Judgment Aggregation June Project: Agenda Characterisation

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(based on the slides of Ulle Endriss)
Goals

We continue with looking into agendas that can be associated with “well-behaved” judgment aggregation.

▶ *Existential Agenda Characterisation*: Fix a class of aggregation rules by means of fixing some axioms. For what kinds of agendas is there a consistent rule in that class?

See two survey papers:


Axioms

We use the following axioms (the last one is new!) for rules $F$:

- **Neutrality**: $N^J_\varphi = N^J_\psi$ implies $\varphi \in F(J) \iff \psi \in F(J)$.
- **Independence**: $N^J_\varphi = N^{J'}_\varphi$ implies $\varphi \in F(J) \iff \varphi \in F(J')$.
- **Monotonicity**: $N^J_\varphi \subset N^{J'}_\varphi$ implies $\varphi \in F(J) \Rightarrow \varphi \in F(J')$.
- **Dictatorship**: There exists an agent $i^*$ (the dictator) such that $F(J) = J_{i^*}$ for every profile $J$. Otherwise, $F$ is nondictatorial.

You see how non-dictatorship is a weakening of anonymity?
An Existential Agenda Char. Theorem

**Theorem (Nehring and Puppe, 2007)**

For $n \geq 3$, there exists a *neutral, independent, monotonic, and nondictatorial* aggregator that is *complete and consistent* for the agenda $\Phi$ *iff* $\Phi$ has the MP.

The right-to-left direction follows from our previous Theorem: Suppose $\Phi$ has the MP. Then:

- The majority rule will be consistent and complete.
- So there exists a rule with all the required properties. ✓

Next we will prove the impossibility direction (left-to-right).

**A very Useful Notion: Winning Coalitions**

$F$ is *independent* iff there is a family of *winning coalitions* of agents $W_\varphi \subseteq 2^N$, one for each $\varphi$, s.t. $\varphi \in F(J) \iff N^J_\varphi \in W_\varphi$.

$F$ is *independent and neutral* if furthermore we have $W_\varphi = W_\psi$ for all formulas $\varphi, \psi \in \Phi$. So we can simply write $W$.

Now suppose $F$ is independent and neutral, and defined by $W$:

- $F$ is *monotonic* iff $W$ is upward closed: $C \in W$ and $C \subseteq C'$ entail $C' \in W$ for all $C, C' \subset N$.
- $F$ is *complete* iff $C \in W$ or $\overline{C} \in W$ for all $C$. (why? ⭐)
- $F$ is *complement-free* iff $C \notin W$ or $\overline{C} \notin W$ for all $C \subseteq N$.

(Note that here we assume that $\Phi$ has at least two atoms.)
Proof Plan: Impossibility Direction

We will show that: If a rule $F$ is neutral, independent, monotonic, complete, and consistent for an agenda $\Phi$ violating the MP, then $F$ must be a dictatorship.

So suppose $F$ has these properties and $\Phi$ violates the MP. By independence and neutrality, there is a (single) family $\mathcal{W} \subseteq 2^\mathcal{N}$ of winning coalitions for $F$: $\varphi \in F(J) \iff N_\varphi^J \in \mathcal{W}$.

We will show that $\mathcal{W}$ is an ultrafilter on $\mathcal{N}$, meaning that:

(i) The empty coalition is not winning: $\emptyset \notin \mathcal{W}$

(ii) Closure under intersections: $C, C' \in \mathcal{W} \Rightarrow C \cap C' \in \mathcal{W}$.

(iii) Maximality: $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$.

In the end, using the finiteness of $\mathcal{N}$, we will show that $\mathcal{W} = \{C \subseteq \mathcal{N} \mid i^* \in C\}$ for some $i^* \in \mathcal{N}$, i.e., $F$ is dictatorial.
(I) **The Empty Coalition is not Winning**

We will use *monotonicity* and *complement-freeness*:

For the sake of contradiction, assume $\emptyset \in \mathcal{W}$.

- From monotonicity (i.e., closure under supersets): $\emptyset \in \mathcal{W}$ implies that $\mathcal{N} \in \mathcal{W}$.
- But now consider some profile $\mathbf{J}$ with $p \in J_i$ for all individuals $i \in \mathcal{N}$. (why can we take such a $\mathbf{J}$? ⭐)
- Then, $N_p^J = \mathcal{N}$ and $N_{\neg p}^J = \emptyset$.
- That is, $p \in F(\mathbf{J})$ and $\neg p \in F(\mathbf{J})$, as both $\mathcal{N}$ and $\emptyset$ are winning coalitions.
- **Contradiction** with complement-freeness. ✓
(III) Maximality (Easy first)

We will use completeness:

- Take any coalition $C$ and formula $\varphi$.
- Construct a profile $J$ with $N^J_\varphi = C$.
- From completeness: $\varphi \in F(J)$ or $\sim \varphi \in F(J)$.
- Then from $\mathcal{W}$-determination of $F$: $N^J_\varphi \in \mathcal{W}$ or $N^J_{\sim \varphi} \in \mathcal{W}$.
- From completeness and complement-freeness of $F$: $N^J_{\sim \varphi} = \overline{N^J_\varphi}$.
- Finally, all this means that $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$. ✓
(II) Closure Under Intersections

We use: MP-violation, monotonicity, consistency, completeness.

MP-violation: there is a mi subset $X = \{\varphi_1, \ldots, \varphi_k\}$ with $k \geq 3$. We can construct a complete and consistent profile $J$ with these properties (everyone accepts $k - 1$ of the propositions in $X$):

- $N_{\varphi_1}^J = C$.
- $N_{\varphi_2}^J = C' \cup (N \setminus C)$.
- $N_{\varphi_3}^J = N \setminus (C \cap C')$, thus $N_{\sim \varphi_3}^J = C \cap C'$.
- $N_{\psi}^J = N$ for all $\psi \in X \setminus \{\varphi_1, \varphi_2, \varphi_3\}$.

Then, suppose that $C, C' \in \mathcal{W}$.

- $C \in \mathcal{W} \Rightarrow \varphi_1 \in F(J)$.
- (monotonicity) $C' \in \mathcal{W} \Rightarrow C' \cup (N \setminus C) \in \mathcal{W} \Rightarrow \varphi_2 \in F(J)$
- (maximality) $\emptyset \notin \mathcal{W} \Rightarrow N \in \mathcal{W} \Rightarrow X \setminus \{\varphi_1, \varphi_2, \varphi_3\} \subseteq F(J)$
- (consistency) $\varphi_3 \notin F(J) \Rightarrow \sim \varphi_3 \in F(J) \Rightarrow C \cap C' \in \mathcal{W}$. ✓
Dictatorship

We have shown that:

(I) The *empty coalition* is not winning: $\emptyset \notin \mathcal{W}$

(II) *Closure under intersections*: $C, C' \in \mathcal{W} \Rightarrow C \cap C' \in \mathcal{W}$.

(III) *Maximality*: $C \in \mathcal{W}$ or $\overline{C} \in \mathcal{W}$.

From (I) and completeness, we have that $\mathcal{N} \in \mathcal{W}$.

*Contraction Lemma*: if $C \in \mathcal{W}$ and $|C| > 2$, then $C' \in \mathcal{W}$ for some $C' \subset C$. Proof: Let $C = C_1 \amalg C_2$. If $C_1 \notin \mathcal{W}$, then $\overline{C_1} \in \mathcal{W}$ by maximality. But then, $C \cap C_1 = C_2 \in \mathcal{W}$ by closure under intersections.

Recall that $\mathcal{N}$ is *finite*. By induction (★): $\{i^*\} \in \mathcal{W}$ for one $i^* \in \mathcal{N}$, i.e., $\mathcal{W} = \{C \subseteq \mathcal{N} \mid i^* \in C\}$. That is, $i^*$ is a *dictator*.

We just used that every ultrafilter on a finite set is *principal*!
Relevance to Preference Aggregation

A similar characterisation result by Dokow and Holzman is particularly interesting since it can be considered a generalisation of the most famous theorem in social choice theory: Arrow’s Theorem for preference aggregation.

To see the relevant result in judgment aggregation and for a comparison with Arrow’s Theorem, consult the papers below:


Existential agenda char. theorems are of the following form: 

*There exists* a nondictatorial complete and consistent *rule* meeting certain axioms ⇔ the agenda has a certain *property*.

Both directions are of interest:

(⇐) *Possibility direction*: If the agenda property holds, then “reasonable” and consistent aggregation is possible.

(⇒) *Impossibility direction*: For structurally rich domains, all seemingly “reasonable” rules are in fact dictatorial.

Possibility is proved by providing a *concrete rule* doing the job. Impossibility is (sometimes) proved using *ultrafilters*. 
SUMMARY OF TODAY

We investigated two big questions, connecting the structure of an agenda with the (im)possibility of consistent aggregation.

▶ First, we focused specifically on the majority rule.
▶ Then, we examined the axioms characterising the majority rule, minus anonymity, and we saw a universal characterisation result, also called a safety result.
▶ In the second part, we took again the majority axioms, weakening anonymity to non-dictatorship, and we saw an existential characterisation result.

Tomorrow, fun (and lighter) stuff is coming! ★★★