Problem Set Nine – Black Holes – March 9, 2015

Question 1

The geodesic equation is equivalent to the Euler-Lagrangian equation

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial(dx^\mu/d\lambda)}\right) - \frac{\partial L}{\partial x^\mu} = 0, \tag{1}$$

where the Lagrangian is given by

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}, \tag{2}$$

as discussed in Carroll (2003), p 107 (or as you may have shown in Homework 1). Apply this equation to the Schwarzschild metric, and derive the geodesic equation.

Question 2

Consider the smallest possible stable circular orbit for a massive particle in the Schwarzschild metric at $r_c = 6GM$.

1. What are the energy E and angular momentum L (both per unit mass) for this particle?

2. Suppose some matter falls into this Schwarzschild black hole from infinity to this orbit at r_c. What is the energy conversion efficiency of this process? How big is it compared with the efficiency of nuclear burning, e.g., release of a maximum of 0.9% of the rest mass (H $\to$ Fe)?

3. The frequency of a photon of momentum p^μ, as seen by an observer with velocity U^μ is $\omega = -g_{\mu\nu}U^\mu p^\nu$; Use conservation of energy in the Schwarzschild metric to show that the frequency of a photon climbing out of a star's gravitational potential (from a radius $r > r_S$ to a radius $R \to \infty$) is given by

$$\frac{\omega_R}{\omega_r} = \sqrt{\frac{1 - r_S/r}{1 - r_S/R}} \tag{3}$$

4. Suppose an object at rest at $r_c = 6GM$ emits a photon of wavelength 410 nm (violet). How many Schwarzchild radii away must an observer stay in order to observe that photon with wavelength 450 nm (blue)? 500 nm (green)? 700 nm (red)? What is this wavelength as the observer approaches the horizon?

Question 3

In order to treat the Schwarzschild solution within the Schwarzschild radius, we perform the following coordinate transformation (Kruskal coordinates):

$$\begin{aligned} T &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right), & (4) \\ R &= \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right), & (5) \\ r &> 2GM & (6) \end{aligned}$$

1. Show that in this new coordinate system, the metric can be written as

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-\mathrm{d}T^2 + \mathrm{d}R^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{7}$$

   where r is implicitly given by $T^2 - R^2 = (1 - r/2GM)e^{r/2GM}$ as a function of T and R.

2. Show that this metric is regular for any values of r except for $r = 0$.

3. Show that $dT/dR = \pm 1$, such that null rays lies live at 45 degree angles in a Kruskal diagram.

Question 4

Why is there no black hole at center of the earth?

Question 5

It is known that in the gravitational field, the light arrives later than it would in the flat spacetime. Let us assume the gravitational fields are weak, and the light trajectory can be expanded as a straight line plus a small perturbation, $x^\mu(\lambda) = x^{(0)\mu} + x^{(1)\mu}$. Show that the time delay according to the second term is given by

$$\Delta t = -2 \int \Phi \, ds \,. \tag{8}$$