1 Problem 13.8

B. Since $X$ and $Y$ are independent, by Prop 13.2 we have that

$$\phi_Z(u) = \phi_X(u)\phi_Y(u) = \phi_X(u)\phi_Y(\sigma u).$$

By easy computation (see Ex 13.4) we get that $\phi_Y(u) = e^{-u^2/2}$. Then,

$$\int_{\mathbb{R}} |\phi_Z(u)| du \leq \int_{\mathbb{R}} e^{-\sigma^2u^2/2} = \frac{\sqrt{2\pi}}{\sigma} < \infty.$$

Hence, $\phi_z \in L^1$. From the proof of Theorem 13.6 we know that $Z$ has density

$$p(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izu} \phi_Z(u) du.$$

Then by substituting $u = -y/\sigma$ we have

$$p(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izu} \phi_X(u)e^{-\sigma^2u^2/2} = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} \phi_X(-y/\sigma)e^{yz/\sigma-y^2/2} dy.$$

A. To prove A we need to show that

$$\frac{1}{\sqrt{2\pi\sigma}} \mathbb{E}e^{-(z-X)^2/2\sigma^2} = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} \phi_X(-y/\sigma)e^{yz/\sigma-y^2/2} dy.$$

Note that by Fubini theorem (since we can bound the absolute value of the functions in the integral by $e^{-y^2/2}$)

$$\frac{1}{2\pi\sigma} \int_{\mathbb{R}} \phi_X(-y/\sigma)e^{yz/\sigma-y^2/2} dy = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iyz/\sigma} \mathbb{P}^X(dx) e^{yz/\sigma-y^2/2} dy dx = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iyz/\sigma} e^{yz/\sigma-y^2/2} dy \mathbb{P}^X(dx) = \frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{iyz/\sigma} \mathbb{P}^X(dx) = \frac{1}{\sqrt{2\pi\sigma}} \mathbb{E}e^{-(z-X)^2/2\sigma^2}.$$

2 Problem 13.9

A. We know that for all $x \in \mathbb{R}$ it holds that $|\phi_n(x)| \leq 1$ for all $n$, as well as $|\phi(x)| \leq 1$. This allows us to make use of dominated convergence theorem. By Ex 13.8, for a fixed $z$:

$$p(z) = \frac{1}{2\pi\sigma} \int \phi(-y/\sigma) \exp \left( iyz/\sigma - y^2/2 \right) dy = \frac{1}{2\pi\sigma} \int \lim_{n \to \infty} \phi_n(-y/\sigma) \exp \left( iyz/\sigma - y^2/2 \right) dy = \lim_{n \to \infty} \frac{1}{2\pi\sigma} \int \phi_n(-y/\sigma) \exp \left( iyz/\sigma - y^2/2 \right) dy = \lim_{n \to \infty} p_n(z).$$

Hence, we have a pointless convergence of $p_n$ to $p$.

B. We have

$$|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| = \left| \int f(z)p_n(z) - f(z)p(z) \right| dz \leq \int |f(z)(p_n(z) - p(z))| dz \leq B \int |p(z) - p_n(z)| dz.$$
Since both $p$ and $p_n$ are densities we can see that

$$0 = \int_{\mathbb{R}} (p(z) - p_n(z))dz = \int_{\mathbb{R}} (p(z) - p_n(z))^+ dz - \int_{\mathbb{R}} (p(z) - p_n(z))^- dz.$$ 

Then, we have

$$\int_{\mathbb{R}} |p(z) - p_n(z)|dz = \int_{\mathbb{R}} (p(z) - p_n(z))^+ dz + \int_{\mathbb{R}} (p(z) - p_n(z))^− dz = 2\int_{\mathbb{R}} (p(z) - p_n(z))^+ dz,$$

which finishes our proof.

C. From B we know that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \leq 2B \int_{\mathbb{R}} (p(z) - p_n(z))^+ dz$. Part A gives us the pointless convergence $p_n \rightarrow p$. We can also notice that $(p(z) - p_n(z))^+ \leq p(z)$. Then, by application of the dominant convergence theorem we get

$$\int_{\mathbb{R}} (p(z) - p_n(z))^+ dz \rightarrow 0.$$ 

Since $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \geq 0$, we deduce that it must converge to zero.

D. Assume that $X_n \rightarrow X$ weakly. Then, by proposition 13.10 we have a pointwise convergence of the characteristic functions.

Conversely, if $\phi_n \rightarrow \phi$ pointwise, then by part C we have for all $f \in C_b(\mathbb{R})$

$$|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \rightarrow 0.$$ 

Hence $\mathbb{E}f(X_n + \sigma Y) \rightarrow \mathbb{E}f(X + \sigma Y)$ for all $f \in C_b(\mathbb{R})$.

That means that $X_n + \sigma Y \rightarrow X + \sigma Y$ weakly. Using Proposition 12.11 we conclude that $X_n \rightarrow X$ weakly.