Exercise 4.2:
Let \( f, g \in \Sigma^+ \) and \( \alpha, \beta > 0 \).

Prove that \( \mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g) \leq \infty \):

(4) Use example 4.13 to get increasing sequences of simple functions, which converge to \( f \) and \( g \): \( f_n \to f \) and \( g_n \to g \). Obviously, the increasing sequence \( h_n := \alpha f_n + \beta g_n \) converges to \( h := \alpha f + \beta g \). Hence the monotone convergence theorem shows:

\[
\mu(h) = \lim_{n \to \infty} \mu(h_n) = \lim_{n \to \infty} \mu(\alpha f_n + \beta g_n) = \alpha \lim_{n \to \infty} \mu(f_n) + \beta \lim_{n \to \infty} \mu(g_n) = \alpha \mu(f) + \beta \mu(g)
\]

Exercise 4.6:
\( \lambda \) the lebesgue measure on \([0,1] \), \( f : [0,1] \to \mathbb{R} \) continuous and \( I \) its Riemann integral.

By replacing \( f \) with \( f - \min(f) \) we can assume without loss of generality that \( f \) is in fact non-negative. Here we use that \( f \) is continuous and hence admits a minimum and we use linearity of the integrals (proven in the previous exercise).

\( f \) is integrable:
(2) Any continuous function is measurable (3.3(iii)). Now note that \( \lambda(f) \leq \lambda(M_{1[0,1]}) = M < \infty \) with \( M := \max(f) \). Hence \( f \) is integrable.

Construct an increasing sequence of simple functions (1) \( h_n \to f \) (1) such that \( \lambda(h_n) \to I(1) \):
(3) For each \( n \in \mathbb{N} \) and \( 1 < i \leq 2^n \) let \( E_{i,n} := (\frac{i-1}{2^n}, \frac{i}{2^n}] \) and let \( E_{1,n} := [0, \frac{1}{2^n}] \). Define \( h_n \) as:

\[
h_n := \sum_{i=1}^{2^n} \min_{x \in E_{i,n}} (f(x)) \cdot 1_{E_{i,n}}
\]

It is obvious that \( h_n \) is increasing and has limit which equals \( f \). Note that \( \lambda(h_n) = \sum_{i=1}^{2^n} \min_{x \in E_{i,n}} (f(x)) \cdot 2^{-n} \), which is in its limit \( n \to \infty \) equal to the lower Riemann/Darboux sum of \( f \). Integrability of \( f \) shows that \( \lambda(h_n) \to I \).

Conclude that \( \lambda(f) = I \):
(1) By monotone convergence we also have that \( \lambda(h_n) \to \lambda(f) \) and hence \( \lambda(f) = I \).

Exercise 4.7:
\( f(x) := \frac{\sin(x)}{x} \) on \([0,\infty)\) with \( f(0) = 1 \).

\( I := \int_0^\infty f(x) \, dx \) exists as an improper Riemann integral:

For \( j \in \mathbb{N}_{>0} \), define \( a_j := \int_{j\pi}^{(j+1)\pi} |f(x)| \, dx \) and let \( a_0 := \int_0^\pi f(x) \, dx \). Note that \( 0 \leq a_j < \infty \) as \( |f| \) is continuous and \([j\pi, (j+1)\pi]\) compact. Then:

\[
\int_0^{(n+1)\pi} f(x) \, dx = a_0 + \sum_{j=1}^{n} (-1)^n a_n
\]
Therefore $I$ exists if and only if the sum $\sum_j (-1)^j a_j$ converges. Using the Leibniz test for alternating series, this is the case if $a_j \to 0$ monotonically, which we will prove now. Firstly:

$$0 \leq a_j = \int_{j\pi}^{(j+1)\pi} \frac{\vert \sin(x) \vert}{x} \, dx \leq \int_{j\pi}^{(j+1)\pi} \frac{1}{x} = \log \left( \frac{j+1}{j} \right) \to \log(1) = 0$$

Hence $a_j \to 0$. Secondly, for $j \geq 2$ note that $\vert \sin(x) \vert = \vert \sin(x+\pi) \vert$. Therefore:

$$a_j = \int_{j\pi}^{(j+1)\pi} \frac{\vert \sin(x) \vert}{x} \, dx = \int_{(j-1)\pi}^{j\pi} \frac{\vert \sin(x+\pi) \vert}{x+\pi} \, dx = \int_{(j-1)\pi}^{j\pi} \frac{\vert \sin(x) \vert}{x+\pi} \, dx$$

$$\leq \int_{(j-1)\pi}^{j\pi} \frac{\vert \sin(x) \vert}{x} \, dx = a_{j-1}$$

Therefore $a_j \to 0$ monotonically and hence $I$ exists as an improper Riemann integral.

$f$ is not Lebesgue measurable:

Note that $f^+$ is equal to:

$$f^+ = \sum_{n \geq 0} f \cdot 1_{[2n\pi,(2n+1)\pi]}$$

Moreover, $f \cdot 1_{[2n\pi,(2n+1)\pi]}$ gives:

$$\lambda(f \cdot 1_{[2n\pi,(2n+1)\pi]}) = \int_{2n\pi}^{(2n+1)\pi} \frac{\sin(x)}{x} \, dx \geq \frac{1}{(2n+1)\pi} \int_{2n\pi}^{(2n+1)\pi} \sin(x) \, dx = \frac{2}{(2n+1)\pi}$$

Therefore: $\lambda(f^+) \geq \frac{2}{\pi} \sum_n \frac{1}{2n+1} = \infty$. We conclude that $f$ is not (Lebesgue) integrable.