Problems 3

1. As an example of definition of manifolds, we shall look at the two dimensional sphere S^2. Embedded in 3D Euclidean space (x^1, x^2, x^3), coordinates on S^2 are

$$S^2 = \{(x^1, x^2, x^3) \in \mathbf{R}^3 | (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}. \tag{1}$$

We then define six hemispherical subsets $U_i^\pm$ $(i = 1, 2, 3)$:

$$U_i^\pm = \{(x^1, x^2, x^3) \in S^2 | \pm x^i > 0\}. \tag{2}$$

We now want to show that these six sets are "sewn smoothly." To do so, we introduce charts $\phi_i^\pm : U_i^\pm \to \mathbf{R}^2$ as $\phi_3^+(x^1, x^2, x^3) = (x^1, x^2)$, etc. Now, write down explicitly maps like $\phi_1^+ \circ (\phi_3^+)^{-1}$, and show that this is indeed C^∞.

2. Given two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$, we define their **commutator** (or **Lie bracket**) $[X, Y]$ by its action on a function $f(x^\mu)$:

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{3}$$

We shall show that $[X, Y]$ is a vector field.

(a) If f and g are functions and a and b are real numbers, show that the commutator is linear:

$$[X, Y](af + bg) = a[X, Y](f) + b[X, Y](g). \tag{4}$$

(b) Show that it obeys the Leibniz rule,

$$[X, Y](fg) = f[X, Y](g) + g[X, Y](f). \tag{5}$$

This quantity is a 'directional derivative' of Y in the direction of the vector X. When suitably generalised, it can be used to find constants of the motion.

(c) Derive an explicit expression for the components of the commutator as

$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu. \tag{6}$$

(d) Show that Eq. (6) indeed transforms as a vector.

3. Consider the transformation from Cartesian, $x^i = (x, y, z)$, to polar coordinates, $x^{i'} = (r, \theta, \phi)$, in 3D Euclidean space. The metric for this space is given by $ds^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, where $\mathrm{d}x^i$ is the basis one-form for the Cartesian coordinate system.

   (a) Express the basis one-froms $\{\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z\}$ and then the metric $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in terms of the basis one-forms of the new coordinate system, $\{\mathrm{d}r$, $\mathrm{d}\theta$, $\mathrm{d}\phi\}$.

   (b) Use the transformation law for (0, 2) tensors, i.e.,

$$g_{i'j'} = g_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}, \tag{7}$$

   to obtain the metric components in the polar coordinate.

4. Consider a trajectory of a particle in 3D Euclidean space as

$$x(\lambda) = \cos\lambda, \;\; y(\lambda) = \sin\lambda, \;\; z(\lambda) = \lambda. \tag{8}$$

   (a) Express the path of the curve in the polar coordinate system (r, θ, ϕ). (b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems. (c) Show the tangent "vectors" obtained in (b) indeed transform as vectors. (d) Find a better coordinate system in which to express this trajectory, and calculate the components of the tangent vector in this coordinate system.