Problem Set Two – Index Gymnastics – Tuesday, 27th January 2015

Question 1

Find the components of the inverse Lorentz transformation, for the case of rotation by angle θ and boost of rapidity ϕ:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{and} \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Question 2

Show that the left hand sides of

$$\begin{aligned} V_{\mu} &= \eta_{\mu\nu} V^{\nu}, \\ \omega^{\mu} &= \eta^{\mu\nu}\omega_{\nu}, \\ T^{\alpha\beta\mu}{}_{\delta} &= \eta^{\mu\gamma} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu}{}^{\beta}{}_{\gamma\delta} &= \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\delta}, \\ T_{\mu\nu}{}^{\rho\sigma} &= \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\delta}, \end{aligned}$$

transform as tensors under Lorentz transformations, given that V^{ν}, ω_{ν}, and $T^{\alpha\beta}{}_{\gamma\delta}$ are tensors.

Question 3

Show explicitly $X^{(\mu\nu)}Y_{[\mu\nu]} = 0$, $X^{(\mu\nu)}Y_{\mu\nu} = X^{(\mu\nu)}Y_{(\mu\nu)}$, and find a similar expression for antisymmetrised tensors.

Question 4

Imagine we have a tensor $X^{\mu\nu}$ and a vector V^{μ}, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^{\mu} = (-1, 2, 0, -2).$$

In Minkowski space and in Minkowski coordinates, find the components of (a) $X^{\mu}{}_{\nu}$, (b) $X_{\mu}{}^{\nu}$, (c) $X^{(\mu\nu)}$, (d) $X_{[\mu\nu]}$, (e) $X^{\lambda}{}_{\lambda}$, (f) $V^{\mu}V_{\mu}$, and (g) $V_{\mu}X^{\mu\nu}$.

Question 5

1. Using the tensor transformation law applied to $F_{\mu\nu}$, show how the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$ transform under: (a) a rotation about the y-axis, (b) a boost along the z-axis.

2. Verify that $\partial_{[\mu}F_{\nu\lambda]} = 0$ is equivalent to $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$, and that these imply

$$\begin{aligned} \tilde{\epsilon}^{ijk}\partial_j E_k + \partial_0 B^i &= 0, \\ \partial_i B^i &= 0. \end{aligned}$$