Problems 2

1. *Supplementary problem of the class:* (a) What's the matrix form of $\Lambda^{\nu}{}_{\mu'}$, if $\Lambda^{\mu'}{}_{\nu}$ is given by Eq. (1.23)

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for rotation, and by Eq. (1.24)

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for boost, respectively? (b) Show the right hand sides of Eqs. (1.54):

$$\begin{aligned} V_\mu &= \eta_{\mu\nu} V^\nu, \\ \omega^\mu &= \eta^{\mu\nu}\omega_\mu, \end{aligned}$$

and Eqs. (1.55):

$$\begin{aligned} T^{\alpha\beta\mu}{}_{\delta} &= \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu}{}^{\beta}{}_{\gamma\delta} &= \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta}, \end{aligned}$$

behave as correct tensors as indicated in their left hand sides. (c) Show explicitly $X^{(\mu\nu)}Y_{\mu\nu} = X^{(\mu\nu)}Y_{(\mu\nu)}$, and discuss also a similar expression for antisymmetric tensors.

2. Imagine we have a tensor $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^\mu = (-1, 2, 0, -2).$$

Find the components of: (a) $X^{\mu}{}_{\nu}$, (b) $X_{\mu}{}^{\nu}$, (c) $X^{(\mu\nu)}$, (d) $X_{[\mu\nu]}$, (e) $X^{\lambda}{}_{\lambda}$, (f) $V^\mu V_\mu$, and (g) $V_\mu X^{\mu\nu}$.

3. Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$ transform under: (a) a rotation about the y-axis, (b) a boost along the z-axis.

4. Verify that Eq. (1.68): $\partial_{[\mu}F_{\nu\lambda]} = 0$, is equivalent to (1.68′): $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$, and that they are both equivalent to the last two equations in (1.65):

$$\begin{aligned} \tilde{\epsilon}^{ijk}\partial_j E_k + \partial_0 B^i &= 0, \\ \partial_i B^i &= 0. \end{aligned}$$