Exercise Measure Theoretic Probability
Korteweg-de Vries Instituut voor wiskunde, UvA

Lecturer: Sonja Cox
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Exercise 1 (4pt)
Let \((X_n)_{n=1}^{\infty}\) be a sequence of independent, identically distributed random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Assume \(\mathbb{E}X_1 = 0\). Prove: \(\lim_{n \to \infty} \prod_{k=1}^{n} e^{X_k/n} = 1\) a.s.

Exercise 2 (12pt)
For \(A \subset \mathcal{B}([-1, 1])\) define \(-A = \{x \in \mathbb{R}: -x \in A\}\). Let \(\lambda\) denote the Lebesgue measure on \(\mathcal{B}([-1, 1])\). Define
\[
\mathcal{A} = \{A \in \mathcal{B}([-1, 1]): \lambda(A) = \lambda(-A)\}.
\]

i. (3pt) Prove that \(\mathcal{A}\) is a \(d\)-system.

ii. (1 pt) Prove that \(\mathcal{A} = \mathcal{B}([-1, 1])\).
iii. (4pt) Prove: if \( f: [-1, 1] \rightarrow \mathbb{R} \) is a simple function and \( f(x) = -f(-x) \) for all \( x \in [-1, 1] \), then \( \int_{-1}^{1} f(x) \, dx = 0 \).

iv. (4pt) Prove: if \( f: [-1, 1] \rightarrow \mathbb{R} \) is Lebesgue integrable and \( f(x) = -f(-x) \) for all \( x \in [-1, 1] \), then \( \int_{-1}^{1} f(x) \, dx = 0 \).

Exercise 3 (14pt)
Let \((X_n)_{n=1}^{\infty}\) be a sequence of independent, identically distributed random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}X_1^2 = 1 \). Define \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and for \( n \in \mathbb{N} \) define \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \) and

\[
M_n = \sum_{k=1}^{n} \frac{1}{k}(X_1 \cdot \ldots \cdot X_k)
\]

(i.e., \( M_1 = X_1, M_2 = X_1 + \frac{1}{2}X_1X_2, M_3 = X_1 + \frac{1}{3}X_1X_2 + \frac{1}{3}X_1X_2X_3 \)).

i. (4pt) Calculate \( \mathbb{E}M_n^2, n \in \mathbb{N} \).

ii. (4pt) Prove that \((M_n)_{n=1}^{\infty}\) is an \((\mathcal{F}_n)_{n=1}^{\infty}\)-martingale.

iii. (4pt) Provide \( \langle M \rangle \).

iv. (2pt) Is \((M_n)_{n=1}^{\infty}\) uniformly integrable?

Exercise 4 (18pt)
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \( A_{n,j} \in \mathcal{F}, n \in \mathbb{N}_0, j \in \{1, 2, 3, \ldots, 2^n\}, \) be such that for all \( n \in \mathbb{N}_0 \) it holds that \( \bigcup_{j=1}^{2^n} A_{n,j} = \Omega \) and

\[
\forall i, j \in \{1, 2, 3, \ldots, 2^n\}, i \neq j: A_{n,i} \cap A_{n,j} = \emptyset \text{ and } A_{n,i} = A_{n+1,2i-1} \cup A_{n+1,2i}.
\]

(1)

For \( n \in \mathbb{N}_0 \) define

\[
\mathcal{F}_n = \sigma(\{A_{n,j}: j \in \{1, 2, 3, \ldots, 2^n\}\}).
\]

i. (2pt) Prove that \((\mathcal{F}_n)_{n=0}^{\infty}\) is a filtration.

ii. (3pt) Let \( \mu: \mathcal{F} \rightarrow [0, \infty) \) be a probability measure on \((\Omega, \mathcal{F})\). Assume:

\[
\text{If } \mathbb{P}(A) = 0 \text{ for some } A \in \mathcal{F}, \text{ then } \mu(A) = 0.
\]

(2)

For \( n \in \mathbb{N}_0 \) define

\[
M_n(\omega) = \begin{cases} \frac{\mu(A_{n,j})}{\mathbb{P}(A_{n,j})}, & \omega \in A_{n,j}, \mathbb{P}(A_{n,j}) \neq 0; \\ 0, & \text{otherwise}. \end{cases}
\]

(3)

Prove that for all \( n \in \mathbb{N}_0 \) and all \( j \in \{1, 2, 3, \ldots, 2^n\} \) it holds that

\[
\int_{A_{n,j}} M_{n+1} \, d\mathbb{P} = \int_{A_{n,j}} M_n \, d\mathbb{P}.
\]

(4)
iii. (1pt) Explain why it follows that \((M_n)_{n=1}^{\infty}\) is \((\mathcal{F}_n)_{n=0}^{\infty}\)-martingale.

iv. (2pt) Prove that for all \(A \in \mathcal{F}_n\) it holds that \(\mu(A) = \int_A M_n \, d\mathbb{P}\).

v. (4pt) Assumption (2) is equivalent to:
\[\forall \varepsilon > 0 \exists \delta > 0: \text{if } \mathbb{P}(A) < \delta \text{ for some } A \in \mathcal{F}_n, \text{ then } \mu(A) < \varepsilon.\]

Prove that \((M_n)_{n=1}^{\infty}\) is uniformly integrable.

vi. (4pt) Define \(\mathcal{F}_{\infty} = \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)\). Prove that there exists a \(M_{\infty} \in L^1(\Omega, \mathcal{F}_{\infty}, \mathbb{P})\) such that \(\mu(A) = \int_A M_{\infty} \, d\mathbb{P}\) for all \(A \in \mathcal{F}_{\infty}\).

Exercise 5 (10pt)

Let \((X_n)_{n=1}^{\infty}\) be a sequence of independent, identically distributed random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying \(\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}\). For \(n \in \mathbb{N}\) let \(\phi_n : \mathbb{R} \to \mathbb{C}\) denote the characteristic function of \(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k\).

In this exercise you may not use the Central Limit Theorem.

You may use that by l'Hopital's rule we have that
\[
\lim_{x \to 0} \frac{\log(\cos(ux))}{x^2} = \lim_{x \to 0} \frac{-u \sin(ux)}{2x \cos(ux)} = \lim_{x \to 0} \frac{-u^2 \cos(ux)}{2(\cos(ux) - x \sin(ux))} = \frac{-u^2}{2}.
\]

i. (4pt) Prove that \(\phi_n(u) = (\cos(n^{-1/2}u))^n\) for \(u \in \mathbb{R}\).

ii. (4pt) Let \(\gamma : \Omega \to \mathbb{R}\) be a standard Gaussian random variable and let \(\phi_\gamma : \mathbb{R} \to \mathbb{R}\) denote its characteristic function, i.e., \(\phi_\gamma(u) = e^{-u^2/2}\) for all \(u \in \mathbb{R}\). Prove that \(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \to \gamma\) weakly as \(n \to \infty\).

Exercise

Let \((X_n)_{n=1}^{\infty}\) be a sequence of independent, identically distributed random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying \(\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}\). For \(n \in \mathbb{N}\) and \(k \in \{1, \ldots, n\}\) let \(a_{n,k} \in [0, \infty)\) be such that \(\sum_{j=1}^{n} a_{n,j}^2 = 1\) and define \(\xi_{n,k} = a_{n,k} X_k\).

i. Formulate the Lindeberg condition for \((\xi_{n,k})_{n \in \mathbb{N}, k \in \{1, \ldots, n\}}\).

ii. Provide explicit values of \(a_{n,k}, n \in \mathbb{N}, k \in \{1, \ldots, n\}\), such that \(\sum_{k=1}^{n} \xi_{n,k} \to \gamma\) weakly as \(n \to \infty\), where \(\gamma\) is a standard normal Gaussian random variable.

Exercise

Let \((S, \mathcal{S})\) be a measurable space and let \(\mu, \nu : \mathcal{S} \to [0, \infty)\) be finite measures on \((S, \mathcal{S})\). Assume that for all \(A \in \mathcal{S}\) it holds that \(\mu(A) = 0\) implies \(\nu(A) = 0\). Prove that for all \(\varepsilon > 0\) there exists a \(\delta > 0\) such that if \(\mu(A) < \delta\), then \(\nu(A) < \varepsilon\).
Exercise

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(A_n \in \mathcal{F}, n \in \mathbb{N}\). Assume \(\mathcal{F} = \sigma(\{A_n: n \in \mathbb{N}\})\). For \(n \in \mathbb{N}\) define \(\mathcal{F}_n = \sigma(\{A_k: k \in \{1, \ldots, n\}\})\). Let \(B \in \mathcal{F}\).

i. Prove that \(\lim_{n \to \infty} \mathbb{E}[1_B|\mathcal{F}_n] = 1_B\) a.s.

ii. Does it hold that \(\lim_{n \to \infty} \mathbb{E}[\mathbb{E}[1_B|\mathcal{F}_n] - 1_B] = 0\)? Motivate your answer.

Exercise

Let \((X_n)_{n=1}^{\infty}\) be a sequence of independent, identically distributed random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying \(\mathbb{E}(X_1^2) < \infty\). Let \(Y_n \to Y\) weakly as \(n \to \infty\). Prove that for all \(a, b \in \mathbb{R}, a < b\), it holds that \(\liminf_{n \to \infty} \mathbb{P}(Y_n \in (a, b)) > \mathbb{P}(Y \in (a, b))\).