**Exercise 5.2:**

$(S_i, \Sigma_i, \mu_i)$ measure spaces, $(S, \Sigma, \mu)$ their product, $f \in \mathcal{L}^1(S, \Sigma, \mu)$.

There exists a $\mu_2$–null-set $N$ such that $I_f^\pm$ is well-defined on $N^c$:

First note that $(I_f^\pm)^\pm = I_f^\pm$. As $f$ is integrable, we know that $\mu(f^\pm) < \infty$. By 5.5(ii), this implies that $\mu_2(I_f^\pm) < \infty$. That is, $I_f^\pm$ are integrable and hence they can only be infinite on a $\mu_2$–null set $N_\pm$. Letting $N := N_+ \cup N_-$ shows that $I_f^\pm = I_2^{\pm} = I_f^{\pm}$ is real-valued on $N^c$ and obviously $N$ is a null-set as well.

The map $g : s_1 \mapsto f(s_1, s_2)$ belongs to $\mathcal{L}^1(S_1, \Sigma_1, \mu_1)$ for all $s_2 \in N^c$:

Let $s_2 \in N^c$. By considering $g^\pm(s_1) = f^\pm(s_1, s_2)$ again, we realize that $\mu_1(g^\pm) = I_2^{\pm}(s_2)$ and we have seen that this is indeed finite. Therefore, $g$ is integrable.

Finally: $\mu(f) = \mu_2(I_f^\pm)$:

We use theorem 5.5(ii):

$$\mu(f) = \mu(f^+) - \mu(f^-) = \mu_2(I_2^+) - \mu_2(I_2^-) = \mu_2(I_f^\pm).$$

**Remark:** Theorem 5.5(iii) now follows from permuting 1 and 2.
Exercise 5.4:

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space, $X_1, X_2$ random variables with (marginal) probability density functions $f_1$.

(6) The following statements are equivalent:

1. $X_1$ and $X_2$ are independent;
2. $\mathbb{P}(X_1, X_2) = \mathbb{P}^{X_1} \times \mathbb{P}^{X_2}$;
3. $F(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$ for all $x_i \in \mathbb{R}$.

- 1 iff 3: By corollary 3.13 we know that $X_1, X_2$ are independent if and only if $\mathbb{P}(X_1 \leq x_1 \cap X_2 \leq x_2) = \mathbb{P}(X_1 \leq x_1) \cdot \mathbb{P}(X_2 \leq x_2)$ for all $x_1, x_2$. By definition of $F$, this equation reads $F(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$. Hence the claim follows.

- 1,3 iff 2: First assume that 2 holds. Then:

$$F(x_1, x_2) = \mathbb{P}(X_1, X_2) = \mathbb{P}(X_1 \leq x_1 \cap X_2 \leq x_2)$$

$$= \mathbb{P}(X_1 \leq x_1) \cdot \mathbb{P}(X_2 \leq x_2)$$

$$= \mathbb{P}^{X_1}(\mathbb{R}) \cdot \mathbb{P}^{X_2}(\mathbb{R})$$

$$= \mathbb{P}^{X_1} \times \mathbb{P}^{X_2}((-\infty, x_1] \times (-\infty, x_2])$$

Now assume that 3 holds. As the rectangles $(-\infty, x_1] \times (-\infty, x_2]$ are a $\pi-$system that generate the sigma-algebra, theorem 1.16 shows that it is enough to prove the claim only for these sets. Essentially it is the same computation as we have already made:

$$\mathbb{P}^{X_1}(\mathbb{R}) \cdot \mathbb{P}^{X_2}(\mathbb{R}) = \mathbb{P}^{X_1}(\mathbb{R}) \cdot \mathbb{P}^{X_2}(\mathbb{R})$$

(4) If $(X_1, X_2)$ has joint probability density $f$, then $X_1$ and $X_2$ are independent if and only if $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ for almost all $(x_1, x_2)$:

Let $E = E_1 \times E_2 := (-\infty, x_1] \times (-\infty, x_2]$. Using the first part, we see that $X_1, X_2$ are independent if and only if:

$$\int_E f \; d\lambda \times \lambda = \mathbb{P}^{X_1, X_2}(E) = F(x_1, x_2) = F_{X_1}(x_1) \cdot F_{X_2}(x_2)$$

$$= \int_{E_1} f_1 \; d\lambda \cdot \int_{E_2} f_2 \; d\lambda = \int_{E_1} \int_{E_2} f_1 \cdot f_2 \; d\lambda \cdot d\lambda \text{ Fubini } \int_E f_1 \cdot f_2 \; d\lambda \times \lambda \quad (1)$$

Now note that if $f_1 \cdot f_2$ almost everywhere, equation (1) certainly holds. Hence this implies that $X_1, X_2$ are independent. The other way around, if $X_1, X_2$ are independent, the story above shows that (1) holds, for all $E = E_1 \times E_2$. By considering $|f - f_1 \cdot f_2|$, lemma 4.11 shows that $1_E f = 1_E (f_1 \cdot f_2)$ almost everywhere. As the sets $E$ exhaust $\mathbb{R}^2$ we conclude that $f = f_1 \cdot f_2$ almost everywhere.