Problems 4

1. Show that the requirement that $\nabla_\mu V^\nu$ is a tensor implies the transformation rule

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{1}$$

2. If the connection is given by the Christoffel symbol, satisfying $\nabla_\mu g_{\alpha\beta} = 0$, prove

$$\nabla_\mu \epsilon_{\alpha\beta\gamma\delta} = 0, \;\; \nabla_\mu g^{\alpha\beta} = 0, \tag{2}$$

where, if necessary, you may use $\Gamma^\mu_{\mu\lambda} = (\partial_\lambda |g|^{1/2})/|g|^{1/2}$, without proving.

3. Consider the metric of the expanding universe:

$$ds^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{3}$$

where $a(t)$ is a scale factor. Show that the Christoffel symbol of this metric is given by

$$\Gamma^0_{00} = \Gamma^0_{i0} = \Gamma^i_{00} = \Gamma^i_{jk} = 0, \;\; \Gamma^0_{ij} = a\dot{a}\delta_{ij}, \;\; \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j. \tag{4}$$

4. (a) Compute the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ for the two-sphere S^2 with the unit radius. (b) On this unit sphere, consider the vector A^μ which is the unit vector in the θ-direction, at the point $(\theta, \phi) = (\pi/2, 0)$ in polar coordinates. What happens to the vector if we parallel transport it around the equator, i.e., along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ for $0 \le \lambda \le 2\pi$? (c) Next, consider a curve which consists of four segments:

$$\begin{aligned}
\gamma_1(\lambda) &= (\pi/2, \lambda) \text{ for } 0 \le \lambda \le \lambda_1, \\
\gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \text{ for } 0 \le \lambda \le \lambda_2, \\
\gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \text{ for } 0 \le \lambda \le \lambda_1, \\
\gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \text{ for } 0 \le \lambda \le \lambda_2,
\end{aligned} \tag{5}$$

where $0 < \lambda_1 < 2\pi$ and $0 < \lambda_2 < \pi/2$. What happens to the vector A^μ once we parallel transport it around this closed path?