Problems 5

1. In an inertial frame $\mathcal{O}$, calculate the components of the energy-momentum tensors of the following systems:

   (a) A group of particles all moving with the same speed v along the x-axis, as seen in $\mathcal{O}$. Let the rest-mass density of these particles to be ρ_0, as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

   (b) A ring of N particles of rest mass m are rotating counter-clockwise in the x-y plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross section of radius $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the energy-momentum of whatever forces keep them in orbit. (Part of the calculation will relate ρ_0 of part (a) to N, a, ω, and δa).

   (c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or interact in any way.

2. Consider variation of the inverse metric $(g^{\mu\nu})$, $\delta g^{\mu\nu}$.

   (a) Show $\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$.

   (b) By noting that $\delta g^{\mu\nu}$ is the tensor and also by using $\nabla_\mu g^{\alpha\beta} = 0$, show that the covariant derivative of the metric variation satisfies

$$\nabla_\mu (\delta g^{\alpha\beta}) = -(\delta\Gamma^{\alpha}_{\mu\lambda}) g^{\lambda\beta} - (\delta\Gamma^{\beta}_{\mu\lambda}) g^{\alpha\lambda}, \tag{1}$$

   where $\delta\Gamma^{\rho}_{\mu\nu}$ is the variation of the connection associated with $\delta g^{\mu\nu}$.

   (c) Using Eq. (1), show

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right], \tag{2}$$

   which is Eq. (4.54) in the lecture.

3. (a) As Eq. (2) shows, $\delta\Gamma^{\sigma}_{\mu\nu}$ is the tensor. Keeping this in mind, show the variation of the Riemann tensor is given as

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \tag{3}$$

(b) Using Eq. (2), show

$$g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^{\lambda}_{\lambda\mu} = g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda}). \tag{4}$$

These results are used to derive Eq. (4.57) in the lecture.