Problems 4

1. Show that the requirement that $\nabla_\mu V^\nu$ is a tensor implies the transformation rule

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{1}$$

2. Consider a coordinate transformation that locally is written near the origin $x^i = 0$ as a Taylor expansion,

$$x^{i'} = a^{i'} + b_j x^j + \frac{1}{2}c^{i'}_{mn}x^m x^n + \mathcal{O}(x^3)$$

where the a, b, c are constant (in the unprimed x coordinates). We take $b^{i'}{}_i$ to be invertible with inverse $b^i{}_{i'}$ so that $b^{i'}{}_j b^j{}_{k'} = \delta^{i'}_{k'}$.

   (a) Show that only the symmetric part $c^i_{(jk)}$ contributes to the transformation above.

   (b) Find the Jacobian $\frac{\partial x^{i'}}{\partial x^i}$ and its inverse, up to $\mathcal{O}(x^2)$.

   (c) Taylor expand the metric g_{ij}, then express the metric in the new coordinates $g_{i'j'}$ using the tensor transformation law.

   (d) Show that the choice $g_{ij}|_0 b^i{}_{i'} b^j{}_{j'} = \eta_{i'j'}$ gives a metric transformation

$$g_{i'j'} = \eta_{i'j'} + \left(b^i{}_{i'} b^j{}_{j'} \frac{\partial g_{ij}}{\partial x^m} - 2c^{p'}_{(mn)} b^n_{(j'}\eta_{i')p'} \right)\Bigg|_0 x^m + \mathcal{O}(x^2)$$

   and show that this transformed metric is symmetric.

   (e) Find the value of $c^i_{(jk)}$ that gives a local inertial frame

$$g_{i'j'} = \eta_{i'j'} + \mathcal{O}(x^2)$$

   (f) Show that $\Gamma^\mu_{\rho\sigma} = \mathcal{O}(x)$ in the local inertial frame. Hence show that $\nabla_\mu v^\nu|_0 = \partial_\mu v^\nu|_0$ in the local inertial frame.

3. If the connection is given by the Christoffel symbol, satisfying $\nabla_\mu g_{\alpha\beta} = 0$, prove

$$\nabla_\mu \epsilon_{\alpha\beta\gamma\delta} = 0, \;\; \nabla_\mu g^{\alpha\beta} = 0, \tag{2}$$

where, if necessary, you may use $\Gamma^\mu_{\mu\lambda} = (\partial_\lambda |g|^{1/2})/|g|^{1/2}$.

4. Consider the metric of the expanding universe:

$$ds^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{3}$$

where $a(t)$ is a scale factor. Show that the Christoffel symbol of this metric is given by

$$\Gamma^0_{00} = \Gamma^0_{i0} = \Gamma^i_{00} = \Gamma^i_{jk} = 0, \;\; \Gamma^0_{ij} = a\dot{a}\delta_{ij}, \;\; \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j. \tag{4}$$

5. (a) Compute the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ for the two-sphere S^2 of unit radius.

   (b) On this unit sphere, consider the vector A^μ which is the unit vector in the θ-direction, at the point $(\theta, \phi) = (\pi/2, 0)$ in polar coordinates. What happens to the vector if we parallel transport it around the equator, i.e., along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ for $0 \leq \lambda \leq 2\pi$?

   (c) Next, consider a curve which consists of four segments:

$$\begin{aligned} \gamma_1(\lambda) &= (\pi/2, \lambda) \text{ for } 0 \leq \lambda \leq \lambda_1, \\ \gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \text{ for } 0 \leq \lambda \leq \lambda_2, \\ \gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \text{ for } 0 \leq \lambda \leq \lambda_1, \\ \gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \text{ for } 0 \leq \lambda \leq \lambda_2, \end{aligned} \tag{5}$$

   where $0 < \lambda_1 < 2\pi$ and $0 < \lambda_2 < \pi/2$. What happens to the vector A^μ once we parallel transport it around this closed path?