Problem Set Three – Equivalence Principle – Tuesday, 27th January 2015

Question 1

Suppose you have two perfectly accurate and synchronized clocks. You keep one of them with you standing still at the surface of the Earth, and throw the second one vertically in the air. When you receive the second clock back in your hand, you compare the time of these two clocks. Which of them are more advanced, and why?

Question 2

Consider a coordinate transformation that locally near the origin $x^i = 0$ is written as a Taylor expansion,

$$x^{i'} = a^{i'} + b^{i'}{}_j x^j + \frac{1}{2} c^{i'}_{mn} x^m x^n + \mathcal{O}(x^3)$$

where the a, b, c are constant (in the unprimed x coordinates). We take $b^{i'}{}_i$ to be invertible with inverse $b^i{}_{i'}$ so that $b^{i'}{}_j b^j{}_{k'} = \delta^{i'}_{k'}$.

1. Show that only the symmetric part $c^i_{(jk)}$ contributes to the transformation above.
2. Find the Jacobian $\frac{\partial x^{i'}}{\partial x^i}$ and its inverse, up to $\mathcal{O}(x^2)$.
3. Taylor expand the metric g_{ij}, then express the metric in the new coordinates $g_{i'j'}$ using the tensor transformation law.
4. Show that the choice $g_{ij}|_0 \, b^i{}_{i'} b^j{}_{j'} = \eta_{i'j'}$ gives a metric transformation

   $$g_{i'j'} = \eta_{i'j'} + \left(b^i{}_{i'} b^j{}_{j'} \frac{\partial g_{ij}}{\partial x^m} - 2 c^{p'}_{(mn)} b^n_{(j'} \eta_{i')p'} \right)\Bigg|_0 x^m + \mathcal{O}(x^2)$$

   and show that this transformed metric is symmetric.
5. Find the value of $c^i_{(jk)}$ that gives a local inertial frame

   $$g_{i'j'} = \eta_{i'j'} + \mathcal{O}(x^2)$$

6. Show that $\Gamma^\mu_{\rho\sigma} = \mathcal{O}(x)$ in the local inertial frame. Hence show that $\nabla_\mu v^\nu|_0 = \partial_\mu v^\nu|_0$ in the local inertial frame. (You may skip this question until you learn about Γ and ∇)

Question 3

In General Relativity, gravity arises as an *inertial force* related to the fact that we try to apply the physical laws of freely falling frames, to frames which are not actually freefalling. This problem investigates the fictitous forces that arise in nonrelativistic systems, when we use the laws of classical mechanics in noninertial frames.

A train accelerates from rest with a uniform acceleration a. A spring of constant k, colinear to the motion, connects a small mass m to the accelerating carriage.

1. Using Hooke's Law and Newton's Law, find an expression for the displacement of the small mass.

2. An observer in the train, unaware of the acceleration, notices that the small mass does not sit at the natural length of the spring, yet does not move relative to the train; the observer concludes (erroneously) that it must be in mechanical equilibrium. Write down the law for mechanical equilibrium that such an observer would (erroneously) use.

3. This observer then peforms an experiment, replacing the mass attached to the spring; using your result from part (a), find the displacement from the natural spring length with this new mass.

4. The observer, still unaware of the acceleration, now has a number of data points $(\Delta x_i, M_i)$ for a selection of masses. Using Hooke's Law, the observer plots F_i vs M_i; what does this observer conclude?

5. An observer on the station platform suspends a series of test masses on a spring, measures the displacement of the test masses under gravity, and plots F_i vs M_i; what does this observer conclude?