Problems 8

1. In order to treat the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \quad (1)$$

within the Schwarzschild radius ($r = R_S = 2GM$), we perform the following coordinate transformation:

$$T = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right), \quad (2)$$

$$R = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right), \quad (3)$$

for $r > 2GM$ (Kruskal coordinates). Show that in this new coordinate system, the metric can be written as

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}R^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \quad (4)$$

where r is implicitly given by $T^2 - R^2 = (1 - r/2GM)e^{r/2GM}$ as a function of T and R. Note that this metric is regular for any values of r except for $r = 0$ (true singularity).

2. *Shapiro time delay.*—It is known that in the gravitational field, the light arrives later than it would in the flat spacetime. Let us assume the gravitational fields are weak, and the light trajectory can be expanded as the straight line and small perturbation, $x^\mu(\lambda) = x^{(0)\mu} + x^{(1)\mu}$. Show that the time delay according to the second term is given by

$$\Delta t = -2 \int \Phi \, ds \, . \quad (5)$$