\ j_t\leftarrow i_1 & \dots & j_t\leftarrow i_1 & j_t\leftarrow i_s & \dots & j_t\leftarrow i_s \\ \vdots & |\theta_{j_t}\theta_{i_1}|\times & \vdots & \vdots & |\theta_{j_t}\theta_{i_s}|\times & \vdots \\ j_t\leftarrow i_1 & \dots & j_t\leftarrow i_1 & j_t\leftarrow i_s & \dots & j_t\leftarrow i_s \end{bmatrix}$$

Now if $W \in \mathsf{Rep}(Q,\alpha)$ then we can substitute each entry in D to its corresponding matrix-value in W. In this way we obtain a block matrix D_W with dimensions $\sum_i \alpha_i|\theta_i| \times \sum_j \alpha_j|\theta_j|$. One can easily check that

$$D_{g\cdot W} = \begin{bmatrix} g_{j_1} & & & & \\ & \ddots & & & \\ & & g_{j_1} & & \\ & & & \ddots & \\ & & & & g_{j_t} \\ & & & & & \ddots \\ & & & & & & g_{j_t} \end{bmatrix} D_W \begin{bmatrix} g_{i_1}^{-1} & & & & \\ & \ddots & & & \\ & & g_{i_1}^{-1} & & \\ & & & \ddots & \\ & & & & g_{i_s}^{-1} \\ & & & & & \ddots \\ & & & & & & g_{i_s}^{-1} \end{bmatrix}$$

So if D_W is a square matrix the determinant of D_W is a θ-semi-invariant:

$$\det D_{g\cdot W} = \det g_{j_1}^{\theta_{j_1}} \cdot \det g_{j_t}^{\theta_{j_t}} \det D_W \det g_{i_1}^{-|\theta_{i_1}|} \cdot \det g_{i_s}^{-|\theta_{i_s}|} = g^\theta \det D_W.$$

We will call these semi-invariants *determinantal semi-invariants*

Theorem 5.2. *As a $\mathbb{C}[\mathsf{Rep}_\alpha Q]^{\mathsf{GL}_\alpha}$-module $\mathbb{C}[\mathsf{Rep}_\alpha Q]_\theta$ is generated by determinantal semi-invariants. As a ring $\mathsf{SI}_\theta[\mathsf{Rep}_\alpha Q]$ is generated by invariants (i.e. traces of cycles) and determinantal $n\theta$-semi-invariants with $n \in \mathbb{N}$.*

Note that this implies that there are only θ-semi-invariants if D_W is a square matrix so $\sum_i \alpha_i|\theta_i| = \sum_j \alpha_j|\theta_j|$ or equivalently $\theta \cdot \alpha = 0$.

Now we can use this special form for the semi-invariants to get a nice interpretation for the covering of $\mathsf{Rep}(Q,\alpha)^s s_\theta /\!\!/ \mathsf{GL}_\alpha$.

As we have seen $\mathsf{Rep}(Q,\alpha)$ describes the α-dimensional representations of the pathalgebra $\mathbb{C}Q$. Now if $W \in \mathsf{Rep}(Q,\alpha)$ is θ-semistable then there exists a $\sum_j |\theta_j| \times \sum_i |\theta_i|$-matrix D with entries in $\mathbb{C}Q$ such that $\det D_W \neq 0$, so D_W is an invertible matrix:

$$\exists E_W : D_W E_W = 1_{\sum |\theta_j|\alpha_j} \text{ and } D_W E_W = 1_{\sum |\theta_i|\alpha_i}$$

So W is also a representation of a new algebra for which D is is indeed an invertible matrix. To be more precise we need a good interpretation a the identity matrices that appeared in the equations above. Recall that for an α-dimensional representation W the vertex i can considered as an idempotent in $\mathbb{C}Q$ and i_W will correspond to the identity matrix on the α_i-dimensional space iW. The identity matrix $\mathsf{1}_{\sum|\theta_j|\alpha_j}$ can hence be considered as the evaluation in W of the matrix

$$\mathsf{1}_j = \begin{bmatrix} j_1 & & & & & \\ & \ddots & & & & \\ & & j_1 & & & \\ & & & \ddots & & \\ & & & & j_t & \\ & & & & & \ddots \\ & & & & & & j_t \end{bmatrix} = \begin{bmatrix} h(D_{11}) & & \\ & \ddots & \\ & & h(D_{pp}) \end{bmatrix}$$

Now we define $E = E_{\mu\nu}$ to be the matrix such that $DE = \mathsf{1}_j$ and $ED = \mathsf{1}_i$ so

$$\sum_\kappa D_{\mu\kappa}E_{\kappa\nu} = \delta_{\mu\nu}h(D_\mu\mu) \text{ and } \sum_\kappa E_{\mu\kappa}D_{\kappa\nu} = \delta_{\mu\nu}t(D_\mu\mu).$$

Now we define the universal localization of $\mathbb{C}Q$ at D to be the algebra

$$\mathbb{C}Q[D^{-1}] = \mathbb{C}Q_E/(\sum_\kappa D_{\mu\kappa}E_{\kappa\nu} - \delta_{\mu\nu}h(D_\mu\mu), \sum_\kappa E_{\mu\kappa}D_{\kappa\nu} = \delta_{\mu\nu}t(D_\mu\mu))$$

Here Q_E is a new quiver consisting of Q together with extra arrows $E_{\mu\nu}$ such that $h(E_{\mu\nu}) = t(D_{\mu\nu})$ and $t(E_{\mu\nu}) = h(D_{\mu\nu})$.

There is a natural map $\mathbb{C}Q \to \mathbb{C}Q[D^{-1}]$ so we also have a map

$$\mathsf{Rep}(\mathbb{C}Q[D^{-1}], \alpha) \to \mathsf{Rep}(Q, \alpha)$$

This map is an (open) embedding because $(D^{-1})_W$ is uniquely defined by D_W, its image consist precisely of these representations of $\mathbb{C}Q$ for which $\det D_W \neq 0$.

Theorem 5.3. $\mathsf{Rep}_\theta^{ss}(Q, \alpha)$ *can be covered by representation spaces of universal localizations of* $\mathbb{C}Q$*. This covering is compatible with the* GL_α*-action, so* $\mathsf{M}_\theta^{ss}(Q, \alpha)$ *can be covered by quotient spaces of universal localizations of* $\mathbb{C}Q$*.*

This theorem also holds for quotients of path algebras. We will work this out in the next section for the preprojective algebras.

5.3 Moduli space for preprojective Algebras

So lets now take a closer look at the case of Kleinian singularities. As we already know we can consider the singularity as the quotient space of the preprojective algebra over the McKay quiver with the standard dimension vector.

$$V /\!\!/ G = \mathsf{iss}(\Pi_G, \alpha_G.$$

In order to construct a nice desingularization of this space we have to find a good character. Let $e_1, \ldots, e_n$ be the vertices of the quiver and let e_1 correspond to the trivial representation. For every vertex $e_i \neq e_1$ there exists a character θ_i mapping e_1 to $-\alpha_{Gi} = -\dim S_i$, e_i to 1 and all the other vertices to zero. We denote the sum of all these θ_i as *theta* and this will be the character under consideration:

$$\theta(e_1) = -|\alpha_G| + 1 \text{ and } \theta_i = 1 \text{ if } i \neq 0.$$

Theorem 5.4. *If $V /\!\!/ G$ is a kleinian singularity and $\tilde{V} /\!\!/ G \to V /\!\!/ G$ is its minimal resolution, then $\tilde{V} /\!\!/ G = \mathsf{M}^{ss}_\theta(\Pi_G, \alpha)$.*

The semi-invariants are constructed using matrices D of which the entries are all paths starting from e_1. If we construct D_W, then every column of D_W corresponds to a column of D because the dimension of e_1 is 1. The determinant is linear in the columns so we can chose D up to linear combinations of the columns. In this way we can turn D into a form such that D_W is block diagonal with the dimension of every block corresponding to the dimension of a vertex. This means that the θ-semi-invariants are generated by products of the θ_i-semi-invariants. Therefore we can conclude that a representation is θ-semistable if and only if it is θ_i-semistable for every i.

The θ_i-semi-invariants are generated over $\mathbb{C}[\mathsf{iss}(\Pi, \alpha)]$ by D's that are $1 \times \dim S_i$-matrices whose entries are paths from e_1 to e_i. Using the preprojective relation and the relations from matrix identities one can find a finite number of generating paths. For instance, these paths cannot run twice through a vertex with dimension 1 otherwise we could split of a trace of a cycle (and this is contained in $\mathbb{C}[\mathsf{iss}(\Pi, \alpha)]$). If there are k_i such paths there are $C^{k_i}_{\alpha_i}$ generators for the semi-invariants.

This means that we can embed $M^{ss}_\theta(\Pi, \alpha)$ in

$$\mathbb{C}^3 \times \mathbb{P}^{C^{k_2}_{\alpha_2}} \times \cdots \times \mathbb{P}^{C^{k_n}_{\alpha_n}}.$$

The first factor is for the 3 invariants, the others for the θ_i-semi-invariants for every $i > 1$.

We will now look at the different cases, separately.

1. A_n Up to multiplication with invariants there are for every e_i exactly two paths from e_1: a clockwise p_i and a counterclockwise q_i. Each of these paths

gives a $theta_i$-semi-invariant. The projection map $\mathsf{Rep}^{ss}_\theta(\Pi, \alpha) \to M^{ss}_\theta(\Pi, \alpha)$ can now be seen as

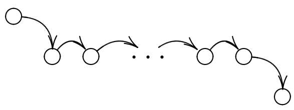
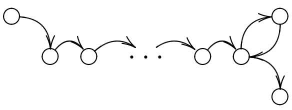
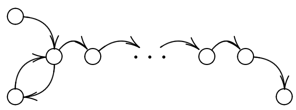
$$W \mapsto [(X_W, Y_W, Z_W), (p_2, q_2)_W, \dots, (p_n, q_n)_W].$$

To calculate the exceptional fiber we must look at the semistable representation that have zero invariants (X, Y, Z). Because X is zero there must be an i such that $p_{iW} \neq 0$ but $p_{i+1W} = 0$. Semistability then implies that $q_{i+1W} \neq 0$. Also q_{i-1W} must be zero otherwise $Z = p_i/p_{i-1}q_{i-1}/q_i$ would not be zero. This means that a point P comes from a point in the exceptional fiber if it is of the form

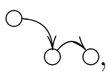
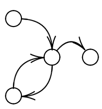
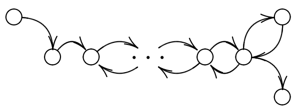
$$[(0, 0, 0), (1, 0), \cdots, (1, 0), (p_i, q_i)_W, (0, 1), \cdots, (0, 1)].$$

From this we can conclude that the exceptional fiber is indeed the union of $n - 1$ $\mathbb{P}_1$ intersection each other consecutively.

D_n For this case we can use the preprojective relations to show that for the vertices of dimension 1 there are always two independant paths: e.g.

 and  = 

For a vertex with dimension 2 we have 3 paths that matter e.g.

,  and 

To calculate the exceptional fiber we first assume that there is a vertex of dimension 1 for which the two paths are nonzero. If this is the case then the longest path for the other vertices with dimension 1 must be zero otherwise the invariant connecting those 2 vertices is nonzero.

to be continued