Midterm Homework Problems 1

Due: February 27 (2012); Maximum score: 50 points
General rule: Choose 5 problems to solve. You may of course solve all 6 problems, but keep in mind that the problem with the worst score does not count in your final score.

1. In the Minkowski spacetime, consider an inertial frame S with coordinates $x^\mu = (t, x, y, z)$, and a frame S' with coordinates $x^{\mu'}$ related to S by a boost with velocity parameter v along the y-axis. Imagine we have a wall at rest in S', lying along the line $x' = -y'$. From the point of view of S, what is the relationship between the incident angle of a ball hitting the wall (traveling in the x-y plane) and the reflected angle? What about the velocity before and after? Assume the collision between the ball and the wall happens elastically. [10 points]

2. (a) Study and report the details of both the Michelson-Morley experiment and the Eötvös experiment, and discuss how these experiments are related to principles of special/general relativity. [7 points]

   (b) Consider conservation law of particles in special relativity,

$$\partial_\mu (nU^\mu) = 0, \tag{1}$$

   where U^μ is the component of the velocity vector of the fluid, and n is the number density of the fluid in its rest frame. In general relativity, we generalize this equation to

$$\nabla_\mu (nU^\mu) = 0. \tag{2}$$

   Now think about equation like

$$\nabla_\mu (nU^\mu) = qR^2, \tag{3}$$

   where q is some constant and R the Ricci scalar. This corresponds to the gravitational particle production. One may argue that Eq. (3) indeed reduces to Eq. (1) in a flat spacetime. Is Eq. (3) allowed in general relativity? Explain the reason of your conclusion, especially in light of the equivalence principle. [3 points]

3. Derive the geodesic equation for massive particles by maximizing the proper time,

$$\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\mu}{d\lambda}}, \tag{4}$$

   i.e., $\delta\tau = 0$, and using the Christoffel symbol for the connection $\Gamma^\mu_{\alpha\beta}$. [10 points]

4. Prove that the Christoffel symbol satisfies

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|}, \tag{5}$$

which is then used to obtain the covariant divergence

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|}V^\mu\right). \tag{6}$$

[10 points]

5. You are familiar with the operations of gradient ($\nabla\phi$), divergence ($\nabla \cdot \mathbf{V}$), and curl ($\nabla \times \mathbf{V}$) in ordinary vector analysis in three dimensional Euclidean space. Using covariant derivatives, derive formulae for these operations in spherical polar coordinates (r, θ, φ), and compare your results to those in, e.g., Jackson (1999). (Hint: In GR, the covariant derivative ∇_μ is defined as a component of one-form. But in the classical electromagnetism, ∇ is defined as a vector. Also note that the coordinate basis vectors $\{\partial_\mu\}$ are not orthonormal in general.) [10 points]

6. A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{7}$$

where $\Phi = -GM/r$ may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the Earth. For this problem Φ may be assumed to be small.

(a) Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster? [3 points]

(b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$? [3 points]

(c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface? [4 points]