**Exercise 7.5** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{C} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$ be uniformly integrable. Show that $\sup_{X \in \mathcal{C}} \|X\|_{L^1(\Omega, \mathcal{F}, \mathbb{P})} < \infty$. Now assume $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], B([0, 1]), \nu)$ where $\nu$ is the Lebesgue measure, and define, for $n \in \mathbb{N}$, $X_n = n1_{(0, 1/n)}$. Show that $\sup_{n \in \mathbb{N}} \|X_n\|_{L^1(\Omega, \mathcal{F}, \mathbb{P})} < \infty$ but that $\{X_n : n \in \mathbb{N}\}$ is not uniformly integrable.

**Exercise 10.2** Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0, 1)$, $\mathcal{F}$ the Borel sets of $[0, 1)$ and $\mathbb{P}$ the Lebesgue measure. For $n \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, 2^n - 1\}$ let $I^n_k = [k2^{-n}, (k + 1)2^{-n})$, let $\mathcal{F}_n = \sigma(\{I^n_k : k \in \{0, 1, \ldots, 2^n - 1\}\}$, and define $X_n = 1_{I^n_0}2^n$. Show that $X = (X_n)_{n=0}^\infty$ is a $(\mathcal{F}_n)_{n=0}^\infty$-martingale and that the conditions of Theorem 10.5 are satisfied for $X$. What is $X_\infty$ in this case? Do we have $X_n \to X_\infty$ in $L^1$?