Exercise 6.4

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $\theta > 0$ and $X, Y_\theta$ independent random variables such that $\mathbb{P}(X = 0) = \frac{1}{2} = \mathbb{P}(X = 1)$ and $\mathbb{P}(Y_\theta \in A) = \frac{1}{\theta} \lambda(A \cap [0, \theta])$, with $\lambda$ the lebesgue measure on $\mathcal{B}(\mathbb{R})$.

(2) Show that $\mathcal{L}_\theta$, the law of $XY_\theta$, is not absolutely continuous with respect to $\lambda$:

Let us compute $\mathcal{L}_\theta(E)$ for $E \in \mathcal{B}(\mathbb{R})$:

$$
\mathcal{L}_\theta(E) = \mathbb{P}(XY_\theta \in E) = 1_E(0) \cdot \mathbb{P}(XY_\theta = 0) + \mathbb{P}(XY_\theta \in E \setminus \{0\}) \\
= 1_E(0) \cdot \left( \mathbb{P}(X = 0) \cdot \mathbb{P}(Y_\theta = 0) + \mathbb{P}(X = 1) \cdot \mathbb{P}(Y_\theta \in (0, \theta]) \right) \\
= \frac{1}{2} \mathbb{P}(0) + \frac{1}{2\theta} \lambda(E \cap [0, \theta])
$$

Using this it is easy to find a set $E \in \mathcal{B}(\mathbb{R})$ such that $\lambda(E) = 0$ but $\mathcal{L}_\theta(E) \neq 0$, which proves the claim. Indeed, we can take $E = \{0\}$ to find that $\mathcal{L}_\theta(E) = \frac{1}{2} \neq 0$.

(2) Find a $\sigma$–finite measure $\mu$ such that $\mathcal{L}_\theta$ is absolutely continuous with respect to $\mu$ for all $\theta$:

In general it holds that the sum of two $\sigma$–finite measures on any $(S, \Sigma)$ is again a $\sigma$–finite measure on $(S, \Sigma)$. Indeed, let $\mu_i$ be $\sigma$–finite measures and let $(E_n)_n$ be disjoint sets:

$$(\mu_1 + \mu_2)(\emptyset) = \mu_1(\emptyset) + \mu_2(\emptyset) = 0 + 0 = 0$$

$$(\mu_1 + \mu_2)(\cup E_n) = \mu_1(\cup E_n) + \mu_2(\cup E_n) = \sum \mu_1(E_n) + \sum \mu_2(E_n) = \sum (\mu_1 + \mu_2)(E_n)$$

Finally, for $\sigma$–finiteness, let $S^i_n \in \Sigma$ such that $\cup_n S^i_n = S$ and $\mu_i(S^i_n) < \infty$. Then $S_{n,m} := S^i_n \cap S^j_m$ is still a countable set, $\cup S_{n,m} = S$ and $(\mu_1 + \mu_2)(S_{n,m}) \leq \mu_1(S^i_n) + \mu_2(S^j_m) < \infty$.

Using this, let $\mu := \delta_0 + \lambda$ with $\delta_0$ the Dirac measure on 0. Now suppose that $\mu(E) = 0$. This implies that both $0 \notin E$ and $\lambda(E) = 0$. Hence, using (1), we see that $\mathcal{L}_\theta(E) = 0$ and we conclude the claim.

(2) Determine the Radon-Nikodym derivatives of $\mathcal{L}_\theta$ with respect to $\mu$:

As $\mu(f) = f(0) + \lambda(f)$, we see that letting $h_\theta$ be defined as:

$$h_\theta(x) := \frac{1}{2} \left( 1_{\{0\}}(x) + \frac{1}{\theta} 1_{[0,\theta]} \right)$$

gives:

$$\mu(1_E h_\theta) = (1_E h_\theta)(0) + \lambda(1_E \theta) = \frac{1}{2} 1_E(\theta) + \frac{1}{2\theta} \lambda(E \cap [0, \theta]) = \mathcal{L}_\theta(E)$$

Note that $h_\theta$ is almost everywhere continuous and hence measurable.
Exercise 6.5

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $X_1, X_2, \ldots$ an iid (independent and identically distributed) sequence of Bernoulli random variables with $\mathbb{P}(X_i = 1) = \frac{1}{2} = \mathbb{P}(X_i = 0)$ and $X := \sum 2^{-k}X_k$, $Y := 3\sum 4^{-k}X_{2k}$.

Find the distribution of $X$:

Note that $X$ has probability 1 to take value inside $[0, 1]$ as $\sum 2^{-k} = 1$. As $X^{-1}(1)$ has measure zero, we can just regard it as a map to $[0, 1)$. Letting $U = X$ in Lemma 3.15, we conclude that $X$ is uniformly distributed.

Show that $F_Y$ is constant on $(\frac{1}{4}, \frac{3}{4})$, that $F_Y(1 - x) = 1 - F_Y(x)$ and that it satisfies $F_Y(x) = 2F_Y(\frac{x}{4})$ for $x < \frac{1}{4}$:

Note that $3\sum 4^{-k} = 1$ as well. Hence if $X_2(\omega) = 0$, we get that $Y(\omega) < 1 - 3 \cdot \frac{1}{4} = \frac{1}{4}$ and if $X_2(\omega) = 1$, we get that $Y(\omega) \geq 3 \cdot \frac{1}{4} = \frac{3}{4}$. Therefore, $\mathbb{P}(Y \in (\frac{1}{4}, \frac{3}{4})) = 0$ and $F_Y$ has to be constant on this interval.

Let $Z := 3\sum 4^{-k}(1 - X_{2k}) = 1 - Y$. Then $Z$ has the same distribution as $Y$, since $1 - X_{2k}$ has the same distribution as $X_{2k}$. Therefore:

$$F_Y(1 - x) = \mathbb{P}(Y \leq 1 - x) = \mathbb{P}(x \leq Z) = 1 - \mathbb{P}(Z \leq x) = 1 - F_Y(x)$$

Finally, for $x < \frac{1}{4}$ we also have:

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}\left(\{X_2 = 0\} \cap \left\{3 \sum_{k=2}^{\infty} 4^{-k}X_{2k} \leq x\right\}\right)$$

$$= \mathbb{P}(X_2 = 0) \cdot \mathbb{P}\left(3 \sum_{k=2}^{\infty} 4^{-k}X_{2k} \leq x\right)$$

$$= \frac{1}{2} \cdot \mathbb{P}\left(3 \sum_{k=2}^{\infty} 4^{1-k}X_{2k} \leq 4x\right)$$

$$= \frac{1}{2} \cdot \mathbb{P}\left(3 \sum_{k=2}^{\infty} 4^{1-k}X_{2k-2} \leq 4x\right)$$

$$= \frac{1}{2} \cdot \mathbb{P}(Y \leq 4x) = \frac{1}{2} F_Y(4x)$$

Here we use that the $X_i$ form a iid sequence and hence we can change $X_{2k}$ for $X_{2k-2}$.

Show that $F$ is continuous:

By proposition 3.8, we know that $F$ is right-continuous. Hence the claim follows if we can show that $F$ is also left-continuous. Using the right continuity and the previous part, we compute:

$$\lim_{x \uparrow c} F(x) = 1 - \lim_{x \uparrow c} F(1 - x) = 1 - \lim_{x \downarrow 1 - c} F(x)$$

$$= 1 - F(1 - c) = F(c).$$

Hence $F$ is continuous.
Show that $F$ is not absolutely continuous with respect to $\lambda$:
Let us define a set $C$, analogous to the Cantor set: iteratively, we remove the inner half. For example, the first step would give us $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. Note that $F$ is constant on these inner halves, just like it is constant on $(\frac{1}{4}, \frac{3}{4})$. Hence $P^Y(C) = 1$. Just like the Cantor set has $\lambda$–measure zero, so does $C$. Therefore, $F$ is not absolutely continuous with respect to $\lambda$. 
Exercise 6.9
\( \nu \ll \mu \) positive \( \sigma \)-finite measures on \((S, \Sigma)\), with Radon-Nikodym derivative \( h \).

(1) Show that \( \nu(h = 0) = 0 \):
Let \( Z \) be the zero function on \( S \). We compute:
\[
\nu(h = 0) = \mu(1_{h=0} \cdot h) = \mu(Z) = 0.
\]

Show that \( \mu(h = 0) \) if and only if \( \mu \ll \nu \) (2) and determine a Radon-Nikodym derivative of \( \mu \) with respect to \( \nu \) in this case (1):

- Suppose that \( \mu \ll \nu \). As the previous part shows that \( \nu(h = 0) = 0 \), we conclude directly that \( \mu(h = 0) = 0 \).
- Suppose \( \mu(h = 0) = 0 \). Define \( g = 1_{h \neq 0} \cdot \frac{1}{h} \) and note that \( g \cdot h = 1_{h \neq 0} \). We have seen that \( \nu(f) = \mu(hf) \) in Exercise 4.10 and hence we compute:
\[
\mu(E) = \mu(E \cap \{ h \neq 0 \}) = \mu(1_E \cdot g \cdot h) = \nu(1_E g)
\]
Therefore, we conclude that \( \mu \ll \nu \). Moreover, the proof gives us a derivative \( g \).