Measure Theoretic Probability
MasterMath course

Final Exam
Date: February 3rd 2016
Time: 14:00-17:00

Number of pages: 2 (including front page)
Number of questions: 7
Maximum number of points to earn: 30
At each question is indicated how many points it is worth.

BEFORE YOU START

- Please wait until you are instructed to open the booklet.
- Check if your version of the exam is complete.
- Write down your name, student ID number, and if applicable the version number on each sheet that you hand in. Also number the pages.
- Your mobile phone has to be switched off and in the coat or bag. Your coat and bag must be under your table.
- Tools allowed: paper, pen, pencil, eraser.

PRACTICAL MATTERS

- The first 30 minutes and the last 15 minutes you are not allowed to leave the room, not even to visit the toilet.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your enrollment or a valid ID.
- During the examination it is not permitted to visit the toilet, unless the proctor gives permission to do so.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, please fill out the evaluation form at the end of the exam.

Good luck!
Final exam MTP.

Remark: the stars give an indication of how difficult the lecturer believes the question to be; (*) means ‘easy’, (**) means ‘medium’, and (***) means ‘hard’.

Question 1 (4 pts, *) Let \( f, g : \mathbb{R} \to \mathbb{R} \) be given by
\[
    f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}; \quad x \in \mathbb{R}
\]
\[
    g(x) = \begin{cases} 
        0; & x \in (-\infty, 0), \\
        e^{-x}; & x \in [0, \infty).
    \end{cases}
\]

Let \( \mu, \nu : B(\mathbb{R}) \to [0, 1] \) be probability measures defined by
\[
    \mu(B) = \int_B f(x) \, dx; \quad \nu(B) = \int_B g(x) \, dx, \quad B \in B(\mathbb{R}).
\]

By the Radon-Nikodym theorem, there exist unique measures \( \mu_a, \mu_s \) on \( B(\mathbb{R}) \) such that \( \mu = \mu_a + \mu_s, \mu_a \ll \nu, \) and \( \mu_s \perp \nu. \) Provide \( \mu_a \) and \( \mu_s \) explicitly.

Question 2 (3 pts, **) Let \( (W_t)_{t \in [0, \infty)} \) be a Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Prove that

\[
    \lim_{n \to \infty} \frac{W_n}{n} = 0 \quad \text{\( \mathbb{P} \)-a.s.}
\]

Question 3 (3 pts, **) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \( \mathcal{I} \subseteq \mathcal{F} \) be a \( \sigma \)-system satisfying \( \Omega \in \mathcal{I} \), let \( \mathcal{G} \subset \mathcal{F} \) be a \( \sigma \)-algebra generated by \( \mathcal{I} \), let \( X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \), and let \( Y \in L^1(\Omega, \mathcal{G}, \mathbb{P}) \) such that for all \( B \in \mathcal{I} \) it holds that \( \int_B Y \, d\mathbb{P} = \int_B X \, d\mathbb{P} \). Prove that \( Y = \mathbb{E}(X|\mathcal{G}) \).

Question 4 (6 pts, **) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space endowed with a filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\), and let \((M_n)_{n \in \mathbb{N}}\) be a \((\mathcal{F}_n)_{n \in \mathbb{N}}\)-adapted process of integrable random variables.

(a) (2 pts, *) Let \( n \in \mathbb{N}_0 \) and \( B \in \mathcal{F}_n \). Define \( \tau : \Omega \to \mathbb{N}_0 \) by \( \tau = n1_B + (n+1)1_{\Omega \setminus B} \). Prove that \( \tau \) is a stopping time.

(b) (4 pts, **) Prove that \((M_n)_{n \in \mathbb{N}_0}\) is a martingale if and only if for every stopping time \( \tau : \Omega \to \mathbb{N} \) taking at most two values it holds that \( \mathbb{E}M_{\tau} = \mathbb{E}M_0 \).

Question 5 (4 pts, ***) Let \( p \in [1, \infty) \), let \((S, \Sigma, \mu)\) be a measure space, and let \((f_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions in \( L^p((S, \Sigma, \mu); \mathbb{R}) \). Assume there exists an \( f \in L^p((S, \Sigma, \mu); \mathbb{R}) \) such that \( \lim_{n \to \infty} f_n = f \) \( \mu \)-almost surely. Prove that \( \lim_{n \to \infty} \|f - f_n\|_{L^p} = 0 \) if and only if \( \lim_{n \to \infty} \|f_n\|_{L^p} = \|f\|_{L^p} \).

Question 6 (4 pts, **) Let \((\mu_n)_{n \in \mathbb{N}}\) be a sequence of probability measures on \( \mathbb{R} \) that are absolutely continuous with respect to the Lebesgue measure, i.e., \( \mu_n \) admits a density \( f_n \in L^1(\mathbb{R}) \), and let \( \mu \) be a measure on \( \mathbb{R} \) that is absolutely continuous with respect to the Lebesgue measure, i.e., \( \mu \) admits a density \( f \in L^1(\mathbb{R}) \). Use Question 5 to prove that if \( f_n \to f \) a.s. then \( \mu_n \overset{\text{w}}{\to} \mu \).

Question 7 (6 pts, ***) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration such that \( \mathcal{F} = \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n) \). Let \( X \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and define, for \( n \in \mathbb{N} \), the random variable \( M_n = \mathbb{E}(X|\mathcal{F}_n) \). Use Questions 3 and 5 to prove\(^1\) that \( \lim_{n \to \infty} \|M_n - X\|_{L^2} = 0 \).

---

\(^1\)No points are given for merely observing that there is a theorem in the Lecture Notes providing this result.