Chapter 1

Representations of finite groups

In this chapter we give the basic definitions of representation theory, and then look a bit closer to the representations of finite groups. We introduce the notion of characters as a basic tool for the study of these representations.

1.1 Definitions

A *representation of a group G* on a finite-dimensional complex vector space V is a group homomorphism $\rho : \mathsf{G} \to \mathsf{GL}(V)$. If G has some additional structure like topological space, complex variety or a real manifold, we ask that ρ is a corresponding morphism: a continuous map, a polynomial map or a smooth map.

A *representation of a complex algebra A* is an algebramorphism $\rho : \mathsf{G} \to \mathsf{End}(V)$ We say that such a map gives V the structure of an A-module. When there is little ambiguity about the map ρ, we sometimes call V itself a representation of A. For any element $x \in A, v \in V$ we will shorten $\rho(x)v$ to $x \cdot v$ or xv. If A is finite dimensional it can be considered as a representation of itself, this representation will be called the *regular representation.*

If G is a discrete topological group (e.g. finite groups) we can construct the group algebra $\mathbb{C}\mathsf{G}$. This is the complex vector space with as basis the group elements and with as multiplication the bilinear extension of the group multiplication. Every representation of such a group extends linearly to a unique representation of the group algebra and vice versa every representation of the group algebra can be restricted to the basis elements to obtain a group representation.

A *morphism* ϕ between two representations ρ_V and ρ_W is a vector space map

$\phi : V \to W$ such that the following diagram is commutative

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho_V(x)"'] & W \arrow[d, "\rho_W(x)"] \\ V \arrow[r, "\phi"] & W. \end{tikzcd}$$

In short we can also write $\phi(xv) = x\phi(v)$. A morphism is also sometimes called an A-linear map. The set of A-linear maps is denoted by $\mathsf{Hom}_A(V, W)$.

A *subrepresentation* of V is a subspace W such that $x \cdot W \subset W$ for all $x \in A$. Note that a morphism maps subrepresentations to subrepresentations so in particular for any morphism ϕ the spaces $\mathsf{Ker}\phi$ and $\mathsf{Im}\phi$ are subrepresentations.

A representation V is called *simple* if its only subrepresentations are 0 and V. If V and W are representations we can construct new representations from them: the direct sum $V \oplus W = \{(v, w)|v \in V, w \in W\}$ has a componentwise action $x(v, w) = (xv, xw)$. A representation that is not isomorphic to the direct sum of two non-trivial representations is called *indecomposable*. If a representation is a direct sum of simple representations it is called *semisimple*.

If we are considering only group representations we can construct even more new representations:

- The tensor product $V \otimes W = \mathsf{Span}(v_i \otimes w_j|(v_i),\ (w_j)$ are bases for V, W) has as action $x(v \otimes w) = gv \otimes gw$.
- The dual space $V^* = \{f : V \to \mathbb{C}|f \text{ is linear}\}$ has a contragradient action: $(g \cdot f)v = f(g^{-1} \cdot v)$.
- The space of linear maps $\mathsf{Hom}(V, W)$ can be identified with $V^* \otimes W$ and hence the action is $(g \cdot f)v = g \cdot (f(g^{-1} \cdot v))$. Note that this means that the elements of $\mathsf{Hom}_A(V, W)$ are in fact the maps that are invariant under the action of A.

The main topic that representation theory is concerned with is to classify all simple and indecomposable representations. In the next sections we will do this for finite groups.

1.2 Reductivity

In this section we will proof that all indecomposable representations of a finite group are in fact simple. This will allow use to decompose every representation as a unique direct sum of simple representations. Using this fact we will also give a description of $\mathsf{Hom}_A(V, W)$ for general representations. A group which has this above property is called a *reductive group*.

Theorem 1.1. *Let* G *be a finite group and* V *a representation of* G*. If* W *is a subrepresentation of* V *then there exists a subrepresentation* $W^\perp$ *such that* $V = W \oplus W^\perp$*.*

Proof. Let $\pi : V \to W$ be a projection of V onto W. This is a surjective map such that $\forall w \in W : \pi(x) = x$. Using this π we define a new map

$$\pi_{\mathsf{G}} : V \to W : v \mapsto \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} g \cdot (\pi(g^{-1}v)).$$

This map is still surjective because

$$\forall w \in W : \pi_{\mathsf{G}}(w) = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} g(\pi(g^{-1}w)) = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} g((g^{-1}w)) = \frac{|G|}{|G|} w = w.$$

This map is a morphism of representations

$$\pi_{\mathsf{G}}(hv) = \frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} g \cdot (\pi(g^{-1}hv)) = h \frac{1}{|\mathsf{G}|} \sum_{h^{-1}g \in \mathsf{G}} h^{-1}g \cdot (\pi(g^{-1}hv)) = h\pi_{\mathsf{G}}(v).$$

So the image and the kernel of this map are subrepresentations and

$$V = \mathsf{Im}\pi_{\mathsf{G}} \oplus \mathsf{Ker}\pi_{\mathsf{G}} = W \oplus \mathsf{Ker}\pi_{\mathsf{G}}.$$

□

Not every group has this property, we explicitely needed the finiteness of G to define π_G. For instance if $\mathsf{G} = \mathbb{Z}$ this is not true any more: The representation

$$\rho : \mathbb{Z} \to \mathsf{GL}_2 : 1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has only one nontrivial subrepresentation and hence it cannot be written as the direct sum of two one dimensional subrepresentations.

So using the previous theorem, we can start from a representation V, take a subrepresentation and split V in a direct sum $V_1 \oplus V_2$. If these components are not simple we can split them up again and again until we end up with a direct sum of simples. We can reorder these simples to put all isomorphic simples together so we can write

$$V \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}.$$

To prove that this sum is unique up to permutation of the factors we need the following important lemma.

Lemma 1.2 (Schur). *Let S and T be simple representations of a group G (or an algebra A) then*

$$\mathsf{Hom}_{\mathsf{G}}(S,T) = \begin{cases} 0 & \text{if } S \neq T \\ \mathbb{C} & \text{if } S = T \end{cases}$$

Proof. If ϕ is a morphism then its image and kernel are subrepresentations so if $\phi \neq 0$ it must be both injective and surjective and hence an isomorphism.

Note that $\mathbb{C} \subset \mathsf{Hom}_{\mathsf{G}}(S,S)$ because every g acts linearly: $g(\lambda v) = \lambda(gv)$. If $\phi \in \mathsf{Hom}_{\mathsf{G}}(S,S)$ we can find an eigenvalue λ of ϕ. The eigenspace with this eigenvalue is a subrepresentation of S.

$$\phi(v) = \lambda v \Rightarrow \phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv).$$

The eigenspace must be the whole of S so ϕ acts as a scalar on S. □

Miniature 1: Issai Schur

Schur was born in Russia, in 1875 but spent most of his youth in Latvia. In 1894 he enrolled in the University of Berlin, studying math and physics. He received his doctorate with honours in 1901 under supervision of Frobenius. In 1913 he was appointed to succeed Felix Hausdorff as professor at the University of Bonn. He was elected as a member in the Berlin Academy in 1922. In 1938 he left and attempted to immigrate to Palestine. He and his family escaped just days before an appointment with the Gestapo. Soon after, on January 10, 1941, he had a heart attack and died in Palestine. Issai Schur was the one of the great mathematicians that contributed significantly to representation and character theory.

We can use this lemma to compute the $\mathsf{Hom}_{\mathsf{G}}(V,W)$ for general semisimple representations:

Theorem 1.3. *If $V \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ and $W \cong V \cong S_1^{\oplus f_1} \oplus \cdots \oplus S_k^{\oplus f_k}$, where some of the e and f's can be zero, then*

$$\mathsf{Hom}_{\mathsf{G}}(V,W) = \mathsf{Mat}_{f_1 \times e_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{f_k \times e_k}(\mathbb{C}).$$

Proof. For every term in the decomposition of V including the multiplicities we have an embedding

$$\iota_{ij} : S_i \to V : v \mapsto 0 \oplus \cdots \oplus 0 \oplus v \oplus \cdots \oplus 0$$

and for every term in the decomposition of V including the multiplicities we have a projection

$$\pi_{ij} : W \to S_i : w_{11} \oplus \dots w_{1f_1} \oplus w_{k1} \oplus \dots w_{kf_k} \mapsto w_{ij}.$$

Every map $\pi_{pq}\phi\iota_{ij}$ is a morphism from $S_i \to S_p$ so it is a scalar and it is 0 if $i \neq k$. For every i we have an $e_i \times f_i$-matrix of scalars $\pi_{il}\phi\iota_{im}$, so ϕ uniquely determines an element of $\mathsf{Mat}_{f_1\times e_1}(\mathbb{C}) \oplus \dots \oplus \mathsf{Mat}_{f_k\times e_k}(\mathbb{C})$.

On the other hand if $\vec{M} : (M^1, \dots, M^k) \in \mathsf{Mat}_{f_1\times e_1}(\mathbb{C}) \oplus \dots \oplus \mathsf{Mat}_{f_k\times e_k}(\mathbb{C})$ then we can define a morphism

$$\phi_M : \oplus_{i=1}^k \oplus_{j=1}^{e_i} v_{ij} \mapsto \oplus_{i=1}^k \oplus_{j=1}^{e_i} \sum_p M^i_{jp} v_{ip}.$$

□

A simple corollary of this theorem is that the decomposition is unique: if $V \cong W \Leftrightarrow \forall i : e_i = f_i$. This is because $\mathsf{Mat}_{f_1\times e_1}(\mathbb{C}) \oplus \dots \oplus \mathsf{Mat}_{f_k\times e_k}(\mathbb{C})$ can only contain invertible elements if $e_i = f_i$.

1.3 Characters

To classify representations up to isomorphism we need to associate certain objects to representations that are invariant under isomorphisms. One example of such objects are characters.

The *character* of a representation V is the map

$$\chi_V : \mathsf{G} \to \mathbb{C} : g \mapsto \mathsf{Tr}\rho_V(g).$$

As the trace is invariant under conjugation it follows immediately that if $V \cong W$ then $\chi_V = \chi_W$. Also the character has the property that it has the same values on the conjugacy classes in G:

$$\chi_V(hgh^{-1}) = \mathsf{Tr}(\rho_V(h)\rho_V(g)\rho_V(h^{-1})) = \mathsf{Tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \mathsf{Tr}\rho_V(g) = \chi_V(g)$$

One can also easily check the following identities:

- $\chi_{V\oplus W} = \chi_V + \chi_W$,
- $\chi_{V\otimes W} = \chi_V \cdot \chi_W$,

- $\chi_{V^*} = \bar{\chi}_V$. This follows because the eigenvalues of $\rho_{V^*}(g) = \rho_V(g^{-1})^T$ are roots of the unity as $g^{|G|} = 1$ and hence the inverses are the complex conjugates.

We can also extend the character linearly such that it is a map from the group algebra to $\mathbb{C}$. Now we can look at the value of the element

$$p = \frac{1}{|G|}\sum_{g\in \mathsf{G}} g.$$

This element is an idempotent

$$p^2 = (\frac{1}{|G|}\sum_{g\in \mathsf{G}} g)^2 = \frac{1}{|G|^2}\sum_{g,h\in \mathsf{G}} gh = \frac{|G|}{|G|^2}\sum_{g\in \mathsf{G}} g = \frac{1}{|G|}\sum_{g\in \mathsf{G}} g = p$$

and for every $h \in \mathsf{G}$

$$ph = \frac{1}{|G|}\sum_{g\in \mathsf{G}} gh = \frac{1}{|G|}\sum_{gh\in \mathsf{G}} gh = \frac{1}{|G|}\sum_{g\in \mathsf{G}} g = p$$

This implies that for every representation V, $\rho_V(p)$ is a G-linear projection and the image of the projection are the G-invariant vectors of V. Recall that the trace of a projection is the dimension of the image.

If we apply this to the representation $\mathsf{Hom}(V, W) = V^* \otimes W$ we get

$$\frac{1}{|G|}\sum_{g\in \mathsf{G}} \chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_{g\in \mathsf{G}} \bar{\chi}_V(g)\chi_W(g) = \dim \mathsf{Hom}_{\mathsf{G}}(V, W).$$

If we define an hermitian product on the space of complex functions on G by

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in \mathsf{G}} \bar{f}(g)f(g)$$

we get for representations in combination with theorem **??**: if $V \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ and $W \cong S_1^{\oplus f_1} \oplus \cdots \oplus S_k^{\oplus f_k}$

$\langle \chi_V, \chi_W\rangle = \mathsf{DimHom}_{\mathsf{G}}(S, T) = \dim \mathsf{Mat}_{f_1\times e_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{f_k\times e_k}(\mathbb{C}) = e_1 f_1 + \ldots e_k f_k.$

So the characters of simple representations are orthogonal with respect to this hermitian product and if S is simple then $\langle \chi_V, S\rangle$ is the multiplicity of S inside V.

The characters all sit inside the subspace of functions $\mathsf{G} \to \mathbb{C}$ that are constant along the conjugacy classes in G, these are called the class functions. The dimension of this space equals the number of conjugacy classes. The orthogonality conditions imply that the number of isomorphism classes of simple representations is not bigger than this dimension. We will now show that this number is equal.

If this number is not equal this means that we can find a class function α that is orthogonal to all the χ_S. Now define the element

$$a := \frac{1}{|G|}\sum_{g\in \mathsf{G}} \bar{\alpha}(g)g \in \mathbb{C}\mathsf{G}.$$

This function commutes with all $g \in \mathsf{G}$ and so for every simple representation S, $\rho_S(a)$ is a morphism and hence a scalar λ. By assumption

$$\dim S\lambda = \mathsf{Tr}\rho_S(a) = \frac{1}{|G|}\sum_{g\in \mathsf{G}} \bar{\alpha}(g)\chi_S(g) = \langle \alpha, \rangle = 0.$$

This means that $\rho_S(a) = 0$ for all simple and hence for all representations.

Now we take as representation the regular representation of $\mathbb{C}\mathsf{G}$. The previous paragraph tells us that $a \cdot x = 0$ for ever element in $\mathbb{C}G$. This is only possible if $a = 0$ and thus $\alpha = 0$.

Finally we can easily determine the character of the regular representation: the identity element has character $\dim \mathbb{C}\mathsf{G} = |G|$, all the other group elements have character zero because they map no basis element to itself. So

$$\langle \chi_{\mathbb{C}\mathsf{G}}, S\rangle = \dim S \text{ and } \mathbb{C}\mathsf{G} = S_1^{\dim S_1} \oplus \cdots \oplus S_k^{\dim S_k}.$$

We can conclude

Theorem 1.4. *Let G be a finite group then we have that*

1. *The number of isomorphism classes simple representations is the same as the number of conjugacy classes in G.*
2. *The characters of the simple representation form an orthogonal basis for the space of class functions.*
3. *If S is simple then $\langle \chi_V, S\rangle$ is the multiplicity of S inside V.*
4. *V is completely determined by its character χ_V.*
5. $|\mathsf{G}| = \dim \mathbb{C}\mathsf{G} = \dim S_1^{\dim S_1} \oplus \cdots \oplus S_k^{\dim S_k} = \dim S_1^2 + \cdots + \dim S_k^2.$

1.4 The group algebra

In this last section we determine the structure of the group algebra.

Theorem 1.5. $\mathbb{C}\mathsf{G}$ *is a direct sum of matrix algebras:*

$$\mathbb{C}\mathsf{G} \cong \mathsf{End}(S_1) \oplus \cdots \oplus \mathsf{End}(S_k)$$

Proof. Every simple representation S_i gives a morphism $\rho_{S_i} : \mathbb{C}\mathsf{G} \to \mathsf{End}(S_i)$, so we have a big morphism

$$\psi = \oplus_i \rho_{S_i} : \mathbb{C}\mathsf{G} \to \oplus_i \mathsf{End}(S_i) : x \mapsto \oplus_i \rho_{S_i}(x).$$

The morphism ψ is an injection because if $\psi(x) = 0$ then $\rho_S(x) = 0$ for every simple representation so also $\rho_{\mathbb{C}\mathsf{G}}(x) = 0$ and this implies that $x = x \cdot 1 = 0$. Due to the last part of the previous theorem we know that both the target and the source of this morphism have the same dimension, so ψ is a bijection. $\square$

We can also calculate the orthogonal idempotents inside $\mathbb{C}\mathsf{G}$. These are the elements that are of the form $d_i = 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0$. The character of d_i on the representation S_i is $\dim S_i$ and it is zero on all other representations. The unique element that satisfies this requirement is $d_i = \frac{\dim S_i}{|G|} \sum_g \bar{\chi}_{S_i} g$ because

$$\mathsf{Tr}\rho_{S_j}(d_i) = \frac{\dim S_i}{|G|} \sum_g \bar{\chi}_{S_i} \mathsf{Tr}\rho_{S_j}(g) = \dim S_i \langle \chi_{S_i}, \chi_{S_j} \rangle.$$

We end with a nice identity

Theorem 1.6. $\mathsf{End}_{\mathbb{C}\mathsf{G}}(\mathbb{C}\mathsf{G}) \cong \mathbb{C}G$

Proof. For every g we can define a morphism φ_g of $\mathbb{C}\mathsf{G}$ as a representation

$$\varphi_g(h) := hg^{-1}.$$

This is indeed a morphism because $\varphi_g(h'h) = h'hg^{-1} = h'(hg^{-1}) = h'\varphi_g(h)$. Now define the following linear map

$$\varphi : \mathbb{C}G \to \mathsf{End}_{\mathbb{C}\mathsf{G}}(\mathbb{C}\mathsf{G}) : \sum_g \alpha_g g \to \sum_g \alpha_g \varphi_g.$$

This map is an algebramorphism because $\varphi_{gh} = \varphi_g \varphi_h$. The dimensions of source and target of φ are again the same and φ is injective because if $\varphi(\sum_g \alpha_g g) = 0$ then $1 \cdot \sum_g \alpha_g g^{-1} = 0$ so $\forall g \in \mathsf{G} : \alpha_g = 0$. $\square$

1.5 Exercises

Calculate the character tables of the following groups:

1. $\mathfrak{S}_3$, $\mathfrak{S}_4$, $\mathfrak{S}_5$ the permutation groups of $3, 4$ and 5 elements.
2. $\mathsf{D}_n = \langle X, Y | X^2 = 1, Y^2 = 1, (XY)^n = 1\rangle$,
3. $\mathsf{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$.

1.5.1 Right or Wrong

Are the following statements right or wrong, if right prove them, if wrong disprove or find a counterexample.

1. The number of one dimensional representations divides the order of the group.
2. A one-dimensional representation of a nonabelian simple group (i.e. a group without normal subgroups) is trivial.
3. A group with only one one-dimensional representation is simple.
4. The number of simple representations of the cartesian product of two groups is the sum of simple representations of both groups apart.
5. It there is a nontrivial conjugacy class in G with one element, then this element acts trivially on every simple representation of dimension > 1.
6. The characters of a simple 2-dimensional representation are real numbers.
7. For every two-dimensional representation there is an element with trace 0.
8. If a group has only one higherdimensional simple representation then it is simple.
9. Nonabelian groups of order 8 have the same number of simple representations.
10. A group with only 2 simple representations is simple.

11. If 3 characters in the affine space $\mathbb{C}^{\mathsf{G}}$ lie on the same line, then at least one of them is not simple.

12. If one conjugacy class of G contains half of the elements of G then there is a representation V such that $V \times V$ is trivial.

13. $\forall g \in \mathsf{G} : \|\chi_V(g)\| \leq \dim V$.

14. If G has a representation such that $\chi_(g) = -1$ if $g \neq 1$ then $\mathsf{G} = \mathbb{Z}_2$.

15. If all simple representations have odd dimension then G is simple.

16. The number of twosided ideals in the group algebra is equal to the number of conjugacy classes in G.

17. Two nonabelian groups with isomorphic group algebras are isomorphic.

18. Conjugation with an element of g is an endomorphism of the regular representation.

19. For every couple of finite groups, there exists an algebra morphism $\mathbb{C}\mathsf{G} \to \mathbb{C}\mathsf{H}$.

20. Every group algebra can be mapped surjectively to an abelian algebra.

Chapter 2

The representation variety

In this chapter we will consider representations of finitely generated algebras. Let A be an algebra defined by a finite number of generators and relations. This means that we can write A as a quotient of a free algebra:

$$A \cong \mathbb{C}\langle Y_1, \ldots, Y_k \rangle / \mathcal{R} \text{ with } \mathcal{R} = (r_1, \ldots, r_l).$$

We will write the generators of A as $y_i := Y_i \mod \mathcal{R}$.

2.1 $\mathsf{Rep}_n A$

An n-dimensional representation of A can be seen as an algebra morphism $\rho : A \to \mathsf{Mat}_{n\times n}(\mathbb{C})$. We will denote the set of n-dimensional representations as $\mathsf{Rep}_n A$. Every representation ρ is uniquely determined by the k-tuple of matrices $(\rho(y_1), \ldots, \rho(y_k))$ and therefore we can view $\mathsf{Rep}_n A$ as a subset of $\mathsf{Mat}_{n\times n}(\mathbb{C})^k = \mathbb{C}^{n^2 k}$. On the other hand if we have a k-tuple of matrices $(A_1 \ldots, A_k)$ that satisfies the relations $r_1, \ldots, r_k$, we can construct a morphism

$$\rho : A \to \mathsf{Mat}_{n\times n}(\mathbb{C}) : y_i \mapsto A_i$$

This implies that we can consider $\mathsf{Rep}_n A$ as the closed subset of $\mathbb{C}^{n^2 k}$ where the functions

$$f_{mij} : \mathbb{C}^{n^2 k} \to \mathbb{C} : (A_1 \ldots, A_k) \mapsto [r_m(A_1 \ldots, A_k)]_{ij}$$

are zero ($[r_m(A_1 \ldots, A_k)]_{ij}$ is the coefficient on the i^{th} and the j^{th} column of the matrix $r_m(A_1 \ldots, A_k)$). So $\mathsf{Rep}_n A$ is in fact an algebraic variety and its ring of polynomial functions is of the form

$$\mathbb{C}[\mathsf{Rep}_n A] = \mathbb{C}[x^s_{rt} | 1 \leq s \leq k \ 1 \leq r, t \leq n] / (f_{mij} | 1 \leq s \leq k \ 1 \leq r, t \leq n).$$

The structure of $\mathsf{Rep}_n A$ does not depend on the specific set of generators of A. Suppose we have another set of generators $z_1, \dots z_m$ we can express these as functions of the $y_1, \dots, y_k$ and vica versa

$$z_i = \zeta_i(y_1, \dots, y_k), \quad y_i = \eta_i(z_1, \dots, z_m).$$

If $\mathsf{Rep}'_n A \subset \mathbb{C}^{n^2 m}$ is the subset of m-tuples $(\rho(z_1), \dots, \rho(z_m))$ we can define polynomial maps

$$\begin{aligned} \phi &: \mathsf{Rep}_n A \to \mathsf{Rep}'_n A : (A_1, \dots, A_k) \mapsto (\zeta_1(A_1, \dots, A_k), \dots, \zeta_m(A_1, \dots, A_k)) \\ \phi' &: \mathsf{Rep}'_n A \to \mathsf{Rep}_n A : (B_1, \dots, B_m) \mapsto (\eta_1(B_1, \dots, B_k), \dots, \zeta_k(B_1, \dots, B_m)) \end{aligned}$$

who are inverses of each other so $\mathsf{Rep}'_n A$ and $\mathsf{Rep}_n A$ are isomorphic varieties.

Example 2.1. Finite group algebras are finitely genereated algebras f.i.

- $\mathbb{C}[\mathbb{Z}_n] \cong \mathbb{C}[X]/(X^n - 1)$,
- $\mathbb{C}[\mathfrak{S}_3] = \mathbb{C}\langle X, Y\rangle/(X^2 - 1, Y^2 - 1, (XY)^3 - 1)$,
- $\mathbb{C}[\mathsf{Q}_8] = \mathbb{C}\langle X, Y\rangle/(X^2 - 1, Y^2 - X, Z^2 - X, XY - YX, XZ - ZX, YZ - XYZ)$.

If we take the simplest example $A = \mathbb{C}[\mathbb{Z}_2] = \mathbb{C}[Y]/(Y^2 - 1)$ and look at the two dimensional representations

$$\mathsf{Rep}_2 A = \left\{ B \in \mathsf{Mat}_{2\times 2}(\mathbb{C}) | B^2 = 1. \right\}$$

then the ring of polynomial functions is

$$\begin{aligned} \mathbb{C}[\mathsf{Rep}_2 A] \cong \mathbb{C}[y_{11}, y_{12}, y_{21}, y_{22}]/(&y_{11}^2 + y_{12}y_{21} - 1, y_{22}^2 + y_{12}y_{21} - 1, \\ &y_{11}y_{12} + y_{12}y_{22}, y_{21}y_{11} + y_{22}y_{21}) \end{aligned}$$

Choosing another generator $Z = \frac{\sqrt{2}}{2}(Y + 1)$ then $A = \mathbb{C}[Z]/(Z^2 = Z)$ we obtain

$$\mathsf{Rep}'_2 A = \left\{ B \in \mathsf{Mat}_{2\times 2}(\mathbb{C}) | B^2 = B. \right\}$$

and

$$\begin{aligned} \mathbb{C}[\mathsf{Rep}'_2 A] \cong \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}]/(&z_{11}^2 + z_{12}z_{21} - z_{11}, z_{22}^2 + z_{12}z_{21} - z_{22}, \\ &z_{11}z_{12} + z_{12}z_{22} - z_{12}, z_{21}z_{11} + z_{22}z_{21} - z_{21}) \end{aligned}$$

The two rings $\mathbb{C}[\mathsf{Rep}_2 A]$ and $\mathbb{C}[\mathsf{Rep}'_2 A]$ are isomorphic by the identification

$$z_{ij} = \begin{cases} \frac{\sqrt{2}}{2}(y_{ij} + 1) & i = j \\ \frac{\sqrt{2}}{2} y_{ij} & i \neq j. \end{cases}$$

A further invesigation shows that $\mathsf{Rep}_n A$ consists of 3 connected components n.l. 2 points $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and a quadric $\{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) | a^2 - bc = 1\}$.

2.2 The GL_n-action

We recall that an action of a group G on a set X is a map $G \times X \to X : (g, x) \mapsto g \cdot x$ such that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. The orbit of a point x in X is defined as the set $\mathcal{O}_x = \mathsf{G}x = \{g \cdot x | g \in \mathsf{G}\}$.

On the set of k-tupels of matrices we have an action of the linear group GL_n by simultaneous conjugation:

$$\alpha \cdot (A_1, \ldots, A_k) := (\alpha A_1 \alpha^{-1}, \ldots, \alpha A_k \alpha^{-1}).$$

As we have seen in the previous lesson two n-dimensional representations ρ_1 and ρ_2 are isomorphic if there is an invertible linear map $\alpha \in \mathsf{GL}_n$ such that $\forall x \in A : \alpha\rho_1(x) = \rho_2(x)\alpha$. It suffices that the last identity holds for the generators so

$$\rho_1 \cong \rho_2 \Leftrightarrow \exists \alpha \in \mathsf{GL}_n : \alpha \cdot (\rho_1(y_1), \ldots, \rho_1(y_k)) = (\rho_2(y_1), \ldots, \rho_2(y_k)).$$

We can conclude that the orbits of the GL_n-action on $\mathsf{Rep}_n A \subset \mathsf{Mat}_{n\times n}(\mathbb{C})^k$ are in fact the isomorphism classes of representations.

Example 2.2. There are 3 isomorphism classes of 2-dimensional representations of $\mathbb{Z}_2$: $\rho_1^{\oplus 2}$, $\rho_{-1}^{\oplus 2}$ and $\rho_1 \oplus \rho_{-1}$. These three correspond to the three components of $\mathsf{Rep}_2\mathbb{C}[\mathbb{Z}_2]$.

For finite groups the number of orbits in $\mathsf{Rep}_n\mathbb{C}[\mathsf{G}]$ exactly coincides with the number of different irreducible components in $\mathsf{Rep}_n\mathbb{C}[\mathsf{G}]$. One can prove this in the following way: consider the function

$$\varphi : \mathsf{Rep}_n\mathbb{C}[\mathsf{G}] \to \mathbb{C}^{|G|} : \rho \mapsto (\chi_\rho(g))_g.$$

because the characters uniquely define the equivalence classes of representations, this function maps every orbit to a different point in $\mathbb{C}^{|G|}$. As φ is a continuous function (it is even polynomial) and $\mathsf{Im}\varphi$ is a discrete set all the orbits must sit in different connected components of $\mathsf{Rep}_n\mathbb{C}[\mathsf{G}]$. For more general groups and algebras this is not true. Let $A = \mathbb{C}[X]$ then $\mathsf{Rep}_2 A = \mathsf{Mat}_{2\times 2}(\mathbb{C})$ and consists of only 1 connected component but there are infinitely many orbits (f.i. all scalar matrices are different orbits).

For every orbit in $\mathsf{Rep}_n\mathbb{C}\mathsf{G}$ we can also calculate the dimension of the orbit:

Theorem 2.3. *If* $S = S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$ *then*

$$\dim \mathcal{O}_S = n^2 - e_1^2 - \cdots - e_k^2.$$

Proof. Define a map:

$$\mathsf{GL}_n \to \mathcal{O}_S : g \mapsto g \cdot S.$$

This map is obviously surjective. The fiber of the point gS is the group

$$\begin{aligned}\mathsf{Stab}_{gS} &= \{h \in \mathsf{GL}_n : hgS = gS\} = \{g^{-1}hg \in \mathsf{GL}_n : hS = S\}\\ &= g^{-1}(\mathsf{End}_{\mathsf{G}}(S) \cap \mathsf{GL}_n)g = g^{-1}(\mathsf{Aut}_{\mathsf{G}}S)g.\end{aligned}$$

As $x \mapsto g^{-1}xg$ is a bijection

$$\dim \mathsf{Stab}_{gS} = \dim Aut_{\mathsf{G}}(S) = \dim \mathsf{End}_{\mathsf{G}}(S) = e_1^2 + \dots e_k^2.$$

All fibers have the same dimension so

$$\dim \mathcal{O}_S = \dim \mathsf{GL}_n - \dim \text{fibers} = n^2 - (e_1^2 + \dots e_k^2)$$

□

Instead of having a variety that classifies all n-dimensional representations, we would like to have a variety that classifies all n-dimensional representations up to isomorphism. By this we mean that in this new variety Q every orbit of $\mathsf{Rep}_n A$ should correspond to one point and there exists a surjective morphism of varieties $\mathsf{Rep}_n A \to Q$ that maps every point of $\mathsf{Rep}_n A$ to the point in Q to its orbit.

For finite groups we have indeed such a variety: take $Q := \mathsf{Im}\varphi$ and as morphism $\varphi : \mathsf{Rep}_n A \to Q$. The variety Q consists of a finite number of points e so the ring of polynomial functions over Q will be

$$\mathbb{C}[Q] \cong \mathbb{C}^{\oplus e}.$$

Every polynomial function on Q will give us a polynomial function on Q if we compose it with φ. Therefore we have an algebramorphism

$$\varphi^* : \mathbb{C}[Q] \to \mathbb{C}[\mathsf{Rep}_n A] : f \mapsto f \circ \varphi.$$

This morphism is an injection because if $f \circ \varphi = 0$, f must also be zero. This means that we can see $\mathbb{C}[Q]$ as a subring of $\mathbb{C}[\mathsf{Rep}_n A]$. This subring consists of the functions that are constant on the orbits.

Another way of putting this is the following. We can use the action of GL_n on $\mathsf{Rep}_n A$ to construct a new action on $\mathbb{C}[\mathsf{Rep}_n A]$:

$$\mathsf{GL}_n \times \mathbb{C}[\mathsf{Rep}_n A] \to \mathbb{C}[\mathsf{Rep}_n A] : (g, f) \mapsto g \cdot f : \mathsf{Rep}_n A \to \mathbb{C} : x \mapsto f(g^{-1} \cdot x).$$

Note that this action is linear: $g \cdot (\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1(g \cdot f_1) + \lambda_2(g \cdot f_2)$ so in fact $\mathbb{C}[\mathsf{Rep}_n A]$ can be seen as an infinite dimensional representation of GL_n.

A function that is constant on the orbits remains invariant under this new action and vice versa. So in fact $\mathbb{C}[Q]$ is the ring of functions that are invariant under the GL_n-action this ring is also denoted by $\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n}$. In short we have the following diagram:

$$\begin{tikzcd} \text{Geometry} & \text{Algebra} \\ \mathsf{Rep}_n A \arrow[r, dotted, dash] \arrow[d, two heads] & \mathbb{C}[\mathsf{Rep}_n A] \\ Q \arrow[r, dotted, dash] & \mathbb{C}[Q] \arrow[u, hook] \end{tikzcd}.$$

For finite group algebras everything works fine: Q is a variety, every point corresponds uniquely to an orbit in $\mathsf{Rep}_n A$. For more general algebras this will be no longer the case, f.i. some points may correspond to more than one orbit. In the next lessons we will investigate these problems.

2.3 Exercises

Give descriptions of the following representation varieties

1. $\mathsf{Rep}_2 \mathbb{Z}_3$
2. $\mathsf{Rep}_2 \mathbb{Z}_4$
3. $\mathsf{Rep}_2 \mathfrak{S}_3$

2.3.1 Right or Wrong

Are the following statements right or wrong, if right prove them, if wrong disprove or find a counterexample.

1. $\mathsf{Rep}_2 \mathbb{C}\mathsf{G}$ of a finite abelian group algebra is a union of points and quadrics.
2. If $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ is connected then $n = 1$.
3. If $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ is connected then G is trivial.
4. $\mathsf{Rep}_n A \oplus B = \mathsf{Rep}_n A \oplus \mathsf{Rep}_n B$.

5. $\dim \mathsf{Rep}_n \mathbb{C}[\mathbb{Z}] = n^2$.
6. If A is a subalgebra of B then $\mathsf{Rep}_n B$ can be projected to $\mathsf{Rep}_n A$.
7. Two simple representations $\rho_1, \rho_2 : \mathbb{C}\mathsf{G} \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ sit in the same connected component if and only if they have the same kernel.
8. $\dim \mathsf{Rep}_n \mathbb{C}G = n^2 - 1$ for a finite number of n.
9. The number of components of $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ increases with n.
10. The number of components of $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ increases with $|\mathsf{G}|$.
11. If a component of $\mathsf{Rep}_n A$ is isomorphic to an affine space then A is a free algebra.
12. If $\dim \mathsf{Rep}_n \mathbb{C}\mathsf{G} > n^2 - n$ for a given $n \geq 2$ then G is not abelian.
13. If $\dim \mathsf{Rep}_n \mathbb{C}\mathsf{G} > n^2 - n$ for a given $n \geq 2$ then G is not abelian.
14. If $\mathsf{Rep}_n \mathbb{C}G$ is the same variety as $\mathsf{Rep}_n \mathbb{C}H$ for any n then the finite groups are isomorphic.
15. The dimension of the component of the regular representation is the same as the dimension of $\mathsf{Rep}_{|}\mathsf{G}|\mathbb{C}\mathsf{G}$.
16. The dimension of $\mathsf{Rep}_n \mathfrak{S}_3$ is $\frac{3}{4}n^2$ if n is even.
17. If $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ has a unique component of maximal dimension then this component corresponds to a simple representation.
18. If the orbit of $V^* \otimes V$ has dimension $\dim V^4 - 1$ then V is trivial.
19. If for a given $n > 1$ all components in $\mathsf{Rep}_n \mathbb{C}\mathsf{G}$ have the same dimension then G is trivial.
20. If $\mathsf{Rep}_n \mathsf{G}$ is a finite set then $n = 1$.

Chapter 3

The ring of invariants

From the previous lesson we remember that GL_n has an action on both $\mathsf{Rep}_n A$ and its ring of functions $\mathbb{C}[\mathsf{Rep}_n A]$. In this chapter we will look at the ring of invariant functions inside $\mathbb{C}[\mathsf{Rep}_n A]$ and we will prove that this ring is allways finitely generated. This means that $\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n}$ can be seen as the ring of functions over a variety: $\mathsf{iss}_n A$. This variety will be called the algebraic quotient of $\mathsf{Rep}_n A$. It will be studied in the next chapter.

First of all we must look at the representations of GL_n because, as we already know, $\mathbb{C}[\mathsf{Rep}_n A]$ is in fact an infinite dimensional representation of GL_n.

3.1 GL_n is a reductive group

We will prove that every finite dimensional representation of GL_n is a direct sum of simple representations.

First we will prove this for the group of unitary matrices $\mathsf{U}_n := \{A \in \mathsf{GL}_n | A^\dagger A = 1\}$. This is easier because U_n is a compact group.

The proof is very similar to the proof for finite groups but as U_n has an infinite number of elements, we cannot calculate the sums of the form

$$\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} f(g)$$

so instead of a sum we should use some kind of integral. This integral should satisfy the following properties:

- it is normed: $\int_{\mathsf{G}} 1 dg = 1$,

- it is right and left invariant: $\forall h \in \mathsf{G} : \int_{\mathsf{G}} f(gh)dg = \int_{\mathsf{G}} f(hg)dg = \int_{\mathsf{G}} f(g)dg$.

Example 3.1. The group U_1 can be seen as the complex numbers of the form $e^{i\phi}$ where $\phi \in [0, 2\pi[$. A function over U_1 is then a function over $\mathbb{R}$ with period 2π, so we can integrate it over the inverval $[0, 2\pi[$. To norm this we must divide this integral by 2π so

$$\int_{\mathsf{G}} f(g)dg := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi})d\phi$$

this integral is indeed left/right invariant because if $h = e^{i\theta}$

$$\begin{aligned}\int_{\mathsf{G}} f(hg)dg &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi+\theta})d\phi = \frac{1}{2\pi}\int_\theta^{2\pi+\theta} f(e^{i\phi})d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\phi})d\phi = \int_{\mathsf{G}} f(g)dg.\end{aligned}$$

One can prove that for every compact topological group there exists indeed such a unique integral and its corresponding measure is called the *Haar Measure*. The prove of this statement is in general quite technical and therefore we will omit it.

Miniature 2: Alfred Haar

Alfred Haar was born on 11 Oct 1885 in Budapest, Hungary. In 1904 Haar travelled to Germany to study at Göttingen and there he studied under Hilbert's supervision, obtaining his doctorate in 1909 with a dissertation entitled Zur Theorie der orthogonalen Funktionensysteme.

Haar is best remembered for his work on analysis on groups. In 1932 he introduced a measure on groups, now called the Haar measure, which allows an analogue of Lebesgue integrals to be defined on locally compact topological groups. It was used by von Neumann, by Pontryagin in 1934 and Weil in 1940 to set up an abstract theory of commutative harmonic analysis.

For U_n we use the standard embedding in $\mathsf{Mat}_{n\times n}(\mathbb{C})$. We can consider this last space as $\mathbb{R}^{2n^2}$ with the standard metric (in this way $\|A\| = \sqrt{\mathsf{Tr}AA^\dagger}$). This allows us to integrate over subvarieties using the standard technique of multiple integrals. This gives us an integral over U_n: choose a parametrization $g(\sigma_1, \dots \sigma_p)$ of U_n where p is the dimension of U_n and define

$$\oint_{\mathsf{U}_n} f(g)dg := \int \dots \int f(g(\sigma_1, \dots, \sigma_p))J(g)d\sigma_1 \dots d\sigma_p$$

where $J(g)$ is the p-dimensional volume of the box made by the vectors $\frac{dg}{d\sigma_1}, \dots, \frac{dg}{d\sigma_p}$. Note that this definition is independent of the parametrization g.

Multiplying the elements of $\mathsf{Mat}_{n\times n}(\mathbb{C})$ on left/right with a unitary matrix $X \in \mathsf{U}_n$ is in fact an orthogonal transformation because $\|AX\| = \sqrt{\mathsf{Tr} AX(AX)^\dagger} = \sqrt{\mathsf{Tr} AXX^\dagger A^\dagger} = \sqrt{\mathsf{Tr} AA^\dagger} = \|A\|$. As orthogonal transformations do not change the metric $J(g) = J(Xg) = J(gX)$, so $dg = d(Xg) = d(gX)$ and

$$\oint_{\mathsf{U}_n} f(Xg)dg = \oint_{\mathsf{U}_n} f(Xg)dXg = \oint_{\mathsf{U}_n} f(g)dg = \oint_{\mathsf{U}_n} f(gX)dg.$$

The integral is left and right invariant. To make it normed we have to divide out the p-volume of U_n.

$$\int_{\mathsf{U}_n} f(g)dg := \frac{\oint_{\mathsf{U}_n} f(g)dg}{\oint_{\mathsf{U}_n} 1dg}.$$

This volume is finite because U_n is compact.

Example 3.2. The set U_2 is a 4-dimensional subset of $\mathbb{R}^8$. We can see this set as the product of a 3-dimensional sphere and a circle: If $A \in \mathsf{U}_2$ then $|A| \in \mathsf{U}_1$ and $A/|A| \in \mathsf{SU}_2 = \{A \in \mathsf{U}_2 | \det A = 1\}$. The last set is a threedimensional sphere because it are the matrices of the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \text{ with } a^2+b^2+c^2+d^2 = 1$$

Now parametrize $|A|$ as $e^{i\theta_0}$ and $A/|A|$ by $a = \cos\theta_1\cos\theta_2$, $b = \cos\theta_1\sin\theta_2$, $c = \sin\theta_1\cos\theta_3$, $d = \sin\theta_1\sin\theta_3$ with $\theta_0, \theta_2, \theta_3 \in [0, 2\pi[$ and $\theta_1 \in [0, \pi/2[$.

If we calculate the derivatives of this parametrization we see that all $\frac{\partial A}{\partial \theta_i}$ are perpendicular and have lengths $\sqrt{2}$,$\sqrt{2}$,$\sqrt{2}\cos\theta_1$,$\sqrt{2}\sin\theta_1$ so $J(A) = \sqrt{2}^4 \sin\theta_1\cos\theta_1 = 2\sin 2\theta_1$. The volume of U_2 is then $4(2\pi)^3$

So

$$\int_{\mathsf{U}_2} f(g)dg = \frac{1}{16\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(A(\vec{\theta}))\sin 2\theta_1 d\theta_0 \dots d\theta_3.$$

Now using the Haar measure we can prove the reductivity of compact groups.

Theorem 3.3 (Reductivity of compact groups). *Let G be a compact group and V, ρ_V a representation of G. If W is a subrepresentation of V then there exist another subrepresentation $W^\perp$ such that $V = W \oplus W^\perp$.*

Proof. Let $\pi : V \to W$ a projection from $V \to W$, and define

$$\pi_{\mathsf{G}} : V \to W : x \to \int_{\mathsf{G}} g\pi(g^{-1}x)dg$$

This map is again a projection onto W because if $w \in W$ then $g^{-1}w \in W$ so $\pi(g^{-1}w) = g^{-1}w$ and

$$\pi_{\mathsf{G}}(w) = \int_{\mathsf{G}} g\pi(g^{-1}w)dg = \int_{\mathsf{G}} gg^{-1}wdg = w\int_{\mathsf{G}} 1dg = w.$$

The kernel of this map is also a subrepresentation because if $\pi_{\mathsf{G}}(x) = 0$ then

$$\begin{aligned}\pi_{\mathsf{G}}(hx) &= \int_{\mathsf{G}} g\pi(g^{-1}hx)dg \\ &= h\int_{\mathsf{G}} (h^{-1}g)\pi((g^{-1}h)x)dg \\ &= w\int_{\mathsf{G}} g\pi(g^{-1}x)dg = h\pi_{\mathsf{G}}(x) = 0.\end{aligned}$$

As $V = \mathsf{Ker}\pi_{\mathsf{G}} \oplus \mathsf{Im}\pi_{\mathsf{G}}$ and $\mathsf{Im}\pi_{\mathsf{G}} = W$ we can choose $W^{\perp} = \mathsf{Ker}\pi_{\mathsf{G}}$. $\square$

We now have the reductivity of compact groups, but GL_n is not compact. However GL_n contains U_n as a subgroup. This implies that if $\rho_V : \mathsf{GL}_n \to \mathsf{GL}(V)$ is a representation of GL_n, $\rho_V|_{\mathsf{U}_n}$ is a representation of U_n. If V has a GL_n-subrepresentation W, this subrepresentation is also a U_n-subrepresentation so we can find a U_n-subrepresentation $W^{\perp}$ such that $V = W \oplus W^{\perp}$. We will prove that $W^{\perp}$ is also a GL_n-subrepresentation.

To do this we need a small lemma:

Lemma 3.4. *If $\phi : \mathsf{GL}_n \to \mathbb{C}^m : x \mapsto (\phi_1, \ldots, \phi_m)$ is a complex polynomial map such that $\phi|_{\mathsf{U}_n} = 0$ then $\phi = 0$.*

Proof. Let x_{ij} be the n^2 standard coordinates for $\mathsf{Mat}_{n\times n}(\mathbb{C})$. Then the functions ϕ_i are rational function in the x_{ij}.

Suppose that A is an antihermitian matrix: $A^{\dagger} = -A$, then e^{tA} is a unitary matrix because $(e^{tA})^{\dagger} = e^{(tA^{\dagger})} = e^{-tA} = (e^{tA})^{-1}$. Therefore

$$\sum_{ij} A_{ij}\frac{\partial}{\partial X_{ij}}\phi|_1 = \lim_{\epsilon\to 0}\frac{\phi(e^{\epsilon A}) - \phi(1)}{\epsilon} = 0.$$

In the formula above the partial derivatives are complex but we can choose the $\epsilon \in \mathbb{R}$.

Now we can choose a complex basis for $\mathsf{Mat}_{n\times n}(\mathbb{C})$ that consist of antihermitian matrices (B^{ij}):

- If $i \leq j$ then B^{ij} is the matrix with i on the entries i, j and j, i, and zero elsewhere.
- If $i > j$ then B^{ij} is the matrix with 1 on the place i, j, -1 on the entry j, i and zero elsewhere.

This implies that
$$\frac{\partial}{\partial X_{ij}}\phi|_1 = 0.$$
Analoguously we can prove that also all higher order derivatives are zero, so $\phi = 0$. $\square$

Theorem 3.5 (Reductivity of GL_n). *Let $\rho_V : \mathsf{GL}_n \to \mathsf{GL}(V)$ be a rational representation (this means that ρ_V is also a morphism of complex varieties). If W is a subrepresentation of V then there exist another subrepresentation $W^\perp$ such that $V = W \oplus W^\perp$.*

Proof. Let π be a projection of V onto W and define π_{U_n} as before. Denote the kernel of π_{U_n} by $W^\perp$. We will prove that $W^\perp$ is a GL_n-subrepresentation.

For a given $x \in W^\perp$ we can construct the map
$$\pi_x : \mathsf{GL}_n \to W : g \mapsto \pi_{\mathsf{U}_n}(\rho_V(g)x).$$
This map is polynomial because both $\rho_V(g)$ and π_{U_n} are. As $\pi_x|_{\mathsf{U}_n} = 0$ we know from the lemma that $\pi_x = 0$ so $\forall g \in \mathsf{GL}_n : gx \in \mathsf{Ker}\pi_{\mathsf{U}_n} = W^\perp$ and hence $W^\perp$ is a GL_n-subrepresentation and $V = W \oplus W^\perp$. $\square$

The extra condition that ρ_V is also a morphism of varieties is a natural condition and it will hold in all cases we will consider.

We now have proved that every representation of GL_n is a direct sum of simple representations. We will not classify these simple representations because this would lead us far away from the scope of the course notes: representations of finitely generated algebras. This subject will be dealt with in the course Lie theory.

3.2 The ring of invariants

Suppose now we have a reductive group G and let Ω be the set of its simple representations up to isomorphism. Let V be a finite dimensional representation with dimension k. The ring of polynomial functions over V is $R = \mathbb{C}[V] \cong \mathbb{C}[X_1, \dots, X_k]$ is a graded polynomial ring if we give the X_i degree 1.

On R we have an action of G:

$$\mathsf{G} \times \mathbb{C}[V] \to \mathbb{C}[V] : (g, f) \mapsto g \cdot f := f \circ \rho_V(g^{-1}).$$

This action is linear and compatible with the algebra structure: $g \cdot f_1 f_2 = (g \cdot f_1)(g \cdot f_2)$. As $g \cdot X_i : \sum_j \rho_V(g^{-1})_{ij} X_j$ is homogeneous of degree 1 the G-action maps homogeneous elements of to homogeneous elements with the same degree. This means that all homogeneous components R_κ are finite dimensional rational representations of G.

We can decompose every R_κ as a direct sum of simple representations

$$R_\kappa = \bigoplus_{\omega \in \Omega} W_\kappa^\omega \text{ with } W_\kappa^\omega \cong \omega^{\oplus e_{\kappa\omega}}.$$

If we define then the *isotopic components* of R as $R^\omega = \oplus_\kappa W_\kappa^\omega$. Now we can regroup the terms in the direct sum of our ring to obtain

$$R \cong \bigoplus_{\kappa=0, \omega \in \Omega}^{\infty} W_\kappa^\omega = \bigoplus_{\omega \in \Omega} R^\omega.$$

In words, the ring R is the direct sum of its isotopic components. Note also that if α is a endomorphism of R as a G-representation then α will map isotopic components inside themselves because of Schur's lemma (try to prove this as an exercise).

One isotopic component interests us specially: the isotopic component of the trivial representation $\mathsf{1}$. This component consists of all functions that are invariant under the G-action. It is not only a vector space but it is also a graded ring: the ring of invariants $S = R^{\mathsf{G}} = \{f \in R | g \cdot f = f\} = \oplus_{\kappa \in \mathbb{N}} W_\kappa^{\mathsf{1}}$.

Consider a function $f \in S$. The map $\mu_f : R \to R : x \to fx$ is an endomorphism of R as a representation: $\mu_f(g \cdot x) = f(g \cdot x) = (g \cdot f)(g \cdot x) = g \cdot fx = g \cdot \mu_f x$. So $fR^\omega \subset R^\omega$ for every ω. Put in another way we can say that all isotopic components are S-modules.

We are now ready to prove the main theorem:

Theorem 3.6. *If G is a reductive group and V a finite dimensional representation then the ring of invariants $\mathbb{C}[V]^{\mathsf{G}}$ is finitely generated.*

Proof. To prove that S is finitely generated we first prove that this ring is noetherian. Suppose that

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots$$

is an ascending chain of ideals in S. Multiplying with R we obtain a chain of ideals in R:

$$\mathfrak{a}_1 R \subset \mathfrak{a}_2 R \subset \mathfrak{a}_3 R \subset \cdots .$$

This chain is stationary because R is a polynomial ring and hence noetherian. Finally we show that $\mathfrak{a}_i R \cap S = \mathfrak{a}_i$. Multiplication with $\mathfrak{a}_i$ maps the isotopic components into themselves so

$$(\mathfrak{a}_i R) \cap S = \left(\mathfrak{a}_i \bigoplus_{\omega \in \Omega} R^\omega\right) \cap S = \left(\bigoplus_{\omega \in \Omega} \mathfrak{a}_i R^\omega\right) \cap S = \mathfrak{a}_i R^1 = \mathfrak{a}_i S = \mathfrak{a}_i.$$

Now let $S_+ = \oplus_{\kappa \geq 1} W^1_\kappa$ denote the ideal of S generated by all homogeneous elements of nonzero degree. Because S is noetherian, S_+ is generated by a finite number of homogeneous elements: $S_+ = f_1 S + \cdots + f_r S$. We will show that these f_i also generate S as a ring.

Now $S = \mathbb{C} + S_+$ so $S_+ = \mathbb{C}f_1 + \cdots + \mathbb{C}f_r + S_+^2$, $S_+^2 = \sum_{i,j} \mathbb{C} f_i f_j + S_+^3$ and by induction

$$S_+^t = \sum_{i_1 \ldots i_t} \mathbb{C} f_{i_1} \cdots f_{i_t} + S_+^{t+1}.$$

So $\mathbb{C}[f_1, \ldots, f_r]$ is a graded subalgebra of S and $S = \mathbb{C} + S_+ = \mathbb{C}[f_1, \ldots, f_r] + S_+^t$ for every t. If we look at the degree d-part of this equation we see that

$$S_d = \mathbb{C}[f_1, \ldots, f_r]_d + (S_+^t)_d.$$

Because S_+^t only contains elements of degree at least t, $(S_+^t)_d = 0$ if $t > d$. As the equation holds for every t we can conclude that

$$S_d = \mathbb{C}[f_1, \ldots, f_r]_d \text{ and thus } S = \mathbb{C}[f_1, \ldots, f_r]$$

$\square$

We now return to the setting of representations of finitely generated algebras. From the previous chapter we know that $\mathsf{Rep}_n A$ can be considered as a closed subset of the vector space $\mathsf{Mat}_{n \times n}(\mathbb{C})^k$. This space has a linear action of GL_n by simultaneous conjugation. By theorem **??** we know that the ring of invariant functions $\mathbb{C}[\mathsf{Mat}_{n \times n}(\mathbb{C})^k]^{\mathsf{GL}_n}$ is finitely generated.

Now as the ring of polynomial function over $\mathsf{Rep}_n A$ is a quotient ring of $\mathbb{C}[\mathsf{Mat}_{n \times n}(\mathbb{C})^k]$:

$$\mathbb{C}[\mathsf{Rep}_n A] = \mathbb{C}[\mathsf{Mat}_{n \times n}(\mathbb{C})^k]/\mathfrak{n} \text{ with } \mathfrak{n} = \{f | f|_{\mathsf{Rep}_n A} = 0\}.$$

Because $\mathsf{Rep}_n A$ is closed under the GL_n-action, $g \cdot \mathfrak{n} = \mathfrak{n}$, so the GL_n-action is compatible with the quotient:

$$g \cdot (f + \mathfrak{n}) = g \cdot f + \mathfrak{n}.$$

This means that $\mathfrak{n}$ is a subrepresentation of $R = \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]$ and we can decompose $\mathfrak{n}$ as a direct sum of isotopical components:

$$\mathfrak{n} = \bigoplus_{\omega} \mathfrak{n} \cap R^{\omega} = \bigoplus_{\omega} \mathfrak{n}^{\omega}.$$

Taking the quotient we get

$$\mathbb{C}[\mathsf{Rep}_n A] = R/\mathfrak{n} = \bigoplus_{\omega} R^{\omega} / \bigoplus_{\omega} \mathfrak{n}^{\omega} = \bigoplus_{\omega} R^{\omega}/n^{\omega}.$$

This shows us that the invariants of $\mathbb{C}[\mathsf{Rep}_n A]$ correspond to the summand $R^{1}/\mathfrak{n}^{1}$ and are the invariants of $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]$ modulo $\mathfrak{n}^1$:

$$\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n} = \frac{\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n}}{\mathfrak{n} \cap \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n}} = \frac{\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n} + \mathfrak{n}}{\mathfrak{n}}.$$

Thus the generators of $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n}$ modulo $\mathfrak{n} \cap \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n}$ are generators of $\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n}$. This allows us to conclude:

Theorem 3.7. *For a finitely generated algebra A, $\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n}$ is a finitely generated ring.*

3.3 The generators of $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$

3.3.1 Polarization and restitution

The action of GL_n on polynomial maps $f \in \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]$ is fully determined by the action on the coordinate functions x_{ij}^k. The action maps every x_{ij}^k to a linear combination of coordinate functions with the same k Hence, we can define a gradation on $\mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]$ by $deg(x_{ij}^k) = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 at place k) and decompose

$$\mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m] = \bigoplus_{(d_1,\dots,d_m)\in\mathbb{N}^m} \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)}$$

where $\mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)}$ is the subspace of all multihomogeneous forms f in the x_{ij}^k of degree $(d_1, \dots, d_m)$, that is, in each monomial term of f there are exactly d_k factors that are coordinate functions labeled by k, for all $1 \le k \le m$. The action of GL_n stabilizes each of the subspaces $\mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)}$, that is,

$$\text{if } f \in \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)} \quad \text{then } g.f \in \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)} \text{ for all } g \in \mathsf{GL}_n.$$

In particular, if f determines a polynomial map on $\mathsf{Mat}_n(\mathbb{C})^m$ which is constant along orbits, that is, if f belongs to the ring of invariants $\mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]^{\mathsf{GL}_n}$, then each of its multihomogeneous components is also an invariant and therefore the ring of invariants is generated by multihomogeneous invariants.

Definition 3.8. If $f \in \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(1,\dots,1)}$ we will call f *multilinear*. Equivalently f is multilinear if for all $i \in \{1,\dots,m\}$

$$f(A_1,\dots,A_i+\lambda B,\dots,A_m) = f(A_1,\dots,A_i,\dots,A_m) + \lambda f(A_1,\dots,B,\dots,A_m)$$

First, we will show that we can derive the multihomogeneous invariants from the multilinear ones.

Let $f \in \mathbb{C}[\mathsf{Mat}_n(\mathbb{C})^m]_{(d_1,\dots,d_m)}$ and take for each $1 \le k \le m$ d_k new variables $t_{1k},\dots,t_{d_k k}$. Expand

$$f(t_{11}A_1^{(1)} + \ldots + t_{d_1 1}A_{d_1}^{(1)}, \dots, t_{1m}A_1^{(m)} + \ldots + t_{d_m m}A_{d_m}^{(m)})$$

as a polynomial in the variables t_{ik}, then we get an expression

$$\sum t_{11}^{s_{11}} \ldots t_{d_1 1}^{s_{d_1 1}} \ldots t_{1m}^{s_{1m}} \ldots t_{d_m m}^{s_{d_m m}} \cdot f_{(s_{11},\dots,s_{d_1 1},\dots,s_{1m},\dots,s_{d_m m})}(A_1^{(1)},\dots,A_{d_1}^{(1)},\dots,A_1^{(m)},\dots,A_{d_m}^{(m)})$$

such that for all $1 \le k \le m$ we have $\sum_{i=1}^{d_k} s_{ik} = d_k$. Moreover, each of the $f_{(s_{11},\dots,s_{d_1 1},\dots,s_{1m},\dots,s_{d_m m})}$ is a multi-homogeneous polynomial function on

$$\underbrace{\mathsf{Mat}_n(\mathbb{C}) \oplus \ldots \oplus \mathsf{Mat}_n(\mathbb{C})}_{d_1} \oplus \ldots \oplus \underbrace{\mathsf{Mat}_n(\mathbb{C}) \oplus \ldots \oplus \mathsf{Mat}_n(\mathbb{C})}_{d_m}$$

of multi-degree $(s_{11},\dots,s_{d_1 1},\dots,s_{1m},\dots,s_{d_m m})$. Observe that if f is an invariant polynomial function on $\mathsf{Mat}_n(\mathbb{C})^m$, then each of these multihomogeneous functions is an invariant polynomial function on $\mathsf{Mat}_n(\mathbb{C})^{\oplus D}$ where $D = d_1 + \ldots + d_m$.

In particular, we consider the function

$$f_{1,\dots,1} : \mathsf{Mat}_n(\mathbb{C})^{\oplus D} = \mathsf{Mat}_n(\mathbb{C})^{\oplus d_1} \oplus \ldots \oplus \mathsf{Mat}_n(\mathbb{C})^{\oplus d_m} \to \mathbb{C}$$

is multilinear and we will call it the *polarization* of the polynomial f and denote it with $\mathsf{Pol}(f)$.

We can recover f back from its polarization. We claim to have the equality

$$\mathsf{Pol}(f)(\underbrace{A_1,\dots,A_1}_{d_1},\dots,\underbrace{A_m,\dots,A_m}_{d_m}) = d_1! \ldots d_m! f(A_1,\dots,A_m)$$

and hence we recover f. This process is called *restitution*. The claim follows from the observation that

$$\begin{aligned} &f(t_{11}A_1 + \ldots + t_{d_1 1}A_1, \dots, t_{1m}A_m + \ldots + t_{d_m m}A_m) = \\ &\qquad f((t_{11} + \ldots + t_{d_1 1})A_1, \dots, (t_{1m} + \ldots + t_{d_m m}A_m) = \\ &\qquad\qquad (t_{11} + \ldots + t_{d_1 1})^{d_1} \ldots (t_{1m} + \ldots + t_{d_m m})^{d_m} f(A_1,\dots,A_m) \end{aligned}$$

and the definition of $\mathsf{Pol}(f)$. Hence we have proved

Theorem 3.9. *Any multi-homogeneous invariant polynomial function f on $\mathsf{Mat}_n(\mathbb{C})^m$ of multidegree $(d_1, \ldots, d_m)$ can be obtained by restitution of a multilinear invariant function over $\mathsf{Mat}_n(\mathbb{C})^{d_1+\ldots+d_m}$.*

So it remains to determine the multilinear invariants of $\mathsf{Mat}_n(\mathbb{C})^m$. But in order to do that we need something from the theory of finite groups.

3.3.2 Intermezzo: the double centralizer theorem

In this intermezzo we consider a finite group G and right $\mathbb{C}\mathsf{G}$-module V. Note that a right module is also a representation of G by the action $g \cdot v = v \cdot g^{-1}$ so the whole theory of representations is also applicable to right modules.

So we know that this module can be seen as a direct sum of simple representations:

$$V \cong S_1^{\oplus e_1} \oplus \cdots \oplus S_k^{\oplus e_k}$$

and we have that

$$\mathsf{End}_{\mathbb{C}\mathsf{G}}(V) \cong \mathsf{Mat}_{e_1 \times e_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{e_k \times e_k}(\mathbb{C}).$$

From now on we will denote the image of $\mathbb{C}\mathsf{G}$ under ρ_V by A and $\mathsf{End}_{\mathbb{C}\mathsf{G}}(V) = B$. A and B are subalgebras from $\mathsf{End}(V)$ and by definition B is the centralizer of A:

$$B = \{\phi \in \mathsf{End}(V) | \forall x \in \mathbb{C}\mathsf{G} : \rho_V(x)\phi = \phi\rho_V(x)\} = Z(A).$$

We will now prove also the opposite

Theorem 3.10. *Under these identifications we have that*

$$A = Z(B) = \mathsf{End}_B(V),$$

where we can see V as a B-representation by the standard injection $B \to \mathsf{End}(V)$.

Proof. From the first chapter we know that $\mathbb{C}\mathsf{G}$ is a direct sum of matrix algebras $\mathsf{End}S$ for every simple representation S. Because the action is on the right every S can be seen as the space of row vectors $\mathbb{C}^{\dim S}$ where $\mathsf{Mat}_{\dim S \times \dim S}(\mathbb{C})$ acts by multiplication on the right. Each of these matrix algebras acts thus cannonically on its own S and as the zero on the others. So

$$A = \mathsf{Mat}_{\dim S_1 \times \dim S_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{\dim S_k \times \dim S_k}(\mathbb{C}),$$

where the summation is over the simples that occur in V. Now let for every S_i, $d_i^j, 1 \leq j \leq \dim S_i$ denote the element in A that has a one on the j^{th} diagonal element of $\mathsf{Mat}_{\dim S_i \times \dim S_i}(\mathbb{C})$ and zero everywhere else. For a given simple S

$$Sd_i^j = \begin{cases} 0 & \text{if } S \ncong S_i \\ \mathbb{C} & \text{if } S \cong S_i \end{cases}$$

So for the representation V we have that

$$Vd_i^j = (S_i d_i^j)^{e_i} = \mathbb{C}^{e_i}$$

On this space the i^{th} acts $B = \mathsf{Mat}_{e_1 \times e_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{e_k \times e_k}(\mathbb{C})$ by multiplication on the left, so Vd_i^j is a simple B-representation. Trivially one can see that $Vd_i^j \cong Vd_k^l$ if and only if $k = i$, Let's denote these simple representations by $T_i = Vd_i^j$.

So as B-representation we can decompose V as

$$V = T_1^{\dim S_1} \oplus \cdots \oplus T_k^{\dim S_k}.$$

This implies that by Schur's lemma $Z(B) := \mathsf{End}_B V = \mathsf{Mat}_{dimS_1 \times dimS_1}(\mathbb{C}) \oplus \cdots \oplus \mathsf{Mat}_{dimS_k \times dimS_k}(\mathbb{C})$ is equal to A. $\square$

3.3.3 The multilinear invariants

Let V denote the standard n-dimensional complex vector space. This space has a natural $\mathsf{GL}_n(\mathbb{C})$-action so we can consider V as a GL_n-representation. With this notation $\mathsf{Mat}_n(\mathbb{C})^m$ is equal to the representation

$$(V^* \otimes V)^{\oplus m}.$$

From linear algebra we know that the set of multilinear maps over a direct sum of vector spaces $V_1 \oplus \cdots \oplus V_k$ can be seen as the tensor product $V_1^* \otimes \cdots \otimes V_k^*$. With this in mind we can identify the set of multilinear maps over $\mathsf{Mat}_n(\mathbb{C})^m$ with the representation

$$(V^* \otimes V)^{*\otimes m} \cong (V \otimes V^*)^{\otimes m} \cong (V^{\otimes m} \otimes V^{\otimes m *}) \cong \mathsf{End}(V^{\otimes m}).$$

The GL_n-invariant multilinear functions can then be seen as the GL_n-linear endomorphisms of $V^{\otimes m}$, so we have to determine $\mathsf{End}_{\mathsf{GL}_n}(V^{\otimes m})$.

On the other hand we have that $V^{\otimes m}$ has also the structure of a right $\mathfrak{S}_m$-module by means of the action

$$(v_1 \otimes \cdots \otimes v_m)^\sigma = v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_m}$$

This representation gives us a map

$$\rho : \mathbb{C}\mathfrak{S}_m \to \mathsf{End}(V^{\otimes m})$$

We will prove that the image of this map is exactly $\mathsf{End}_{\mathsf{GL}_n}(V^{\otimes m})$.

For this we will use the double centralizer theorem: from the fact that $V^{\otimes m}$ is a GL_n-representation we have a map from GL_n to $\mathsf{End}(V^{\otimes m})$. The image of this map is not an algebra but we can consider the smallest algebra that contains this image and denote it by B. Of course we then have that $\mathsf{End}_{\mathsf{GL}_n}(V^{\otimes m}) = \mathsf{End}_B(V^{\otimes m})$. If we can prove that B is equal to $\mathsf{End}_{\mathbb{C}\mathfrak{S}_m} V^{\otimes m}$ then we are done by the double centralizer theorem.

Under the identification $\mathsf{End}(V^{\otimes m}) = \mathsf{End}(V)^{\otimes m}$ an element $g \in GL_n$ is mapped to the symmetric tensor $g \otimes \ldots \otimes g$. So by definition B is the span of all tensors $g \otimes \ldots \otimes g,\ g \in \mathsf{GL}_n$.

On the other hand, the image of $\mathsf{End}_{\mathfrak{S}_m}(V^{\otimes m})$ in $\mathsf{End}(V)^{\otimes m}$ is the subspace of all symmetric tensors in $\mathsf{End}(V)^{\otimes m}$. We can give a basis of this subspace as follows.

Let $\{e_{11}, \ldots, e_{nn}\}$ be a basis of $\mathsf{End}(V)$, then the vectors $e_{i_1 j_1} \otimes \ldots \otimes e_{i_m j_m}$ form a basis of $\mathsf{End}(V_n)^{\otimes m}$. The $\mathfrak{S}_m$-action permutes these basis elements and two basis elements sit in the same orbit if they contain the e_{ij} with the same multiplicity. So for every map

$$h : \{e_{11}, \ldots, e_{nn}\} \to \mathbb{N} \text{ with } \sum_{i,j} f(e_{ij}) = m$$

we have a basis element for $\mathsf{End}_{\mathfrak{S}_m}(V^{\otimes m})$

$$e_h = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} (\underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{f(e_{11})\times} \otimes \cdots \otimes \underbrace{e_{nn} \otimes \cdots \otimes e_{nn}}_{f(e_{nn})\times})^{\sigma}.$$

If B is not the whole space $\mathsf{End}_{\mathfrak{S}_m}(V^{\otimes m})$ there would be a nontrivial element in the dual space $\mathsf{End}_{\mathfrak{S}_m}(V^{\otimes m})^*$ that is zero on all elements of B. Denote this element by

$$\alpha := \sum_h \alpha_h e_h^*.$$

Embedding $\mathsf{End}(V)$ diagonally in $\mathsf{End}(V)^{\otimes m}$ we can construct a homogeneous polynomial function on $\mathsf{End}(V)$:

$$f_\alpha : \mathsf{End}(V) \to \mathbb{C} : v \mapsto \sum_h \alpha_h e_h^*(v \otimes \cdots \otimes v).$$

If $x_{ij} \in \mathbb{C}[\mathsf{End}(V)]$ is the coordinate function of $\mathsf{End}(V)$ corresponding to the basis element e_{ij} then

$$\alpha := \sum_h \alpha_h \prod_{i,j} x_{ij}^{h(e_{ij})}.$$

For every h the monomial $\prod_{i,j} x_{ij}^{h(e_{ij})}$ is different so $\alpha = 0 \Leftrightarrow f_\alpha = 0$.

If $\alpha|_B = 0$ then we know that $f_\alpha|_{\mathsf{GL}_n} = 0$ and as GL_n is a dense subspace of $\mathsf{End}(V)$ we can conclude that $f_\alpha = 0$. So $\alpha|_B = 0 \Leftrightarrow \alpha = 0$ and hence $B = \mathsf{End}_{\mathfrak{S}_m}(V^{\otimes m})$.

By the double centralizer theorem we can now conclude that $\mathsf{End}_{\mathsf{GL}_n}(V^{\otimes m})$ is spanned by elements of the form

$$f_\sigma : V^{\otimes m} \to V^{\otimes m} : (v_1 \otimes \cdots \otimes v_m) \mapsto (v_1 \otimes \cdots \otimes v_m)^\sigma,\ \sigma \in \mathfrak{S}_m$$

The only thing that we still have to do is to reinterpret these elements as multilinear functions over $\mathsf{Mat}_{n\times n}(\mathbb{C})^{\oplus m}$. Let $w_1, \dots, w_n$ be the standaard basis for V and let $w_1^*, \dots, w_n^*$ be its dual basis with this notation we can write

$$\begin{aligned} f_\sigma &= \sum_{i_1,\dots i_m} (w_{i_1}^* \otimes \cdots \otimes w_{i_m}^*) \otimes (w_{i_{\sigma_1}} \otimes \cdots \otimes w_{i_{\sigma_m}}) \\ &= \sum_{i_1,\dots i_m} (w_{i_1}^* \otimes w_{i_{\sigma_1}}) \otimes \cdots \otimes (w_{i_m}^* \otimes w_{i_{\sigma_m}}) \end{aligned}$$

The element $w_i^* \otimes w_j \in \mathsf{Mat}_{n\times n}(\mathbb{C})^*$ returns from a given matrix M the coefficient M_{ij}. So for a given m-tuple $(M^1, \dots, M^m)$ we have that

$$f_\sigma(M^1, \dots, M^m) = \sum_{i_1,\dots i_m} M^1_{i_1 i_{\sigma_1}} \cdots M^m_{i_m i_{\sigma_m}}.$$

we can now split up the permutation in cycles:

$$\sigma = (1\sigma(1)\sigma^2(1)\dots)\dots(k\sigma(k)\sigma^2(k)\dots) = (c_1^1 \dots c_l^1)\dots(c_1^k \dots c_s^k).$$

Using these cycles we regroup the product and rename the indices such that the sumations become matrix products

$$\begin{aligned} f_\sigma(M^1, \dots, M^m) &= \sum_{j_1,\dots j_m} (M^{c_1^1}_{j_1 j_2} \cdots M^{c_l^1}_{j_l j_1}) \cdots (M^{c_1^k}_{j_{m-s+1} j_{m-k+2}} \cdots M^{c_s^k}_{j_m j_{m-s+1}}) \\ &= \sum_j (M^{c_1^1} \cdots M^{c_l^1})_{jj} \times \cdots \times \sum_j (M^{c_1^k} \cdots M^{c_s^k})_{jj} \\ &= \mathsf{Tr}(M^{c_1^1} \cdots M^{c_l^1}) \cdots \mathsf{Tr}(M^{c_1^k} \cdots M^{c_s^k}) \end{aligned}$$

So f_σ is a product of traces of products of matrices in such a way that every matrix M^l occurs exactly once. These functions form a basis for the multilinear invariants of $\mathsf{Mat}_{n\times n}(\mathbb{C})^{\oplus m}$. To obtain all invariants we simply have to apply restitution. In this case this means that we allow some matrices to occur more than once and some nowhere. Also because we are looking for generators and not for a basis we can take as generators just the traces and not the products of traces. We can conclude with

Theorem 3.11 (First fundamental theorem of matrix invariants). *The ring* $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$ *is generated by functions of the form* $\mathsf{Tr}M_1\cdots M_k$ *where* $k\in\mathbb{N}$ *and* $M_1,\ldots,M_k$ *are matrices out of the set* $\{X^1,\ldots,X^m\}$, $X^l := (x^l_{ij})$ *which consists of the matrices of coordinate functions (i.e. the generic matrices).*

Of course not all these generators are needed because there are an infinite number of them and as we know $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$ is finitely generated. One can prove that one needs only the generators of degree at most 2^n but we will omit this proof.

3.4 Exercices

1. Determine all simple (polynomial) representations of $\mathsf{GL}_1=\mathbb{C}^*$, and more general for the complex tori $\mathbb{T}^i=\mathbb{C}^*\times\cdots\times\mathbb{C}^*$.
2. Show that $\mathbb{C}[\mathsf{Mat}_{2\times 2}(\mathbb{C})^2]^{\mathsf{GL}_2}$ is generated by five invariants, and show that there are no relations between them.
3. Show that $\mathbb{C}[\mathsf{Mat}_{2\times 2}(\mathbb{C})^3]^{\mathsf{GL}_2}$ is generated by ten invariants.

3.4.1 Right or Wrong

Are the following statements right or wrong, if right prove them, if wrong disprove or find a counterexample.

1. If dg is a Haar measure on G and dh on H then $dgdh$ is a Haar measure on $\mathsf{G}\times H$.
2. Let dg be a Haar measure on a compact Lie group G. If H is subgroup of H then $(\int_{\mathsf{H}} dg)$ is zero or the inverse of a natural number.
3. Every topological group that has a Haar measure is compact.
4. Every compact subgroup of GL_n is contained in U_n.
5. A subgroup of a reductive group is also reductive.
6. The product of two reductive groups is reductive.

7. If every non-trivial subgroup of G is reductive then G is finite.

8. $\mathsf{Hom}_{\mathsf{GL}_n}(V, W) = \mathsf{Hom}_{\mathsf{U}_n}(V, W)$.

9. Let $\rho : \mathsf{GL}_n \to \mathsf{GL}(V)$ be a GL_n-representation and let B be the smallest subalgebra of $\mathsf{End}(V)$ containing $\rho(\mathsf{U}_n)$, then B also contains $\rho(\mathsf{GL}_n)$.

10. If W is a GL_n-subrepresentation of V then $\mathbb{C}[W]^{\mathsf{GL}_n}$ is a quotient ring of $\mathbb{C}[V]^{\mathsf{GL}_n}$.

11. If V is a GL_n-representation and $\mathbb{C}[V]^{\mathsf{GL}_n}$ is finite dimensional then it is isomorphic to $\mathbb{C}$.

12. If V is a simple representation of a finite group G then $\mathbb{C}[V]^{\mathsf{G}} = \mathbb{C}$.

13. If V, W are GL_n-representations then $\mathbb{C}[V \oplus W]^{\mathsf{GL}_n} \cong \mathbb{C}[V]^{\mathsf{GL}_n} \oplus \mathbb{C}[W]^{\mathsf{GL}_n}$.

14. If V is a representation of a finite group G then the number of generators of $\mathbb{C}[V]^{\mathsf{G}}$ is at least the dimension of V.

15. If G is a finite group and V is a representation then the dimension of the space of invariants of degree k is k times the dimension of the linear invariants.

16. If G is a finite group then $\mathbb{C}[\mathsf{Rep}_n \mathbb{C}\mathsf{G}]^{\mathsf{GL}_n}$ is generated by one generator.

17. If the polarization of f and the polarization of g are the same then f and g are the same.

18. $\dim \mathsf{End}_{\mathsf{GL}(V)}(V^{*\otimes k}) = k!$.

19. The number of generators of $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^k]^{\mathsf{GL}_n}$ is smaller than the dimension of $\mathsf{End}_{\mathsf{GL}(V)}(V^{*\otimes k})$ (with $V = \mathbb{C}^n$).

20. The span of $v_1 \otimes v_2 - v_2 \otimes v_1 \in V \otimes V$ forms a simple $\mathsf{GL}(V)$-representation.

Chapter 4

The algebraic quotient

Because $\mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n}$ is finitely generated without nilpotent elements, it can be seen as the ring of polynomial functions over a certain variety. We will call this variety the algebraic quotient of $\mathsf{Rep}_n A$ and we will denote it by $\mathsf{iss}_n A$.

$$\mathsf{iss}_n A = \{\mathfrak{m} \lhd \mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n} | \mathfrak{m} \text{ is a maximal ideal}\}$$

The injection $\mathbb{C}[\mathsf{iss}_n A] = \mathbb{C}[\mathsf{Rep}_n A]^{\mathsf{GL}_n} \hookrightarrow \mathbb{C}[\mathsf{Rep}_n A]$ will give us a map

$$\pi : \mathsf{Rep}_n A \to \mathsf{iss}_n A : \mathfrak{m} \mapsto \mathfrak{m} \cap \mathbb{C}[\mathsf{iss}_n A].$$

Because all elements of $\mathbb{C}[\mathsf{iss}_n A]$ are invariant under the GL_n-action, points of $\mathsf{Rep}_n A$ in the same orbit are mapped to the same point in $\mathsf{iss}_n A$.

The reverse implication is however not true, points that are mapped to the same point do not need to lie in the same orbit. Because π is a continuous map $\pi^{-1}(x)$ must be a closed subset, so if there exists an orbit $\mathcal{O}$ that is not closed and $w = \pi(\mathcal{O})$ we know that $\pi^{-1}(w)$ must contain points outside $\mathcal{O}$.

Example 4.1. Let $A = \mathbb{C}[X]$ then $\mathsf{Rep}_2 A = \mathsf{Mat}_{2\times 2}(\mathbb{C})$. The orbit of the zero matrix consist of a single point but the orbit of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ contains the zero matrix in its closure because

$$\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix}\right)$$

and we can chose ϵ as small as we like.

In this chapter we will describe the geometrical nature of the algebraic quotient: what orbits are contained in $\pi^{-1}(x)$ and when are two orbits mapped to the same point under π.

We will also investigate the connection between the geometrical picture and the representation theory of the algebra.

4.1 Jordan-Hölder and 1-parameter subgroups

Let A be a finite-dimensional algebra. A chain of representations is a sequence of representations that are contained in each other:

$$\mathcal{C} : V_0 \subset V_1 \subset \cdots \subset V_n.$$

This chain is also called a filtration of V_n. The factors of a chain are the quotient representations

$$V_i/V_{i-1} := \{x + V_{i-1} | x \in V_i\}.$$

To a chain one can associate the representation $V_{\mathcal{C}} = \oplus_i V_i/V_{i-1}$. Note that for a semisimple representation $V_{\mathcal{C}} \cong V$ because for every subrepresentation $W \subset V$, W is also semisimple and $V \cong W \oplus V/W$ (prove this as an exercise). By repeated application of this we get

$$V \cong V_n/V_{n-1} \oplus V_{n-1} \cong V_n/V_{n-1} \oplus V_{n-1}/V_{n-2} \oplus V_{n-2} \cong \cdots \cong \oplus_i V_i/V_{i-1} = V_{\mathcal{C}}.$$

If $V_0 = 0$ and all factors are non-trivial simple representations the chain $\mathcal{C}$ is called a *composition series of* V_n and the corresponding $V_{\mathcal{C}}$ is called the *semisimplification* of V and is denoted by V^{ss}.

Every finite-dimesional representation has a composition series: Let V be a representation in $\mathsf{Rep}_n A$, this representation is not neccesarily simple or semisimple but it contains at least one simple subrepresentation V_1. We can take the quotient of V by this representation and we obtain a new representation of smaller dimension. This representation contains again a simple representation S, define now $V_2 = \pi^{-1}(S)$ where π is the projection of V onto V/V_1. The quotient V_2/V_1 is then isomorphic to S and hence simple. In a similar way we can proceed: find a simple representation in V/V_2 and construct V_3. As the dimension of the $V_i \subset V$ increases we get for some n that $V_n = V$. The coresponding chain will then be a decomposition chain.

A decomposition chain is however not unique: it is possible that V contains several simple subrepresentations and each of them will give different decomposition chains. However one can prove the following

Theorem 4.2 (Jordan-Hölder). *The factors of a composition series of V are unique up to permutation. In other words: the semisimplification of V*

$$V^{ss} := \bigoplus_{i=1}^{n} V_i/V_{i-1}$$

doesn't depend on the composition series $(V_i)_{i \leq n}$.

Proof. Let $(V_i)_{i\leq n}$ and $(W_j)_{i\leq m}$ be two decomposition series of V. We proof this by induction on n. If $n = 1$ then V is simple, so the only possible chain is $0 \subset V$.

If $n > 1$, we consider the new chains

$$(V_{i+1}/V_1)_{i\leq n-1} \text{ and } ((W_j + V_1)/V_1)_{j\leq m}$$

The first is a composition series of V/V_1 because

$$\frac{V_{i+1}/V_1}{V_i/V_1} \cong \frac{V_{i+1}}{V_i}$$

is simple. Because V_1 is simple it is either contained in W_j or $V_1 \cap W_j = 0$, so $W_j + V_1$ is either $W_j \oplus V_1$ or W_j. If l is the first index such that W_{l+1} contains V_1 then $W_{l+1}/W_l \supset (W_l \oplus V_1)/W_l = V_1$. As W_{l+1}/W_l is simple is must be equal to V_1 and $W_{l+1} \cong W_l \oplus V_1$.

This allows us to conclude that

$$\frac{(W_{j+1} + V_1)/V_1}{(W_j + V_1)/V_1} = \begin{cases} W_{j+1}/W_j & V_1 \subset W_j \subset W_{j+1} \\ \frac{(W_{j+1} \oplus V_1)/V_1}{(W_j \oplus V_1)/V_1} = W_{j+1}/W_j & V_1 \not\subset W_j, V_1 \not\subset W_{j+1} \\ \frac{W_j \oplus V_1/V_1}{(W_j \oplus V_1)/V_1} = 0 & V_1 \not\subset W_j, V_1 \subset W_{j+1} \text{i.e. } j = l \end{cases}$$

So the chain $((W_j + V_1)/V_1)_{j\leq m, j\neq l}$ is a composition for V/V_1. By induction the composition factors of $((W_j + V_1)/V_1)_{j\leq m, j\neq l}$ are a permutation of those of $(V_{i+1}/V_1)_{i\leq n-1}$. And hence the composition factors of $(V_i)_{i\leq n}$ and $(W_j)_{i\leq m}$ are also equal up to a permutation. □

Miniature 3: Otto Hölder (1859-1937)

Otto Hölder became a lecturer at Göttingen in 1884 and at first he worked on the convergence of Fourier series. Shortly after be began working at Göttingen he discovered the inequality now named after him. Hölder was offered a post in Tbingen in 1889 but unfortunately he suffered a mental collapse. After a year he made a steady recovery, giving his inaugural lecture in 1890. He began to study the Galois theory of equations and from there he was led to study composotion series of groups. Hölder proved the uniqueness of the factor groups in a composition series, the theorem now called the Jordan-Hölder theorem.

Miniature 4: Camille Jordan (1838-1922)

Jordan was a french mathematician who worked in a wide variety of different areas essentially contributing to every mathematical topic which was studied at that time. In topology, Jordan is best remembered today among analysts and topologists for his proof that a simply closed curve divides a plane into exactly two regions, now called the Jordan curve theorem. He also introduced the notion of homotopy. In group theory he proved the Jordan-Hölder theorem and a second major piece of work on finite groups was the study of the general linear group over the field with p elements, p prime.

Now let $\mathcal{C} : V_0 \subset V_1 \subset \cdots \subset V_m$ be a chain with $V_0 = 0$, $V_m = V$ and $\dim V = n$. choose a basis in $e_1, \ldots e_n \in V$ such that there exist numbers $i_1 < \cdots < i_m$ and $e_1 \ldots e_{i_k}$ is a basis for V_k. According to this basis the representation V has the form

$$\forall a \in A : \rho_V(a) = \begin{bmatrix} \boxed{\rho_{V_1/V_0}(a)} & * & \cdots & * \\ & \boxed{\rho_{V_2/V_1}(a)} & \cdots & * \\ & & \ddots & * \\ & & & \boxed{\rho_{V_m/V_{m-1}}(a)} \end{bmatrix}$$

Now one can choose a one parameter subgroup of GL_n (i.e. an n-dimensional representation of $\mathbb{C}^*$):

$$\Lambda(\epsilon) = \begin{bmatrix} \boxed{\epsilon^1 \mathbb{1}_{i_1}} & & & \\ & \boxed{\epsilon^2 \mathbb{1}_{i_2 - i_1}} & & \\ & & \ddots & \\ & & & \boxed{\epsilon^m \mathbb{1}_{i_m - i_{m-1}}} \end{bmatrix}$$

The action of this one parameter subgroup is the following

$$\Lambda(\epsilon) \cdot \rho_V(a) = \begin{bmatrix} \boxed{\rho_{V_1/V_0}(a)} & \epsilon * & \cdots & \epsilon^{m-1} * \\ & \boxed{\rho_{V_2/V_1}(a)} & \cdots & \epsilon^{m-2} * \\ & & \ddots & * \\ & & & \boxed{\rho_{V_m/V_{m-1}}(a)} \end{bmatrix}.$$

The limit for ϵ to 0 is the blockdiagonal matrix

$$\lim_{\epsilon \to 0} \Lambda(\epsilon) \cdot \rho_V(a) = \begin{bmatrix} \boxed{\rho_{V_1/V_0}(a)} & & & \\ & \boxed{\rho_{V_2/V_1}(a)} & & \\ & & \ddots & \\ & & & \boxed{\rho_{V_m/V_{m-1}}(a)} \end{bmatrix}$$

So $\lim_{\epsilon\to 0}\Lambda(\epsilon)\cdot\rho_V(a)$ corresponds to the representation $V_{\mathcal{C}}$.

On the other hand let $\Lambda : \mathbb{C}^* \to \mathsf{GL}_n$ be a general one parameter subgroup then we know that $\mathbb{C}^*$ is a reductive group and the simple representations of $\mathbb{C}^*$ are of the form $\epsilon \mapsto \epsilon^z$ for $z \in \mathbb{Z}$. We can use this to diagonalize λ:

$$\Lambda(\epsilon) = g \begin{bmatrix} \boxed{\epsilon^{z_1}1_{n_1}} & & & \\ & \boxed{\epsilon^{z_2}1_{n_2}} & & \\ & & \ddots & \\ & & & \boxed{\epsilon^{z_k}1_{n_k}} \end{bmatrix} g^{-1}$$

with $z_1 < z_2 < \cdots \le z_k$. Now if the limit of $\lim_\epsilon \Lambda(\epsilon)\cdot\rho_V$ exists, the entries a_{ij} of $g\rho_V g^{-1}$ with $n_1+\cdots+n_s < j \le n_1+\cdots+n_{s+1}$ and $i > n_1+\cdots+n_{s+1}$ are zero because they transform according with a negative exponent of ϵ. Otherwise $\lim_{\epsilon\to 0}\epsilon^{-k}a_{ij} = \infty$.

This means that there is a chain of submodules of $g\cdot V$:

$$\mathcal{C} : (V_i = g\cdot \mathsf{Span}(e_1,\ldots,e_{n_1+\cdots+n_i})$$

where (e_i) is the standardbasis of $g\cdot V$.

Taking the limit to zero we see that also the entries a_{ij} with $n_1+\cdots+n_s < j \le n_1+\cdots+n_{s+1}$ and $i < n_1+\cdots+n_s$ making the limit of $g\Lambda(\epsilon)g^{-1}g\rho_V g^{-1}$ into a block diagonal matrix. This implies that the representation $\lim_{\epsilon\to 0} g\Lambda(\epsilon)g^{-1}\cdot gV = g\lim_{\epsilon\to 0}\Lambda(\epsilon)V$ is isomorphic to

$$V_1/V_0 \oplus V_2/V_1 \oplus \cdots \oplus V_k/V_{k-1}.$$

Out all of this we can conclude

Theorem 4.3. *A representation V has a filtration $\mathcal{C}$ with $V_{\mathcal{C}} \cong W$ if and only if there is a one-parameter subgroup $\Lambda : \mathbb{C}^* \to \mathsf{GL}_n$ such that $\lim_{\epsilon\to 0}\Lambda(\epsilon)V \cong W$.*

This implies that every point in the closure of $\mathcal{O}_V$ that can be reached using a one-parameter subgroup is of the from $V_{\mathcal{C}}$ where $\mathcal{C}$ is a chain. However it might be possible that there are other points in the closure that can not be reached using this procedure.

4.2 Hilbert's criterium

If we want to look at the closure of an orbit $\mathcal{O}_x$ we must take care of what closure we mean, because we have two different topologies: the Zariski topology and the complex topology. As the Zariski topology has fewer open and closed sets, the closure according the Zariski topology can be bigger than the one according to the complex topology. We have already seen an example of this: the unitary group is Zariski-dense in GL_n but it is a complex-closed set so its complex closure is U_n itself.

However the above example was a bit tricky because U_n has not the structure of a complex variety. In this special case of complex varieties we can use a well known fact in algebraic geometry (for the proof see [?])

Lemma 4.4. *If X is a subvariety of $\mathbb{C}^n$ and $\bar{X}$ is the Zariski closure of X then X contains a subset Z that is open in $\bar{X}$ such that $\bar{Z} = \bar{X}$.*

We know that Z is the intersection between an open subset $U \subset \mathbb{C}^n$ and a closed subset $\bar{X}$. The complex closure of Z is then equal to the complex closure of U intersected with $\bar{X}$. As $\bar{U}^{\mathbb{C}}$ is the whole $\mathbb{C}^n$, we have that $\bar{Z}^{\mathbb{C}} = \bar{X} = \bar{Z}$. Because $Z \subset X$, we also have that $\bar{Z}^{\mathbb{C}} \subset \bar{X}^{\mathbb{C}} \subset \bar{X}$, so for subvarieties the complex and the Zariski closure coincide: $\bar{X}^{\mathbb{C}} = \bar{X}$.

In the cases we are considering now this is the case: every orbit can be considered as a complex subvariety of $\mathbb{C}^n$, nl. the image of the map

$$\phi : \mathsf{GL}_n \to \mathsf{Rep}_n A \subset \mathbb{C}^{mn^2} : g \mapsto g \cdot x.$$

The complex closure of X in Y can be seen as all points in Y that can be reached as the limit of a curve in X. This means that if $y \in \bar{\mathcal{O}}_x$ that there exists a smooth function

$$\gamma : \mathbb{R} \to \mathsf{GL}_n \text{ s.t. } \lim_{t \to 0} \gamma(t) \cdot x = y.$$

However the limit of γ itself needs not to exist in GL_n. This means that the coefficients γ_{ij} may be functions that have poles in 0. In general we can assume that we can expand them to Laurent series (the proof of this is not completely trivial but we omit it here, see also [?])

$$\gamma_{ij}(t) = a_{-d}t^{-d} + a_{-d+1}t^{-d+1} + \cdots .$$

So we can conclude that there exists a matrix $\gamma \in \mathsf{GL}_n(\mathbb{C}((t)))$ such that the coordinates of $\gamma \cdot x$ all sit in $\mathbb{C}[[t]]$ (because the limit $t \to 0$ exists). and $\gamma \cdot x|_{t=0} = y$.

Lemma 4.5. *Let γ be an $n \times n$ matrix with coefficients in $\mathbb{C}((t))$ and* $\det\ \gamma \neq 0$. *Then there exist $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that*

$$\gamma = u_1 . \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix} . u_2$$

with $r_i \in \mathbb{Z}$ and $r_1 \leq r_2 \leq \ldots \leq r_n$.

Proof. By multiplying γ with a suitable power of t we may assume that $\gamma = (\gamma_{ij}(t))_{i,j} \in \mathsf{Mat}_{n\times n}(\mathbb{C}[[t]])$. If $f(t) = \sum_{i=0}^{\infty} f_i t^i \in \mathbb{C}[[t]]$ define $v(f(t))$ to be the minimal i such that $a_i \neq 0$. Let (i_0, j_0) be an entry where $v(g_{ij}(t))$ attains a minimum, say r_1. That is, for all (i,j) we have $g_{ij}(t) = t^{r_1} t^r f(t)$ with $r \geq 0$ and $f(t)$ an invertible element of $\mathbb{C}[[t]]$.

By suitable row and column interchanges we can take the entry (i_0, j_0) to the $(1,1)$-position. Then, multiplying with a unit we can replace it by t^{r_1} and by elementary row and column operations all the remaining entries in the first row and column can be made zero. That is, we have invertible matrices $a_1, a_2 \in \mathsf{GL}_n(\mathbb{C}[[t]])$ such that

$$\gamma = a_1. \begin{bmatrix} t^{r_1} & \underline{0}^\tau \\ \underline{0} & \boxed{g_1} \end{bmatrix} .a_2$$

Repeating the same idea on the submatrix g_1 and continuing gives the result. $\square$

We can now state and prove the *Hilbert criterium* which allows us to study orbit-closures by one parameter subgroups.

Theorem 4.6 (Hilbertcriterium). *Let V be a GL_n-representation and $X \hookrightarrow V$ a closed GL_n-stable subvariety. Let $\mathcal{O}_x = \mathsf{GL}_n.x$ be the orbit of a point $x \in X$. Let $Y \hookrightarrow \overline{\mathcal{O}(x)}$ be a closed GL_n-stable subset. Then, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathsf{GL}_n$ such that*

$$\lim_{t\to 0} \lambda(t).x \in Y$$

Proof. It suffices to prove the result for $X = V$. By lemma **??** there is an invertible matrix $\gamma \in \mathsf{Mat}_{n\times n}(\mathbb{C}((t)))$ such that

$$(\gamma.x)_{t=0} = y \in Y$$

By lemma **??** we can find $u_1, u_2 \in GL_n(\mathbb{C}[[t]])$ such that

$$\gamma = u_1.\lambda'(t).u_2 \quad \text{with} \quad \lambda'(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

a one-parameter subgroup. There exist $x_i \in V$ such that $u_2.x = \sum_{i=0}^{\infty} z_i t^i$ in particular $u_2(0).x = x_0$. But then,

$$\begin{aligned}(\lambda'(t).u_2.x)_{t=0} &= \sum_{i=0}^{\infty} (\lambda'(t).x_i t^i)_{t=0} \\ &= (\lambda'(t).x_0)_{t=0} + (\lambda'(t).x_1 t)_{t=0} + \dots\end{aligned}$$

For every i we have that the j^{th} coordinate of $(\lambda'(t).x_i t^i)_{t=0}$ is zero if $r_j + i > 0$. So if $i > 0$ only the coordinates corresponding with a negative power r_i are nonzero. this implies thatn

$$\lim_{s\to 0} \lambda'^{-1}(s).(\lambda'(t)x_i t^i)_{t=0} = \begin{cases} (\lambda'(t).x_0)_{t=0} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

We know that $(\lambda'(t).u_2.x)_{t=0} \in Y$ because Y is closed under the GL_n-action, moreover because Y is also a closed subset we have that

$$\lim_{s\to 0} \lambda'^{-1}(s).(\lambda'(t).u_2.x)_{t=0} \in Y \quad \text{that is,} \quad (\lambda'(t).x_0)_{t=0} \in Y$$

We have for the one-parameter subgroup $\lambda(t) = u_2(0)^{-1}.\lambda'(t).u_2(0)$ that

$$\lim_{t\to 0} \lambda(t).x \in Y$$

finishing the proof. □

Miniature 5: David Hilbert (1862-1943)

Hilbert was a german mathematician and he set forth the first rigorous set of geometrical axioms in Foundations of Geometry. He also proved his system to be self-consistent. He invented a simple space-filling curve known as the hilbert curve, and demonstrated the basis theorem in invariant theory. At the Paris International Congress of 1900, Hilbert proposed 23 outstanding problems in mathematics to whose solutions he thought twentieth century mathematicians should devote themselves. These problems have come to be known as Hilbert's problems, and a number still remain unsolved today.

In the statement of theorem **??** it is important that Y is closed. In particular, it does *not* follow that any orbit $\mathcal{O}(y) \hookrightarrow \overline{\mathcal{O}(x)}$ can be reached via one-parameter subgroups.

4.3 $\mathsf{iss}_n A$

Now we will put the two previous paragraphs together to obtain an explicit description of the quotient space.

First of all by the Jordan Holder theorem and the theory of one parameter subgroups we see that every representation $V \in \mathsf{Rep}_n A$ contains its semisimplification in the closure of its orbit. So if π is the projection of $\mathsf{Rep}_n A$ onto $\mathsf{iss}_n A$ we know that every fiber $\pi^{-1}\pi(V)$ contains at least one semisimple representation nl. V^{ss}.

Secondly using Hilbert's criterium we can prove

Theorem 4.7. *An orbit $\mathcal{O}_M$ is closed if and only if M is a semisimple representation*

Proof. If $\mathcal{O}_M$ is a closed orbit its semisimplification is contained in the orbit because it is the limit of a one-parameter subgroup. I.e. $M \cong M^{ss}$.

To prove the converse we need some more steps

We first claim that every orbit $\mathcal{O}_M = \mathsf{GL}_n \cdot M$ is Zariski open in its closure $\overline{\mathcal{O}_M}$. By lemma **??** we can take a subset $U \subset \mathcal{O}(M)$ that is Zariski open in $\overline{\mathcal{O}_M}$ and consider the map $\phi : \mathsf{GL}_n \to \mathsf{Rep}_n A : g \mapsto g \cdot M$. But then,

$$\mathcal{O}_M = GL_n.M = \cup_{g \in GL_n} g.U$$

is also open in $\overline{\mathcal{O}(M)}$.

Next, we claim that $\overline{\mathcal{O}_M}$ contains a closed orbit. Indeed, assume $\mathcal{O}_M$ is not closed, then the complement $C_M = \overline{\mathcal{O}(M)} - \mathcal{O}(M)$ is a proper Zariski closed subset whence $\dim C_M < \dim \overline{\mathcal{O}(M)}$. But, C_M is the union of GL_n-orbits $\mathcal{O}_{M_i}$ with $\dim \overline{\mathcal{O}_{M_i}} < \dim \overline{\mathcal{O}_M}$. Either one of the $\mathcal{O}_{M_i}$ is closed or we can split up C_{M_i} in orbits. If the latter always holds, we can continue this way to come to the point where $\dim C_{M_i} = 0$ and hence a point (because it is connected) which is closed.

So if M is semisimple, let $\mathcal{O}_{M'}$ be the closed orbit in $\mathcal{O}_M$. By the Hilbert criterium there is a one parameter subgroup λ such that $\lim_{t\to 0} \lambda(t)M \subset \mathcal{O}_{M'}$. But because M is semisimple $\lim_{t\to 0} \lambda(t)M \cong M$ and hence $\mathcal{O}_M = \mathcal{O}'_M$. □

Finally we will prove that every fiber contains exactly one closed orbit. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two closed GL_n-orbits in $\mathsf{Mat}_{n\times n}(\mathbb{C})^m$ then the defining ideals

$$\mathfrak{p}_i = \{f \in \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m] : f|_{\mathcal{O}_i} = 0\}$$

are GL_n-invariant: $\mathsf{GL}_n \cdot \mathfrak{p}_i = \mathfrak{p}_i$.

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are mapped to the same point then this means that their isotopical components corresponding to the trivial GL_n-representation are equal:

$$(\mathfrak{p}_1)_1 = \mathfrak{p}_1 \cap \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n} = \mathfrak{p}_2 \cap \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n} = (\mathfrak{p}_2)_1 = \mathfrak{m},$$

where $\mathfrak{m}$ is the maximal ideal in $\mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$ corresponding to that point. As the orbits do not intersect we have that $\mathfrak{p}_1+\mathfrak{p}_1 = \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]$. But looking at the component at the isotopical components of this equation we see that

$$(\mathfrak{p}_1)_1 + (_2)_1 = (p_1)_1 = \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$$

But this is impossible because then $\mathfrak{m} = \mathbb{C}[\mathsf{Mat}_{n\times n}(\mathbb{C})^m]^{\mathsf{GL}_n}$.

we can conclude with the following theorem

Theorem 4.8. *Let A be an affine $\mathbb{C}$-algebra and $M \in \mathsf{Rep}_n A$.*

1. *The orbit $\mathcal{O}_M$ is closed in $\mathsf{Rep}_n A$ if and only if M is an n-dimensional semisimple A-representation.*
2. *The orbitclosure $\overline{\mathcal{O}_M}$ contains exactly one closed orbit, corresponding to the semisimplification of M.*
3. *The points of the quotient variety of $\mathsf{Rep}_n A$ under GL_n parameterize the isomorphism classes of n-dimensional semisimple A-representations. We will denote the quotient variety by $\mathsf{iss}_n A$.*
4. *The map $\pi : \mathsf{Rep}_n A \to \mathsf{iss}_n A$ is the best continuous approximation to the orbit space. That is if $\phi : \mathsf{Rep}_n A \to Y$ is a map of varieties that is constant on the orbits, then there is a map $\phi' : \mathsf{iss}_n A \to Y$ such that $\phi = \phi' \circ \pi$*

Proof. We will only prove (4), $(1)-(3)$ follow easily from the previous theorems. Let $p \in \mathsf{iss}_n A$ then $\pi^{-1}(p)$ containes a unique closed orbit $\mathcal{O}$ such that every orbit in $\pi^{-1}(p)$ has $\mathcal{O}$ in its closure. Because ϕ is continuous $\phi\pi^{-1}(p)$ is the same as $\phi(\mathcal{O})$. So we can define $\phi'(p) = \phi(\mathcal{O})$. ∎

Chapter 5

Some examples

5.1 The polynomial algebra $\mathbb{C}[X]$

From chapter 2 we know that the representations of the polynomial algebra correspond to conjugacy classes of matrices. In this section we look for a particularly nice representative in a given conjugacy class. The answer to this problem is, of course, given by the *Jordan normal form* of the matrix.

From the 3rd chapter we know that the ring of invariants $\mathbb{C}[\mathsf{Mat}_{n\times n}]^{\mathsf{GL}_n}$ are generated by the functions $T_j : A \mapsto \mathsf{Tr}(A^j)$.

We recall that the *characteristic polynomial* of A is defined to be the polynomial of degree n in the variable t

$$\chi_A(t) = \det(t\mathbb{1}_n - A) \in \mathbb{C}[t].$$

As $\mathbb{C}$ is algebraically closed, $\chi_A(t)$ decomposes as a product of linear terms

$$\prod_{i=1}^{e}(t-\lambda_i)^{d_i}$$

where the $\{\lambda_1, \dots, \lambda_e\}$ are called the *eigenvalues* of the matrix A. Observe that λ_i is an eigenvalue of A if and only if there is a non-zero *eigenvector* $v \in V_n = \mathbb{C}^n$ with eigenvalue λ_i, that is, $A.v = \lambda_i v$. In particular, the *rank* r_i of the matrix $A_i = \lambda_i \mathbb{1}_n - A$ satisfies $n - d_i \leq r_i < n$.

The coefficients of $\chi_A(t) = \sum \alpha_k t^k$ (with $\alpha_n = 1$) are homoegeneous invariants under the the GL_n-action and hence can be written in function of of the $T_j,\ j \leq n$ (because the degree of the coefficients is also not bigger than n).

Because of the Cayley hamilton identity we know that A satisfies its own characteristic polynomial so $\sum \alpha_k A^k = 0$. This can be used to write all powers of A higher then $n-1$ as a linear combination of the $A^j, j < n$ and coefficients that are polynomial functions in the $T_j,\ j \leq n$. Taking the trace we can rewrite the $T_j, j > n$ as polynomial functions of the $T_j,\ j \leq n$. So the ring of invariants is generated by the $T_j,\ j \leq n$.

There are no relations between the $T_j,\ j \leq n$. If there would be relations $\dim \mathsf{iss}_n \mathbb{C}[X]$ would have dimension smaller than n. This would mean that the map $\phi : \mathsf{iss}_n \mathbb{C}[X] \to \mathbb{C}^n \cong \{X^n + \mathbb{C}X^{n-1} + \cdots + \mathbb{C}\}$ mapping A to its characteristic polynomial would not be surjective. This is not true: take any point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the matrix $A \in M_n$

$$A = \begin{bmatrix} 0 & & & & a_n \\ -1 & 0 & & & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & a_2 \\ & & & -1 & a_1 \end{bmatrix} \tag{5.1}$$

then we will show that $\pi(A) = (a_1, \ldots, a_n)$, that is,

$$det(t\mathbb{1}_n - A) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \ldots + (-1)^n a_n.$$

Indeed, developing the determinant of $t\mathbb{1}_n - A$ along the first column we obtain

$$\begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix} - \begin{vmatrix} t & 0 & 0 & \cdots & 0 & -a_n \\ 1 & t & 0 & & 0 & -a_{n-1} \\ 0 & 1 & t & & 0 & -a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & t & -a_2 \\ 0 & 0 & & & 1 & t-a_1 \end{vmatrix}$$

Here, the second determinant is equal to $(-1)^{n-1}a_n$ and by induction on n the first determinant is equal to $t.(t^{n-1} - a_1 t^{n-2} + \ldots + (-1)^{n-1}a_{n-1})$, proving the claim.

So we can conclude that

Theorem 5.1. *The ring of invariants of the $n \times n$-matrices under conjugation is a polynomial ring in n variables (the $T_j, j \leq n$. Therefore $\mathsf{iss}_n \mathbb{C}[X] \cong \mathbb{C}^n$.*

Now we take a look at the orbits. The theorem of Jordan-Weierstrass gives us a nice representant for every orbit

Theorem 5.2 (Jordan-Weierstrass). *Let $A \in \mathsf{Mat}_{nn}$ with characteristic polynomial $\chi_A(t) = \prod_{i=1}^{e}(t-\lambda_i)^{d_i}$. Then, A determines unique partitions*

$$p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i}) \quad \textit{of} \quad d_i$$

associated to the eigenvalues λ_i of A such that A is conjugated to a unique (up to permutation of the blocks) block-diagonal matrix

$$J_{(p_1,\ldots,p_e)} = \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \ddots & \\ & & & \boxed{B_m} \end{bmatrix}$$

with $m = m_1 + \ldots + m_e$ and exactly one block B_l of the form $J_{a_{ij}}(\lambda_i)$ for all $1 \le i \le e$ and $1 \le j \le m_i$ where

$$J_{a_{ij}}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in M_{a_{ij}}(\mathbb{C})$$

The proof that we can bring A in such a form uses basic linear algebra an can be found in most undergraduate books of linear algebra.

Let us prove uniqueness of the partitions p_i of d_i corresponding to the eigenvalue λ_i of A. Assume A is conjugated to another Jordan block matrix $J_{(q_1,\ldots,q_e)}$, necessarily with partitions $q_i = (b_{i1}, \ldots, b_{im'_i})$ of d_i. To begin, observe that for a Jordan block of size k we have that

$$rk\ J_k(0)^l = k - l \quad \text{for all } l \le k \text{ and if } \mu \neq 0 \text{ then} \quad rk\ J_k(\mu)^l = k$$

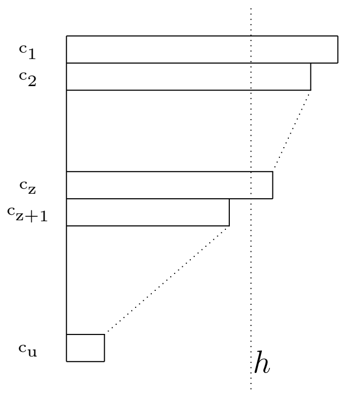
for all l. As $J_{(p_1,\ldots,p_e)}$ is conjugated to $J_{(q_1,\ldots,q_e)}$ we have for all $\lambda \in \mathbb{C}$ and all l

$$rk\ (\lambda 1_n - J_{(p_1,\ldots,p_e)})^l = rk\ (\lambda 1_n - J_{(q_1,\ldots,q_e)})^l$$

Now, take $\lambda = \lambda_i$ then only the Jordan blocks with eigenvalue λ_i are important in the calculation and one obtains for the ranks

$$n - \sum_{h=1}^{l} \#\{j \mid a_{ij} \ge h\} \quad \text{respectively} \quad n - \sum_{h=1}^{l} \#\{j \mid b_{ij} \ge h\}.$$

Now, for any partition $p = (c_1, \ldots, c_u)$ and any natural number h we see that the number $z = \#\{j \mid c_j \geq h\}$



is the number of blocks in the h-th row of the dual partition p^* which is defined to be the partition obtained by interchanging rows and columns in the *Young diagram* of p. Therefore, the above rank equality implies that $p_i^* = q_i^*$ and hence that $p_i = q_i$. As we can repeat this argument for the other eigenvalues we have the required uniqueness. Hence, the Jordan normal form shows that the classification of GL_n-orbits in M_n consists of two parts : a discrete part choosing

- a partition $p = (d_1, d_2, \ldots, d_e)$ of n, and for each d_i,
- a partition $p_i = (a_{i1}, a_{i2}, \ldots, a_{im_i})$ of d_i,

determining the sizes of the Jordan blocks and a continuous part choosing

- an e-tuple of distinct complex numbers $(\lambda_1, \lambda_2, \ldots, \lambda_e)$.

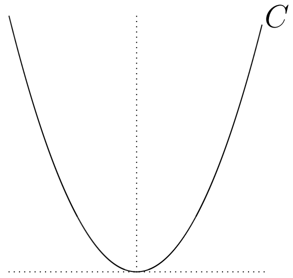
fixing the eigenvalues. Moreover, this e-tuple $(\lambda_1, \ldots, \lambda_e)$ is determined only up to permutations of the subgroup of all permutations π in the symmetric group S_e such that $p_i = p_{\pi(i)}$ for all $1 \leq i \leq e$.

Example 5.3. Orbits in $\mathsf{Rep}_2\mathbb{C}[X]$.

A 2×2 matrix A can be conjugated to an upper triangular matrix with diagonal entries the eigenvalues λ_1, λ_2 of A.

The matrix A has two equal eigenvalues if and only if the discriminant of the characteristic polynomial is zero, that is when $2\mathsf{Tr}(A^2) = \mathsf{Tr}(A)^2$. This condition determines a closed curve C in $\mathbb{C}^2$ where

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - 2y = 0\}.$$



Observe that C is a smooth 1-dimensional submanifold of $\mathbb{C}^2$. We will describe the *fibers* of the surjective map $\pi : \mathsf{Rep}_n \mathbb{C}[X] \rightarrow \mathsf{iss}_n \mathbb{C}[X] = \mathbb{C}^2$.

If $p = (x, y) \in \mathbb{C}^2 - C$, then $\pi^{-1}(p)$ consists of precisely one orbit (which is then necessarily closed in M_2) namely that of the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{where} \qquad \lambda_{1,2} = \frac{x \pm \sqrt{2y - x^2}}{2}$$

If $p = (x, y) \in C$ then $\pi^{-1}(p)$ consists of two orbits,

$$\mathcal{O}_{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}} \qquad \text{and} \qquad \mathcal{O}_{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}$$

where $\lambda = \frac{1}{2}x$. We have seen that the second orbit lies in the closure of the first. Observe that the second orbit reduces to one point in M_2 and hence is closed. Hence, also $\pi^{-1}(p)$ contains a unique closed orbit.

To describe the fibers of π as closed subsets of M_2 it is convenient to write any matrix A as a linear combination

$$A = u(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + v(A) \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + w(A) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z(A) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Expressed in the coordinate functions u, v, w and z the fibers $\pi^{-1}(p)$ of a point $p = (x, y) \in \mathbb{C}^2$ are the common zeroes of

$$\begin{cases} u & = x \\ v^2 + 4wz & = 2y - x^2 \end{cases}$$

The first equation determines a three dimensional affine subspace of M_2 in which the second equation determines a *quadric*. If $p \notin C$ this quadric is non-degenerate and thus $\pi^{-1}(p)$ is a smooth 2-dimensional submanifold of M_2. If $p \in C$, the quadric is a cone with top lying in the point $\frac{x}{2} \mathbb{1}_2$. Under the GL_2-action, the unique singular point of the cone must be clearly fixed giving us the closed orbit of dimension 0 corresponding to the diagonal matrix. The other orbit is the complement of the top and hence is a smooth 2-dimensional (non-closed) submanifold of M_2. The graphs in figure **??** represent the orbit-closures and the dimensions of the orbits.

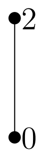


Figure 5.1: Orbit closures of 2×2 matrices

Example 5.4. Orbits in M_3.

We will describe the fibers of the surjective map $M_3^{\pi}\mathbb{C}^3$. If a 3×3 matrix has multiple eigenvalues then the *discriminant* $d = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2$ is zero. Clearly, d is a symmetric polynomial and hence can be expressed in terms of T_1, T_2 and T_3. More precisely,

$$d = \frac{4}{3}T_1^3T_3 - \frac{3}{2}T_1^4T_2 + \frac{1}{6}T_1^6 - \frac{1}{2}T_2^3 + \frac{7}{2}T1^2T_2^2 + 3T_3^2 - 6T_1T_2T_3$$

The set of points in $\mathbb{C}^3$ where d vanishes is a surface S with *singularities*. These singularities are the common zeroes of the $\frac{\partial d}{\partial \sigma_i}$ for $1 \leq i \leq 3$. One computes that these singularities form a twisted cubic curve C in $\mathbb{C}^3$, that is,

$$C = \{(3c, 3c^2, 3c^3) \mid c \in \mathbb{C}\}.$$

The description of the fibers $\pi^{-1}(p)$ for $p = (x, y, z) \in \mathbb{C}^3$ is as follows. When $p \notin S$, then $\pi^{-1}(p)$ consists of a unique orbit (which is therefore closed in M_3), the conjugacy class of a matrix with paired distinct eigenvalues. If $p \in S - C$, then $\pi^{-1}(p)$ consists of the orbits of

$$A_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Finally, if $p \in C$, then the matrices in the fiber $\pi^{-1}(p)$ have a single eigenvalue $\lambda = \frac{1}{3}x$ and the fiber consists of the orbits of the matrices

$$B_1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_2 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad B_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

We observe that the *strata* with distinct fiber behavior (that is, $\mathbb{C}^3 - S$, $S - C$ and C) are all submanifolds of $\mathbb{C}^3$, see figure **??**.

The dimension of an orbit $\mathcal{O}(A)$ in M_n is computed as follows. Let C_A be the subspace of all matrices in M_n commuting with A. Then, the *stabilizer* subgroup of A is a dense open subset of C_A whence the dimension of $\mathcal{O}(A)$ is equal to $n^2 - \dim C_A$.

Performing these calculations for the matrices given above, we obtain the following graphs representing orbit-closures and the dimensions of orbits

$$
\begin{array}{ccc}
\bullet 6 & \mathcal{O}_{A_1} \bullet 6 & \mathcal{O}_{B_1} \bullet 6 \\
 & | & | \\
 & \mathcal{O}_{A_2} \bullet 4 & \mathcal{O}_{B_2} \bullet 4 \\
 & & | \\
 & & \mathcal{O}_{B_3} \bullet 0 \\
\hline
\mathbb{C}^3 - S & S - C & C
\end{array}
$$

5.2 $\mathbb{C}\langle X, Y\rangle$ and $\mathbb{C}[X, Y]$

In this section we will study the 2-dimensional representations of $\mathbb{C}\langle X, Y\rangle$ and $\mathbb{C}[X, Y]$.

First we look at the invariants. For $(A, B) \in M_2^2 = M_2 \oplus M_2$ we will show that the polynomial functions $\mathsf{Tr}(A), \mathsf{Tr}(A^2)$, $\mathsf{Tr}(B), \mathsf{Tr}(B^2)$ and $\mathsf{Tr}(AB)$ generate all invariants.

First of all because of the Cayley-Hamilton identity the we can rewrite A^2 in function of A, $\mathsf{Tr}A$ and $\mathsf{Tr}A^2$. The same holds for B^2 and $(AB)^2$. Using the Cayley-Hamilton identity for $(A+B)^2$ we can rewrite BA in function of AB, A, B and the 5 invariants:

$$
\begin{aligned}
(A+B)^2 &= (A+B)\mathsf{Tr}(A+B) - \frac{1}{2}((\mathsf{Tr}(A+B))^2 - \mathsf{Tr}(A+B)^2) \\
A^2 + AB + BA + B^2 &= A\mathsf{Tr}A + B\mathsf{Tr}B + A\mathsf{Tr}B + B\mathsf{Tr}A - \frac{1}{2}\mathsf{Tr}A^2 + \mathsf{Tr}B^2 + 2\mathsf{Tr}A\mathsf{Tr}B - \mathsf{Tr}A^2 - \mathsf{Tr}B^2 \\
AB + BA &= A\mathsf{Tr}B + B\mathsf{Tr}A - \mathsf{Tr}A\mathsf{Tr}B + \mathsf{Tr}AB
\end{aligned}
$$

This implies that we can rewrite every trace of every product of $A's$ and $B's$ in terms of the 5 basic invariants. Remark that we can interchange $\mathsf{Tr}A^2$, $\mathsf{Tr}B^2$ for $\det A$ and $\det B$ as generators because we can express those traces in terms of the determinants and vice versa.

Here, we will show that the map $M_2^2 = M_2 \oplus M_2^{\pi}\mathbb{C}^5$ defined by

$$
(A, B) \mapsto (\mathsf{Tr}(A), \det(A), \mathsf{Tr}(B), \det(B), \mathsf{Tr}(AB))
$$

is surjective.

For every $(x_1, \dots, x_5) \in \mathbb{C}^5$ we will construct a couple of 2×2 matrices (A, B) (or rather its orbit) such that $\pi(A, B) = (x_1, \dots, x_5)$. Consider the open set where $x_1^2 \neq 4x_2$. We have seen that this property characterizes those $A \in M_2$ such that A has distinct eigenvalues and hence diagonalizable. Hence, we can take a representative of the orbit (A, B) to be a couple

$$(\quad \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \quad , \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \quad)$$

with $\lambda \neq \mu$. We need a solution to the set of equations

$$\begin{cases} x_3 = & c_1 + c_4 \\ x_4 = & c_1 c_4 - c_2 c_3 \\ x_5 = & \lambda c_1 + \mu c_4 \end{cases}$$

Because $\lambda \neq \mu$ the first and last equation uniquely determine c_1, c_4 and substitution in the second gives us $c_2 c_3$. Analogously, points of $\mathbb{C}^5$ lying in the open set $x_3^2 \neq x_4$ lie in the image of π. Finally, for a point in the complement of these open sets, that is when $x_1^2 = x_2$ and $x_3^2 = 4x_4$ we can consider a couple (A, B)

$$(\quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad , \quad \begin{bmatrix} \mu & 0 \\ c & \mu \end{bmatrix} \quad)$$

where $\lambda = \frac{1}{2} x_1$ and $\mu = \frac{1}{2} x_3$. Observe that the remaining equation $x_5 = tr(AB) = 2\lambda\mu + c$ has a solution in c.

Now, we will describe the fibers of π. Assume (A, B) is such that A and B have a common eigenvector v. Simultaneous conjugation with a $g \in GL_n$ expressing a basechange from the standard basis to $\{v, w\}$ for some w shows that the orbit (A, B) contains a couple of upper-triangular matrices. We want to describe the image of these matrices under π. Take an upper triangular representative in (A, B)

$$(\quad \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \quad , \quad \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} \quad).$$

with π-image $(x_1, \dots, x_5)$. The coordinates x_1, x_2 determine the eigenvalues a_1, a_3 of A only as an unordered set (similarly, x_3, x_4 only determine the set of eigenvalues $\{b_1, b_3\}$ of B). Hence, $tr(AB)$ is one of the following two expressions

$$a_1 b_1 + a_3 b_3 \quad \text{ or } \quad a_1 b_3 + a_3 b_1$$

and therefore satisfies the equation

$$(tr(AB) - a_1 b_1 - a_3 b_3)(tr(AB) - a_1 b_3 - a_3 b_1) = 0.$$

Recall that $x_1 = a_1 + a_3, x_2 = a_1 a_3$, $x_3 = b_1 + b_3, x_4 = b_1 b_3$ and $x_5 = tr(AB)$ we can express this equation as

$$x_5^2 - x_1 x_3 x_5 + x_1^2 x_4 + x_3^2 x_2 - 4x_2 x_4 = 0.$$

This determines an hypersurface $H\mathbb{C}^5$. If we view the left-hand side as a polynomial f in the coordinate functions of $\mathbb{C}^5$ we see that H is a four dimensional subset of $\mathbb{C}^5$ with singularities the common zeroes of the partial derivatives

$$\frac{\partial f}{\partial x_i} \text{ for } \ 1 \leq i \leq 5$$

These singularities for the 2-dimensional submanifold S of points of the form $(2a, a^2, 2b, b^2, 2ab)$. We now claim that the smooth submanifolds $\mathbb{C}^5 - H$, $H - S$ and S of $\mathbb{C}^5$ describe the different types of fiber behavior. In chapter 6 we will see that the subsets of points with different fiber behavior (actually, of different representation type) are manifolds for m-tuples of $n \times n$ matrices.

If $p \notin H$ we claim that $\pi^{-1}(p)$ is a unique orbit, which is therefore closed in M_2^2. Let $(A, B) \in \pi^{-1}$ and assume first that $x_1^2 \neq 4x_2$ then there is a representative in (A, B) of the form

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \quad , \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right)$$

with $\lambda \neq \mu$. Moreover, $c_2 c_3 \neq 0$ (for otherwise A and B would have a common eigenvector whence $p \in H$) hence we may assume that $c_2 = 1$ (eventually after simultaneous conjugation with a suitable diagonal matrix $diag(t, t^{-1})$). The value of λ, μ is determined by x_1, x_2. Moreover, c_1, c_3, c_4 are also completely determined by the system of equations

$$\begin{cases} x_3 & = c_1 + c_4 \\ x_4 & = c_1 c_4 - c_3 \\ x_5 & = \lambda c_1 + \mu c_4 \end{cases}$$

and hence the point $p = (x_1, \ldots, x_5)$ completely determines the orbit (A, B). Remains to consider the case when $x_1^2 = 4x_2$ (that is, when A has a single eigenvalue). Consider the couple $(uA + vB, B)$ for $u, v \in \mathbb{C}^*$. To begin, $uA + vB$ and B do not have a common eigenvalue. Moreover, $p = \pi(A, B)$ determines $\pi(uA + vB, B)$ as

$$\begin{cases} tr(uA + vB) & = utr(A) + vtr(B) \\ det(uA + vB) & = u^2 det(A) + v^2 det(B) + uv(tr(A)tr(B) - tr(AB)) \\ tr((uA + vB)B) & = utr(AB) + v(tr(B)^2 - 2det(B)) \end{cases}$$

Assume that for all $u, v \in \mathbb{C}^*$ we have the equality $tr(uA+vB)^2 = 4det(uA+vB)$ then comparing coefficients of this equation expressed as a polynomial in u and

v we obtain the conditions $x_1^2 = 4x_2$, $x_3^2 = 4x_4$ and $2x_5 = x_1x_3$ whence $p \in SH$, a contradiction. So, fix u, v such that $uA + vB$ has distinct eigenvalues. By the above argument $(uA + vB, B)$ is the unique orbit lying over $\pi(uA + vB, B)$, but then (A, B) must be the unique orbit lying over p.

Let $p \in H - S$ and $(A, B) \in \pi^{-1}(p)$, then A and B are simultaneous upper triangularizable, with eigenvalues a_1, a_2 respectively b_1, b_2. Either $a_1 \neq a_2$ or $b_1 \neq b_2$ for otherwise $p \in S$. Assume $a_1 \neq a_2$, then there is a representative in the orbit (A, B) of the form

$$\left(\begin{bmatrix} a_i & 0 \\ 0 & a_j \end{bmatrix} , \begin{bmatrix} b_k & b \\ 0 & b_l \end{bmatrix} \right)$$

for $\{i, j\} = \{1, 2\} = \{k, l\}$. If $b \neq 0$ we can conjugate with a suitable diagonal matrix to get $b = 1$ hence we get at most 9 possible orbits. Checking all possibilities we see that only three of them are distinct, those corresponding to the couples

$$\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}\right) \quad \left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\right) \quad \left(\begin{bmatrix} a_2 & 0 \\ 0 & a_1 \end{bmatrix}, \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}\right)$$

Clearly, the first and last orbit have the middle one lying in its closure. Observe that the case assuming that $b_1 \neq b_2$ is handled similarly. Hence, if $p \in H - S$ then $\pi^{-1}(p)$ consists of three orbits, two of dimension three whose closures intersect in a (closed) orbit of dimension two.

Finally, consider the case when $p \in S$ and $(A, B) \in \pi^{-1}(p)$. Then, both A and B have a single eigenvalue and the orbit (A, B) has a representative of the form

$$\left(\begin{bmatrix} a & x \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & y \\ 0 & b \end{bmatrix}\right)$$

for certain $x, y \in \mathbb{C}$. If either x or y are non-zero, then the subgroup of GL_2 fixing this matrix consists of the matrices of the form

$$Stab \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} = \{ \begin{bmatrix} u & v \\ 0 & u \end{bmatrix} \mid u \in \mathbb{C}^*, v \in \mathbb{C} \}$$

but these matrices also fix the second component. Therefore, if either x or y is nonzero, the orbit is fully determined by $[x : y] \in \mathbb{P}^1$. That is, for $p \in S$, the fiber $\pi^{-1}(p)$ consists of an infinite family of orbits of dimension 2 parameterized by the points of the projective line $\mathbb{P}^1$ together with the orbit of

$$\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} , \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \right)$$

which consists of one point (hence is closed in M_2^2) and lies in the closure of each of the 2-dimensional orbits.

Concluding, we see that each fiber $\pi^{-1}(p)$ contains a unique closed orbit (that of minimal dimension). The orbitclosure and dimension diagrams have the following shapes