Problem Set Five – Curvature Tools – Tuesday, 27th January 2015

Question 1

1. Show that given the transformation laws of ∂_μ and V^ν, the requirement that $\nabla_\mu V^\nu$ be a tensor implies the transformation rule

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{1}$$

2. If the connection is given by the Christoffel symbol, satisfying $\nabla_\mu g_{\alpha\beta} = 0$, prove

$$\nabla_\mu \epsilon_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu g^{\alpha\beta} = 0, \tag{2}$$

where, if necessary, you may use $\Gamma^\mu_{\mu\lambda} = (\partial_\lambda |g|^{1/2})/|g|^{1/2}$.

Question 2

1. Compute the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ for the two-sphere S^2 of unit radius.

2. On this unit sphere, consider the vector A^μ which is the unit vector in the θ-direction, at the point $(\theta, \phi) = (\pi/2, 0)$ in polar coordinates. What happens to the vector if we parallel transport it around the equator, i.e., along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ for $0 \le \lambda \le 2\pi$?

3. Next, consider a curve which consists of four segments:

$$\begin{aligned} \gamma_1(\lambda) &= (\pi/2, \lambda) \text{ for } 0 \le \lambda \le \lambda_1, \\ \gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \text{ for } 0 \le \lambda \le \lambda_2, \\ \gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \text{ for } 0 \le \lambda \le \lambda_1, \\ \gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \text{ for } 0 \le \lambda \le \lambda_2, \end{aligned} \tag{3}$$

where $0 < \lambda_1 < 2\pi$ and $0 < \lambda_2 < \pi/2$. What happens to the vector A^μ once we parallel transport it around this closed path?

Question 3

1. Recall that the commutator of two vectors is bilinear, to prove that

$$\nabla_{\mathbf{u}+\mathbf{n}}\mathbf{v} = \nabla_{\mathbf{u}}\mathbf{v} + \nabla_{\mathbf{n}}\mathbf{v}$$

2. Derive the chain rule

$$\nabla_{\mathbf{u}}(\mathbf{v} \otimes \mathbf{w}) = (\nabla_{\mathbf{u}}\mathbf{v}) \otimes \mathbf{w} + \mathbf{v} \otimes (\nabla_{\mathbf{u}}\mathbf{w})$$

   and write down the component version of this equation. Based on this, extrapolate the chain rule for generic (k, l) tensors.

3. Express $\nabla_{\mathbf{u}}\mathbf{u}$ in component notation. Show that even if the connection had an antisymmetric part $\Gamma^{\alpha}_{[\mu\nu]} \neq 0$, then it would not contribute to this quantity.