Exercise 9.10: Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\sigma^2_n = \mathbb{E}[X^2_n] < \infty$ and $\mathbb{E}[X_n] = 0$ for all $n$. Let $(F_n)_{n \in \mathbb{N}}$ be the filtration generated by $X$ and define the martingale $M$ by $M_n = \sum_{k=1}^n X_k$. Determine $\langle M \rangle$

Proof: We want to prove that $M$ is square integrable to apply corollary 9.17, then we write

$$M^2_n = \sum_{1 \leq i < j \leq n} 2X_i X_j + \sum_{1 \leq i \leq n} X^2_i$$

and then the same for expectation

$$\mathbb{E}[M^2_n] = \mathbb{E}\left[ \sum_{1 \leq i < j \leq n} 2X_i X_j \right] + \mathbb{E}\left[ \sum_{1 \leq i \leq n} X^2_i \right]$$

since the $\mathbb{E}[X_n] = 0$ for each $n$ and $X_n$ is an independent sequence the first summand is equal to zero, while the second is equal to $\sum_{k=1}^n \sigma^2_k < \infty$. Then by corollary 9.17 we have that $\Delta \langle M \rangle_0 = 0$ and for $n \geq 1$ we have that $\Delta \langle M \rangle_n = \mathbb{E}[(M_n - M_{n-1})^2 \mid F_{n-1}]$, by definition we can write $\langle M \rangle_n = \sum_{k=0}^n \Delta \langle M \rangle_k$ we obtain

$$\langle M \rangle_n = \sum_{k=1}^n \mathbb{E}[(M_n - M_{n-1})^2 \mid F_{n-1}] = \sum_{k=1}^n \mathbb{E}[X^2_i \mid F_{n-1}],$$

since $X_n$ is independent from $F_{n-1}$ the last term is equal to $\sum_{k=1}^n \sigma^2_k$.

\qed

Exercise 9.14: Let $M$ and $N$ be two square integrable martingales. Show that there exists a unique predictable process $\langle M, N \rangle$ with $\langle M, N \rangle_0 = 0$ such that $MN - \langle M, N \rangle$ is a martingale. Show also that for $n \in \mathbb{N}$

$$\Delta \langle M, N \rangle_n = \mathbb{E}[\Delta M_n \Delta N_n \mid F_{n-1}].$$

Proof: Since both $M$ and $N$ are adapted $MN$ is adapted as well, moreover by Holder inequality, theorem 4.44, we have that $M_n N_n$ is integrable, that is $\mathbb{E}[M_n N_n] < \infty$, hence we can apply proposition 9.16 to $MN$ and we get that there exist a martingale $X$ and a predictable process, we call it $\langle M, N \rangle$, such that $MN - \langle M, N \rangle = X$, since $\langle M, N \rangle$ is predictable $\langle M, N \rangle_0 = 0$. Now we apply corollary write

$$\mathbb{E}[\Delta M_n \Delta N_n \mid F_{n-1}] = \mathbb{E}[(M_n - M_{n-1})(N_n - N_{n-1}) \mid F_{n-1}] =$$

$$= \mathbb{E}[M_n N_n \mid F_{n-1}] - 2M_{n-1} N_{n-1} + M_{n-1} N_{n-1} = \mathbb{E}[M_n N_n \mid F_{n-1}] - M_{n-1} N_{n-1}$$

$$= \mathbb{E}[(M, N)_n + X_n \mid F_{n-1}] - M_{n-1} N_{n-1} = \langle M, N \rangle_n + X_n - M_{n-1} N_{n-1}$$
\[ M_n N_n - X_n + X_{n-1} - M_{n-1} N_{n-1} = M_n N_n - M_{n-1} N_{n-1} - (X_n - X_{n-1}) \]

and same holds for \( \Delta \langle M, N \rangle_n \) since

\[ \Delta \langle M, N \rangle_n = \Delta (MN - X)_n = \Delta (MN)_n - \Delta X_n = M_n N_n - M_{n-1} N_{n-1} - (X_n - X_{n-1}) \]

where through these equalities we used that \( X \) is a martingale, that \( \langle M, N \rangle \) is predictable and that \( MN = \langle M, N \rangle + X \).

\[ \square \]

**Exercise 10.5:** Let \( Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) and a filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be given. Define for all \( n \in \mathbb{N} \) the random variable \( X_n = E[Y \mid \mathcal{F}_n] \). We know that there is \( X_\infty \) such that \( X_n \to X_\infty \) a.s. Show that for \( Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \), we have \( X_n \to X_\infty \). Find a condition such that \( X_\infty = Y \). Give also an example in which \( \mathbb{P}(X_\infty = Y) = 0 \).

**Proof:** We use Jensen’s inequality \( E[Y \mid \mathcal{F}_n]^2 \leq E[Y^2 \mid \mathcal{F}_n] \) to prove that \( X_n \) is in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and that it is bounded in \( L^2 \),

\[ E[X_n^2] = E[E[Y \mid \mathcal{F}_n]^2] \leq E[E[Y^2 \mid \mathcal{F}_n]] = E[Y^2] < \infty. \]

Moreover \( X_n \) is a martingale since \( \mathcal{F}_{n-1} \subset \mathcal{F}_n \) implies

\[ E[E[Y \mid \mathcal{F}_n] \mid \mathcal{F}_{n-1}] = E[Y \mid \mathcal{F}_{n-1}], \]

hence we can apply theorem 10.15 which says that \( X_n \overset{\mathcal{L}}{\to} X_\infty \). A condition for \( X_\infty = Y \) to be \( \mathcal{F}_\infty \) measurable, since theorem 10.10 guarantees that under condition \( Y \in L^1 \) it holds \( X_\infty = E[Y \mid \mathcal{F}_\infty] \) one representant of the equivalence class, but since we can replace \( X_\infty \) with \( X'_\infty \) such that \( \mathbb{P}(X_\infty = X'_\infty) = 1 \) without loosing convergence we can assume \( Y = X_\infty \). Let us consider \( Y = 1 \) a constant random variable over \((\Omega, \mathcal{F}, \mathbb{P})\). We take \( B = (1, b_1, b_2, \ldots) \) an orthogonal base for \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and we set \( \mathcal{F}_n = \sigma(b_1, \ldots, b_n) \), hence, since \( E[Y \mid \mathcal{F}_n] \) is the projection over \( \langle b_1, \ldots, b_n \rangle \) and this is orthogonal to \( \langle Y \rangle \) for each \( n \) it holds \( E[Y \mid \mathcal{F}_n] = 0 \) and hence \( X_\infty = 0 \) in \( L^2 \), while \( Y = 1 \) hence \( \mathbb{P}(Y = X_\infty) = 0 \).

\[ \square \]