

Problems 3

1. As an example of definition of manifolds, we shall look at two dimensional sphere S^2 . Embedded in 3D Euclidean space (x^1, x^2, x^3) , coordinates on S^2 are

$$S^2 = \{(x^1, x^2, x^3) \in \mathbf{R}^3 | (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}. \quad (1)$$

We then define six hemispherical subsets U_i^\pm ($i = 1, 2, 3$):

$$U_i^\pm = \{(x^1, x^2, x^3) \in S^2 | \pm x^i > 0\}. \quad (2)$$

We now want to show that these six sets are “sewn smoothly.” To do so, we introduce charts $\phi_i^\pm : U_i^\pm \rightarrow \mathbf{R}^2$ as $\phi_3^+(x^1, x^2, x^3) = (x^1, x^2)$, etc. Now, write down explicitly maps like $\phi_1^+ \circ (\phi_3^+)^{-1}$, and show that this is indeed C^∞ .

2. Given two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$, we define their **commutator** (or **Lie bracket**) $[X, Y]$ by its action on a function $f(x^\mu)$:

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \quad (3)$$

We shall show $[X, Y]$ forms the vector field. (a) If f and g are functions and a and b are real numbers, show that the commutator is linear:

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g). \quad (4)$$

(b) Show that it obeys the Leibniz rule,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f). \quad (5)$$

(c) Derive an explicit expression for the components of the commutator as

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu. \quad (6)$$

(d) Show Eq. (6) indeed transforms as a vector.

3. Consider transformation from Cartesian, $x^i = (x, y, z)$, to polar coordinates, $x^{i'} = (r, \theta, \phi)$, in 3D Euclidean space. The metric for this space is given by $ds^2 = g_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2$, where dx^i is the basis one-form for the Cartesian coordinate system. (a) Express the basis one-forms $\{dx, dy, dz\}$ and then the metric $ds^2 = dx^2 + dy^2 + dz^2$ in terms of the basis one-forms of the new coordinate system, $\{dr, d\theta, d\phi\}$. (b) Use the transformation law for $(0, 2)$ tensors, i.e.,

$$g^{i'j'} = g_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}, \quad (7)$$

to obtain the metric components in the polar coordinate.

4. Consider a trajectory of a particle in 3D Euclidean space as

$$x(\lambda) = \cos \lambda, \quad y(\lambda) = \sin \lambda, \quad z(\lambda) = \lambda. \quad (8)$$

(a) Express the path of the curve in the polar coordinate system (r, θ, ϕ) . (b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems. (c) Show the tangent “vectors” obtained in (b) indeed transform as vectors.