Problems 5

1. (a) Show that if particles are not conserved but are generated locally at a rate ε particles per unit volume per unit time in the fluid rest frame, then the conservation law of particles becomes

$$\partial_\mu N^\mu = \varepsilon. \tag{1}$$

(b) Generalize (a) to show that if the energy and momentum of a body are not conserved (e.g., because it interacts with other systems), then there is a nonzero relativistic force four-vector F, whose components are given by

$$\partial_\nu T^{\mu\nu} = F^\mu. \tag{2}$$

Interpret these components F^μ in the fluid rest frame.

2. In an inertial frame $\mathcal{O}$, calculate the components of the energy-momentum tensors of the following systems:

(a) A group of particles all moving with the same speed v along the x-axis, as seen in $\mathcal{O}$. Let the rest-mass density of these particles to be ρ_0, as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

(b) A ring of N particles of rest mass m are rotating counter-clockwise in the x-y plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross section of radius $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the energy-momentum of whatever forces keep them in orbit. (Part of the calculation will relate ρ_0 of part (a) to N, a, ω, and δa).

(c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or interact in any way.

3. One way to obtain the energy-momentum tensor is to use the result of Noether's theorem. In a flat spacetime, the symmetry over spacetime translation leads to the conservation law, $\partial_\nu T^{\mu\nu} = 0$, and $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^i)} \partial^\nu \Phi^i - \eta^{\mu\nu} \mathcal{L}, \tag{3}$$

where $\mathcal{L}$ is the Lagrangian density and Φ^i is some field.

Now consider the Lagrangian density of electromagnetism (in vacuum):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{4}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the vector potential. Using Eq. (3) but by replacing Φ^i with A_α, compute $T^{\mu\nu}$ for this theory and compare the result with what you have learnt from the class of classical electromagnetism.