Problems 6

1. Geodesic equation is equivalent to the Euler-Lagrangian equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial(dx^\mu/d\lambda)}\right) - \frac{\partial L}{\partial x^\mu} = 0, \tag{1}$$

   where the Lagrangian is given by

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{2}$$

   as discussed in Carroll (2003), p 107 (or as you may have shown in Homework 1, problem 3). Apply this equation to the Schwarzschild metric, and derive the geodesic equation. This is indeed an easier way to compute the Christoffel symbol.

2. Consider the smallest possible stable circular orbit for a massive particle in the Schwarzschild metric at $r_c = 6GM$. What are the energy E and angular momentum L (both per unit mass) for this particle? Suppose some matter fall onto this Schwarzschild black hole from infinity to this orbit at r_c. What is the energy conversion efficiency of this process? How big is it compared with the efficiency of nuclear burning, e.g., release of a maximum of 0.9% of the rest mass (H $\to$ Fe)?

3. [Kepler motion] Consider spherically symmetric Lagrangian for a unit mass particle in the Newtonian mechanics:

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \frac{GM}{r}. \tag{3}$$

   By solving the Euler-Lagrange equation, show that the orbit for bound objects is given by

$$r = \frac{(1-e^2)a}{1+e\cos\phi}, \tag{4}$$

   where e is the eccentricity and a is the semi-major axis of the orbit.