Mirror Symmetry
Master Mathematical Physics

Take Home Exam
Date: Somewhere in June
Time: Whenever it suits you

Number of pages: 1
Number of questions:
Maximum number of points to earn: 50
At each question is indicated how many points it is worth.

BEFORE YOU START

- Please wait until you are instructed to open the booklet.
- Check if your version of the exam is complete.
- Write down your name, student ID number, and if applicable the version number on each sheet that you hand in. Also number the pages.
- Your mobile phone has to be switched off and in the coat or bag. Your coat and bag must be under your table.
- Tools allowed: Whatever suits you.

PRACTICAL MATTERS

- The first 30 minutes and the last 15 minutes you are not allowed to leave the room, not even to visit the toilet.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your enrollment or a valid ID.
- During the examination it is not permitted to visit the toilet, unless the proctor gives permission to do so.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, please fill out the evaluation form at the end of the exam.

Good luck!
Question 1. The Grassmannian $G(m, n)$ is the set of all $m$-dimensional subspaces of $\mathbb{R}^n$. Use Morse theory to determine the homology of $G(m, n)$. Start with $m = 1, n = 2, 3 \ldots$. If that works try $G(2, 4)$ and finally you can have a look at the general case.

Question 2. Let $A$ be an $A_\infty$-algebra. Show that the multiplication for the minimal model $\mu_H(A)$ that uses the sum over all trees, satisfies the $A_\infty$-axioms (use the $A_\infty$-axioms for the original $\mu$). Apply this minimal model construction to determine the $A_\infty$-structure on $\text{Ext}_A^\bullet(M, M)$ where $A = \mathbb{C}[X]/X^4$ and $M = A/(X) \cong \mathbb{C}$.

Question 3. Consider the sphere $S^2 = \{(\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta) | \phi, \theta \in \mathbb{R}\}$ with its standard symplectic structure $\cos \theta d\phi \wedge d\theta$ (i.e. the standard volume form).

1. Show that every closed lagrangian submanifold is Hamiltonianly isotopic to a circle.
2. Show that two circles on the sphere are Hamiltonianly isotopic if they are congruent.
3. Show that $HF(C_1, C_2) = 0$ if $C_1$ and $C_2$ are circles and the radius of $C_1$ or $C_2$ is smaller than 1.
4. Show that $HF(C_1, C_2) = \mathbb{C}^2$ if $C_1$ and $C_2$ are both great circles (i.e. the radii are both 1).

Question 4. Consider the graded ring $R = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ where $X, Y, Z$ all have degree 1. Show that $\text{Proj} R$ is equivalent with $\text{Coh} \mathbb{P}^1$. Give for each indecomposable object in $\text{Coh} \mathbb{P}^1$ a graded $R$-module that represents it.

Now consider the category $\text{MF}(\mathbb{C}[X, Y, Z], XY - Z^2)$. Use the fact that $XY - Z^2$ looks like a determinant to construct a matrix factorization of rank 2 (i.e. the matrices are 2 by 2 matrices). Determine the endomorphism ring of this matrix factorization.

Question 5. Consider the torus $T^2$ as a symplectic variety and let $L_1, L_2$ be a lagrangian submanifold with first homology classes $(a_1, b_1)$ and $(a_2, b_2)$ with trivial local systems. Assume that $L_1$ intersects $L_2$ normally in precisely one point $p$.

- Show that $a_1b_2 - b_1a_2 = \pm 1$.
- Consider the twisted complex $(L_1 \oplus L_2, ( 0 \ p \ 0 \ 0))$. Show that it is isomorphic to a lagrangian with homology class $(a_1 + a_2, b_1 + b_2)$.
- Show that there is a diffeomorphism $\psi : T^2 \to T^2$ that maps $L_1$ to $L_2$.

Consider the torus $E : X^3 + Y^3 + Z^3 + 2XYZ = 0 \subset \mathbb{P}^2$ as a complex variety and let $S$ be a skyscraper sheaf and $\mathcal{K}$ be the trivial line bundle.

- Show that $\text{Hom}_{D^b \text{Coh} E}(\mathcal{K}, S) = \mathbb{C}$ and let $p$ be a basis for this space.
- Consider the twisted complex $(\mathcal{K} \oplus S, ( 0 \ p \ 0 \ 0))$. Show that it is isomorphic to a line bundle.
- Is there an automorphism of $E$ that turns $S$ into $\mathcal{K}$?