Problems 5

1. (a) By considering the components of $T^{\nu\mu}{}_{,\mu} = 0$ for a perfect fluid in the classical limit, derive the Euler equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 \\ \frac{\partial (\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot (\vec{v} \otimes (\rho \vec{v})) + \nabla p &= \vec{0} \end{aligned}$$

and argue that particle conservation $\partial_\mu N^\mu = 0$ also results in a continuity equation.

(b) Show that if particles are not conserved but are generated (locally) at a rate $\varepsilon(x)$ in the fluid rest frame, then the conservation law of particle number becomes

$$\partial_\mu N^\mu = \varepsilon \tag{1}$$

(c) Generalize the above to show that if the energy and momentum of a body are not conserved (e.g., because it interacts with other systems), then there is a nonzero four-vector F^μ sourcing the conservation law:

$$\partial_\nu T^{\mu\nu} = F^\mu \tag{2}$$

Interpret these components F^μ in the fluid rest frame.

2. In an inertial frame $\mathcal{O}$, calculate the components of the energy-momentum tensors of the following systems:

(a) A group of particles all moving with the same speed v along the x-axis, as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0, as measured in their comoving frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

(b) A ring of N particles of rest mass m are rotating counter-clockwise in the x-y plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus of circular cross section of radius $\delta a \ll a$, within which the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the energy-momentum of whatever forces keep them in orbit. (Part of the calculation will relate ρ_0 of part (a) to N, a, ω, and δa).

(c) Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or interact in any way.

3. The conservation law $\nabla_\mu T^{\nu\mu} = 0$ is local, i.e. valid at any point. However, in general, there are no conservation laws for $T^{\mu\nu}$ between separate points in curved spacetime (e.g. energy is not conserved along worldlines).

   (a) Show that this is the case by considering the parallel-transport equation for the stress-energy tensor.

   (b) Argue that this is the case by considering the symmetries that generate the classically conserved quantities (Noether's Theorem).

   (c) Show that we recover the classical conservation laws when the metric has these symmetries.

   (d) Show that we recover the classical conservation laws from parallel transport of $T^{\mu\nu}$, at the origin of a local inertial frame (i.e. locally).

4. In a flat spacetime, the conservation law is $\partial_\nu T^{\mu\nu} = 0$ and $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi^i)} \partial^\nu \Phi^i - \eta^{\mu\nu} \mathcal{L}, \tag{3}$$

   where $\mathcal{L}$ is the Lagrangian density and Φ^i is some field. Now consider the Lagrangian density of electromagnetism (in vacuum):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{4}$$

   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the vector potential.

   (a) Using Eq. (3) but by replacing Φ^i with A_α, compute $T^{\mu\nu}$ for this theory. You should find

$$T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

   which is neither symmetric nor gauge invariant.

   (b) Show that the partial derivative of $\partial_\lambda (F^{\mu\lambda} A^\nu)$ vanishes, such that adding this term to the expression above doesn't ruin the conservation law.

   (c) Show that the stress-energy tensor is then

$$T^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda - \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$