Problem Set Ten – Cosmology – March 12, 2015

Question 1

The FRW spacetime can be foliated into surfaces of simultaneity, i.e. the metric can be written in "synchronous" form:

$$ds^2 = -\mathrm{d}t^2 + g^{(3)}_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

where $g^{(3)}$ is a three-dimensional (space-only) metric – to characterise the 3-geometry of these spacelike hypersurfaces as a function of time, is to characterise the geometry of the universe.

1. Assume we know the initial conditions on some surface of simultaneity, $g^{(3)}_{ij}(t = t_0, x^k) = \gamma_{ij}(x^k)$. If the distance between the worldlines of two stationary particles at $t = t_0$ is $\Delta^{(3)}s(t_0) = \sqrt{\gamma_{ij}\Delta x^i\Delta x^j}$, show that the spatial metric of an isotropic and homogenous space can be written as $g^{(3)}_{ij}(x^\mu) = a^2(t)\gamma_{ij}(x^k)$.

2. In order that there do not exist preferred directions or locations, we must construct the three-Riemann tensor using only invariant tensors (Kronecker δ, Levi-Civita ϵ, metric γ). Show that
$$R^{(3)}_{ijkl} \propto \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk} \tag{1}$$
satisfies the index symmetries of the Riemann tensor, and the Bianchi identities.

3. Show that the following 3-metric admits (1) as its curvature tensor:
$$\gamma_{ij} = \left(1 + \frac{K}{4}\delta_{kl}x^k x^l\right)^{-2}\delta_{ij}$$
where K is some (possibly time-dependent) constant on each hypersurface of simultaneity.

4. By transforming the above three-metric to spherical polar coordinates and then rescaling the radial direction, express this metric in the form
$$d\sigma^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega^2$$

Question 2

Show that the two Friedmann Equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \tag{2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3pc^{-2}\right) \tag{3}$$

are not independent of the fluid conservation equation

$$\dot{\rho} + 3\left(\rho + pc^{-2}\right)\frac{\dot{a}}{a} = 0$$

Question 3

Repeat the derivation of the Friedmann Equations from the Einstein Field Equations, in the presence of a cosmological constant.

Question 4

Consider the following passage from MTW:

> "Of all the disturbing implications of "the expansion of the universe", none is more upsetting to many a student on first encounter than the nonsense of this idea. The universe expands, the distance between one cluster of galaxies and another cluster expands, the distance between sun and earth expands, the length of a meter stick expands, the atom expands? Then how can it make any sense to speak of any expansion at all? Expansion relative to what? Expansion relative to nonsense!"

1. Identify the many misunderstandings in the reasoning above, and
2. explain them away.