Exercise 8.2:

$X, Y, Z$ random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Z$ is $\sigma(Y)$-measurable and $X$ integrable.

Show that there exists a measurable $h : \mathbb{R} \to \mathbb{R}$ such that $Z = h \circ Y$:

We apply the standard machinery:

- (1) Suppose that $Z = 1_A$ for $A \in \mathcal{F}$. As $Z$ is $\sigma(Y)$-measurable, we need $A = Y^{-1}(B) \in \sigma(Y)$ for a measurable $B$. Letting $h := 1_B$ shows that $h \circ Y = Z$ and since $B$ is measurable, so is $h$.

- (1) Suppose that $Z = \sum_{i=1}^n a_i 1_{A_i}$ is simple. Then, there exist $h_i$ such that $h_i \circ Y = 1_{A_i}$. Hence if we let $h := \sum_{i=1}^n a_i h_i$, we get:

$$Z = \sum_{i=1}^n a_i 1_{A_i} = \sum_{i=1}^n a_i \cdot h_i \circ Y = h \circ Y$$

As all the $h_i$ are measurable, so is $h$.

- (2) Suppose that $Z$ is positive and measurable. Like usual let $Z_n$ be the sequence of simple functions such that $\lim_n Z_n = Z$, constructed in example 4.13. It follows from this construction that all the $Z_n$ are $\sigma(Y)$ measurable as well, and hence there exist $h_n$ measurable such that $Z_n = h_n \circ Y$. Let $h$ be defined as $h := \limsup h_n$ which is measurable by 3.5(ii). Clearly, we get for all $\omega \in \Omega$:

$$(h \circ Y)(\omega) = \limsup_n h_n(Y(\omega)) = \limsup_n Z_n(\omega) = Z(\omega).$$

Note that we can still have that $\limsup_n h_n(y) = \pm \infty$ for some $y \in \mathbb{R}$. The above computation shows that this is not the case on the image of $Y$ and hence we can alter $h$ by a indicator function to get a well-defined measurable function $h' := h \cdot 1_{\{h = \pm \infty\}^c}$ which satisfies $Z = h' \circ Y$.

- Finally, let $Z = Z^+ - Z^-$ be any measurable function. By the previous part, we find $h^+$ and $h^-$ and $h := h^+ - h^-$ satisfies $Z = h \circ Y$.

(1) Conclude that $\mathbb{E}[X|Y] = h \circ Y$ for a measurable function $h$:

By definition, $\mathbb{E}[X|Y]$ is a $\sigma(Y)$-measurable function and hence the previous part shows that there exists a $h$ such that $\mathbb{E}[X|Y] = h \circ Y$. 

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Exercise 8.3:
\((\Omega,\mathcal{F},\mathbb{P})=([-1,1],\mathcal{B}[-1,1],\frac{1}{2}\lambda)\), \(X: \Omega \rightarrow \mathbb{R}, \omega \mapsto \omega^2\), \(-E := \{-\varepsilon | \varepsilon \in E\}\), \(Y \in L^1(\Omega,\mathcal{G},\mathbb{P})\) and \(2\hat{Y}(x) := Y(x) + Y(-x)\)

(2) Show that \(\sigma(X) = \{E \in \mathcal{B}[-1,1] \mid E = -E\} =: A\):

- Let \(F \in \mathcal{B}(\mathbb{R})\) and let \(E = X^{-1}(F)\). Then \(\omega \in E\) if and only if \(\omega^2 \in F\). However \((-\omega)^2 = \omega^2\) and hence this happens if and only if \(-\omega \in F\). Therefore \(E = -E\), which gives \(\sigma(X) \subset A\).

- Let \(E \in A\) be given. Note that \(F := X(E)\) satisfies \(F \subset [0,\infty)\). On \([0,\infty)\) the mapping \(X\) has a continuous inverse \(\xi: y \mapsto \sqrt{y}\) and \(F = \xi^{-1}(E \cap [0,1])\) is therefore measurable. Now note that \(E = X^{-1}(F)\) and hence belongs to \(\sigma(X)\). We conclude that \(A \subset \sigma(X)\).

(1)+(2) Show that \(\hat{Y}\) is a representative of \(\mathbb{E}[Y|X]\):

Note that \(\hat{Y} \in L^1(\Omega,\sigma(X),\mathbb{P})\). Indeed, since \(\hat{Y}(x) = \hat{Y}(-x)\) it easily follows that \(\hat{Y}^{-1}(F) = -\hat{Y}^{-1}(F)\) for any \(F \in \mathcal{B}(\mathbb{R})\). Hence we are left to show that for all \(E \in \sigma(X)\):

\[
\int_E \hat{Y} \, d\mathbb{P} = \int_E Y \, d\mathbb{P}
\]

Recall that any such \(E\) satisfies \(E = -E\). Let \(\xi: [-1,1] \rightarrow [-1,1] \) be given by \(\xi(x) = -x\). Then we compute:

\[
2 \int_E \hat{Y} \, d\mathbb{P} = \int_E Y \, d\mathbb{P} + \int_E Y \circ \xi \, d\mathbb{P}
\]

\[
= \int_E Y \, d\mathbb{P} + \int_{-E} Y \, d\mathbb{P}
\]

\[
= 2 \int_E Y \, d\mathbb{P}
\]

Where we use exercise 4.11, together with \(\mathbb{P}^\xi(B) = \mathbb{P}(-B)\) for any \(B\). We conclude the claim.
Exercise 8.9:

$X, Y$ random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $f$ the density of $(X, Y)$ with respect to $\lambda^2$, $f_Y$ the marginal density of $Y$ and $\hat{f}(x|y) := \mathbf{1}_{\{f_Y > 0\}}(y) \cdot \frac{f(x,y)}{f_Y(y)}$.

For $h$ such that $\mathbb{E}|h \circ X| < \infty$ show that $\hat{h} \circ Y$ is a version of $\mathbb{E}[h \circ X|Y]$ with $\hat{h}(y) := \int_{\mathbb{R}} h(x) \hat{f}(x|y) \, dx$.

As $\hat{h}$ is measurable, it is enough to show that:

$$\int_E \hat{h} \circ Y \, d\mathbb{P} = \int_E h \circ X \, d\mathbb{P}$$

for all $E = Y^{-1}(B) \in \sigma(Y)$. Let $A := \{f_Y > 0\}$. We compute:

$$\int_E \hat{h} \circ Y \, d\mathbb{P} = \int_B \hat{h} \cdot f_Y \, d\lambda = \int_B \int_{\mathbb{R}} h(x) \hat{f}(x|y) \cdot f_Y(y) \, dx \, dy$$

$$= \int_{A \cap B} \int_{\mathbb{R}} h(x) f(x, y) \, dx \, dy$$

$$= \int_{\mathbb{R}^2} h(x) \cdot 1_{A \cap B}(y) \cdot f(x, y) \, d\lambda^2 = \int_{\mathbb{R}^2} h(x) \cdot 1_B(y) \cdot f(x, y) \, d\lambda^2$$

$$= \int_\Omega 1_B \circ Y \cdot h \circ X \, d\mathbb{P} = \int_{Y^{-1}(B)} h \circ X \, d\mathbb{P} = \int_E h \circ X \, d\mathbb{P}$$

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