Problems 6

1. Consider variation of the inverse metric ($g^{\mu\nu}$), $\delta g^{\mu\nu}$.

   (a) Show $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$.

   (b) By noting that $\delta g^{\mu\nu}$ is the tensor and also by using $\nabla_\mu g^{\alpha\beta} = 0$, show that the covariant derivative of the metric variation satisfies

$$\nabla_\mu (\delta g^{\alpha\beta}) = -(\delta \Gamma^\alpha_{\mu\lambda}) g^{\lambda\beta} - (\delta \Gamma^\beta_{\mu\lambda}) g^{\alpha\lambda}, \tag{1}$$

   where $\delta \Gamma^\rho_{\mu\nu}$ is the variation of the connection associated with $\delta g^{\mu\nu}$.

   (c) Using Eq. (1), show

$$\delta \Gamma^\sigma_{\mu\nu} = -\frac{1}{2} \left[g_{\lambda\mu} \nabla_\nu (\delta g^{\lambda\sigma}) + g_{\lambda\nu} \nabla_\mu (\delta g^{\lambda\sigma}) - g_{\mu\alpha} g_{\nu\beta} \nabla^\sigma (\delta g^{\alpha\beta}) \right], \tag{2}$$

   which is Eq. (4.54) in the lecture.

2. (a) As Eq. (2) shows, $\delta \Gamma^\sigma_{\mu\nu}$ is the tensor. Keeping this in mind, show the variation of the Riemann tensor is given as

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda (\delta \Gamma^\rho_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\rho_{\lambda\mu}) \tag{3}$$

   (b) Using Eq. (2), show

$$g^{\mu\nu} \delta \Gamma^\sigma_{\mu\nu} - g^{\mu\sigma} \delta \Gamma^\lambda_{\lambda\mu} = g_{\mu\nu} \nabla^\sigma (\delta g^{\mu\nu}) - \nabla_\lambda (\delta g^{\sigma\lambda}). \tag{4}$$

   These results are used to derive Eq. (4.57) in the lecture.