Midterm Homework Problems 2

Due: March 21 (2012); Maximum score: 50 points

1. The action of electromagnetic fields in the curved spacetime are given by

$$S_{\rm EM} = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}. \qquad (1)$$

   Vary this action with respect to $g^{\mu\nu}$ and show the energy momentum tensor is given by

$$T^{\rm EM}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm EM}}{\delta g^{\mu\nu}} = F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \qquad (2)$$

   With this, explicitly show that in the flat spacetime, T^{00} and T^{0i} reduce to the energy density of the electromagnetic fields and Poynting flux, respectively. [10 points]

2. Einstein once introduced the cosmological constant Λ in his equation. How was it introduced in the equation? Doesn't this violate any of various requirements that the original equation was supposed to satisfy? What was the original motivation for introducing that constant? What is the recent status of it, in terms of its existence and if so the value? People think that the cosmological constant is an extremely hard problem for physics (not just for cosmology!). Discuss the reason for that. [10 points]

3. An easy way of finding conserved quantities such as energy and angular momentum in the Schwarzschild spacetime is to use **Killing vectors**. We shall see how these are introduced and used.

   (a) Show that the geodesic equation $p^\rho\nabla_\rho p_\mu = 0$ can be written as

$$\frac{dp_\mu}{d\lambda} = \frac{1}{2}g_{\nu\rho,\mu}p^\rho p^\nu. \qquad (3)$$

   Therefore, if $\partial_{\sigma_*}g_{\mu\nu}$ = 0, the momentum component p_{σ_*} is conserved, $dp_{\sigma_*}/d\lambda = 0$. [3 points]

   (b) Define the Killing vector as a basis vector of this direction σ_*, $K = \partial_{\sigma_*}$. Then prove that the condition $dp_{\sigma_*}/d\lambda = 0$ is equivalent to

$$p^\mu\nabla_\mu(K_\nu p^\nu) = 0, \qquad (4)$$

$$\nabla_{(\mu}K_{\nu)} = 0, \qquad (5)$$

where the second equation is known as Killing's equation. [4 points]

(c) Apply this to the Schwarzschild metric. I.e., notice that the metric is independent of t and ϕ coordinates, and you can find the Killing vectors for each. Compute $K_\mu p^\mu$, which are constants along geodesics, and find that they are the same as what you have found (in the lecture) by solving the geodesic equation. [3 points]

4. A massive particle falls radially toward the horizon of a Schwarzschild black hole of mass M. Consider a geodesic with the conserved energy (per unit mass) of $E = (1 - 2GM/r)(dt/d\tau) = 0.95$. Since we consider the radial trajectory, the angular momentum is of course zero, $L = 0$.

   (a) Find the proper times (in units of GM) required to reach $r = 2GM$ from $r = 3GM$, and also to reach $r = 0$ from $r = 2GM$. [3 points]

   (b) Find on the Schwarzschild coordinate basis, its four-velocity components $U^\mu = dx^\mu/d\tau$, at $r = R = 2.001GM$. [3 points]

   (c) As the particle passes $2.001GM$, it sends a photon out radially to a distant stationary observer. Its frequency is ω in the rest frame of the falling particle. Using the result of (b) and

   $$\omega = -g_{\mu\nu} U^\mu k^\nu, \tag{6}$$

   where k^ν is the wave vector of the photon [apply this to the Schwarzschild coordinate system (t, r, θ, ϕ)], compute the relation between ω and $\omega_{\rm rec}$. The latter is the frequency of the photon that an observer at infinity ($r = \infty$) receives. [4 points]

5. Gravitational waves can be detected by monitoring the distance between two freely falling masses. If one of the masses is equipped with a laser and an accurate clock, and the other with a good mirror, the distance between the masses can be measured by timing how long it takes for a pulse of laser light to make the round-trip journey. Now consider the gravitational waves propagating along z-direction, of the plus mode in TT gauge:

   $$ds^2 = -\mathrm{d}t^2 + [1 + h_+ \cos(\omega(t - z))]\mathrm{d}x^2 + [1 - h_+ \cos(\omega(t - z))]\mathrm{d}y^2 + \mathrm{d}z^2. \tag{7}$$

   What is the relation between the time when the laser is emitted ($t_{\rm em}$) and the time when the same laser is received after the round-trip to the mirror ($t_{\rm rec}$)? Write $t_{\rm rec}$ as a function of h_+, $t_{\rm em}$, ω, and L (coordinate distance between two masses. Discuss this when the masses are located along x, y, and z axes, and also when they are on $y = x$ line in the $z = \mathrm{const}$ plane. (Note that, as discussed in the class, the masses are always at fixed coordinates in the TT gauge, even in the presence of the gravitational waves.) What frequency ω goes undetected? [10 points]