Problem Set Seven – Einstein-Hilbert Action – March 2, 2015

It is possible to derive the Einstein Equations from an Action Principle,

$$\delta I = \frac{c^4}{16\pi G} \delta \int R \, d(\text{proper fourvolume}) + \delta I_{\text{matter}} = 0,$$

where R is the Ricci scalar, and the variation is performed over the ten independent components of the metric.

This problem set explores this construction: the first question gets rid of the surface terms of the variation, the second question constructs the Einstein Tensor, and the third question adds matter sources for the gravitational fields, as well as a cosmological constant.

Question 1

1. The Ricci scalar can be written as $R = R^* - L$, where

$$R^* = g^{\mu\nu} \left(\Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\nu,\sigma} \right) \tag{1}$$

$$L = g^{\mu\nu} \left(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \right) \tag{2}$$

   Show that $\int R^*\sqrt{-g}d^4x \sim \int 2L\sqrt{-g}d^4x$ after integration by parts (with perfect differentials that do not contribute to the integral), so that our gravitational action is

$$I = \int R\sqrt{-g}d^4x = \int L\sqrt{-g}d^4x \tag{3}$$

   You may use, without proof,

$$(g^{\mu\nu}\sqrt{-g})_{,\sigma} = (-g^{\nu\beta}\Gamma^\mu_{\beta\sigma} - g^{\mu\alpha}\Gamma^\nu_{\alpha\sigma} + g^{\mu\nu}\Gamma^\beta_{\sigma\beta})\sqrt{-g}$$

2. Show that

$$\delta(L\sqrt{-g}) = \Gamma^\alpha_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})_{,\alpha} - \Gamma^\beta_{\alpha\beta}\delta(g^{\alpha\nu}\sqrt{-g})_{,\nu} + (\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{\mu\nu})\delta(g^{\mu\nu}\sqrt{-g})$$

3. Hence, show that

$$\delta \int (L\sqrt{-g})d^4x \sim \int R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})d^4x \tag{4}$$

   up to perfect differentials.

4. Hence, conclude that $R_{\mu\nu} = 0$ in the absence of a matter term.

Question 2

1. Show that the trace-reversion tensor $T^{\mu\nu}_{\alpha\beta} = (\delta^\mu_\alpha \delta^\nu_\beta - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta})$ turns into the (symmetrised) identity tensor when contracted with another copy of itself.

2. Show that $\delta(g^{\mu\nu}\sqrt{-g}) = T^{\mu\nu}_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}$.

3. Hence, conclude that $G^{\mu\nu} = 0$ (Einstein Equation in a vacuum).

Question 3

In the context of the Einstein field equations, 'matter' is anything that isn't gravity – including gauge bosons like photons. The action of electromagnetic fields (in curved spacetime, with permeability $\mu_0 = 1$) is given by

$$I_{\text{EM}} = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{5}$$

1. Vary this action with respect to $g^{\mu\nu}$ and show that the energy momentum tensor is given by

$$T^{\text{EM}}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta I_{\text{EM}}}{\delta g^{\mu\nu}} = F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \tag{6}$$

2. With this, explicitly show that in flat spacetime, T^{00} and T^{0i} reduce to the energy density of the electromagnetic fields and Poynting flux, respectively.

3. Show that the variation of the entire action gives the full Einstein-Maxwell equations.

4. Add a cosmological constant to the field equations by considering the variations of an extra term

$$I_\Lambda \propto \int \Lambda\sqrt{-g}d^4x. \tag{7}$$

For which equation of state $f(\rho, p) = 0$ can the resulting contribution to the field equations be interpreted as the stress-energy of a perfect fluid?