Algebraic modal logic
Summer 2013

Homework 2
(due Friday, 21 June at the beginning of the lecture)

1. Show that the set of filters of Boolean algebra \( \mathcal{B} \) form a distributive lattice (under \( \subseteq \)) where, for filters \( F_1, F_2 \)
   (a) \( F_1 \land F_2 = F_1 \cap F_2 \),
   (b) \( F_1 \lor F_2 = \{ a \in B : a \geq a_1 \land a_2 \text{ for some } a_1 \in F_1, a_2 \in F_2 \} \). [10 pts]

2. If \( U \) is an ultrafilter of a Boolean algebra \( \mathcal{B} \), show that \( \bigwedge U \) exists, and is an atom \( b \) or equals 0. In the former case show \( U = \{ b \} \uparrow \) (principal ultrafilter generated by \( b \)). [8 pts]

3. If \( \mathcal{B} \) is the Boolean algebra of finite and cofinite subsets of an infinite set \( I \), show that there is exactly one non-principal ultrafilter of \( \mathcal{B} \). [8 pts]

4. Show that a finite topological space \((X, \tau)\) is a Stone space iff it is discrete (i.e., every subset of \( X \) is open, or \( \tau = \mathcal{P}(X) \)). [8 pts]

5. MacNeille completions and Canonical extensions

   **Definition 0.1 (MacNeille completion).** A MacNeille completion of a lattice \( L \) is any complete lattice \( C \) containing \( L \) as a sublattice, with \( L \) both join-dense and meet-dense in \( C \) (that is, each element of \( C \) is both a join of elements of \( L \) and a meet of elements of \( L \)), and such that if \( C' \) is another such lattice then there is a unique isomorphism from \( C \) to \( C' \) fixing \( L \).

   **Definition 0.2 (Canonical extension).** The canonical extension of a lattice \( A \) is a complete lattice \( A^\delta \) containing \( A \) as a sublattice, such that
   (a) Every element of \( A^\delta \) can be expressed both as a join of meets and as a meet of joins of elements from \( A \) (
       denseness);
   (b) For all \( S, T \subseteq A \) with \( \bigwedge S \leq \bigvee T \) in \( A^\delta \), there exist finite sets \( F \subseteq S \) and \( G \subseteq T \) such that \( \bigwedge F \leq \bigvee G \) (compactness).

   Consider the infinite chains \( L = \mathbb{N} \oplus \mathbb{N}^\partial, \bar{L} \text{ and } L^\delta \), in Figure 1 on the next page. Show that
   (a) \( \bar{L} \) and \( L^\delta \) are MacNeille completions of \( L \) \([4 pts]\)
   (b) \( \bar{L} \) is not a canonical extension of \( L \). \([4 pts]\)
   (c) \( L^\delta \) is a canonical extension of \( L \) (define a dense and compact embedding \( \eta : L \hookrightarrow L^\delta \)) \([4pts]\)

6. (Exercise 5.2.3 B, deR, V) Let \( A \) be a collection of finite and co-finite subsets of \( \mathbb{N} \). Define \( f : A \to A \) by
   \[
   f(X) = \begin{cases} 
   \{ y \in \mathbb{N} \mid y + 1 \in X \} & \text{if } X \text{ is finite} \\
   \mathbb{N} & \text{if } X \text{ is co-finite}
   \end{cases}
   \]

   Prove that \((A, \cup, -, \emptyset, f)\) is a boolean algebra with operators. [10 pts]
7. Let $\mathbb{N}_\infty$ be the set of natural numbers with an additional point $\infty$. Define $\mathcal{T} \subseteq \mathcal{P}(\mathbb{N}_\infty)$ as follows: a subset $U$ of $\mathbb{N}_\infty$ belongs to $\mathcal{T}$ if, either (1) $\infty \notin U$, or (2) $\infty \in U$ and $\mathbb{N}_\infty \setminus U$ is finite.

(a) Show that $(\mathbb{N}_\infty, \mathcal{T})$ is a topological space. [5 pts]

(b) Show that $(\mathbb{N}_\infty, \mathcal{T})$ is a Stone space, that is, compact and totally disconnected. (Hint: the clopen subsets of $(\mathbb{N}_\infty, \mathcal{T})$ are finite sets not containing $\infty$, and their complements.) [5 pts]

8. (Bonus exercise) **Algebraic completeness of modal mu-calculus** [16 pts]

For a lattice $L$ and a map $f : L \to L$, an element $x \in L$ is a fixed point of $f$ if, $f(x) = x$. The least fixed point is the least element in the set of fixed points of $f$. The following theorem gives a method to compute the least fixed point of a monotone map on a complete lattice.

**Theorem 0.3 (Knaster-Tarski Theorem).** Let $(L, \leq)$ be a complete lattice and $f : L \to L$ be a monotone map, that is, for each $a, b \in L$, we have $f(a) \leq f(b)$. The Knaster-Tarski theorem states that $f$ has a least fixed point $\text{LFP}(f)$, which can be computed as

$$\text{LFP}(f) = \bigwedge \{a \in L : f(a) \leq a\}$$

Modal mu-calculus is an extension of basic modal logic with a least fixed point operator, which interprets the least fixed point of a formula, seen as a map on a modal algebra. The syntax of the logic is given as

$$\varphi ::= \bot \mid p \mid \neg \varphi \mid \varphi \land \varphi \mid \Diamond \varphi \mid \mu x \varphi$$

where $p \in \text{Prop}$ and $x$ occurs positively in $\varphi$. The formulas of the modal mu-calculus are interpreted over a modal algebra. The interpretation of $\mu x \varphi$ is given as $\text{LFP}(\varphi)$ (read section 3.3 of Yde’s lecture notes on algebraic semantics of mu-calculus available at http://staff.science.uva.nl/~yde/teaching/ml.)
Definition 0.4 (Modal mu-algebra). A modal algebra is a modal mu-algebra if the interpretation of $\mu x \varphi$ exists for all formulas $\varphi$ (where $x$ occurs positively in $\varphi$ and all algebra assignments.

Definition 0.5. Kozen’s axiomatization of mu-calculus consists of the following axiom and rule for the least fixed point operator, in addition to the axioms and rules of modal logic

$$\vdash \varphi[\mu x \varphi/x] \rightarrow \mu x \varphi$$  \hspace{1cm} (Fixed point axiom)

If $\vdash \varphi[\psi/x] \rightarrow \psi$, then $\vdash \mu x \varphi \rightarrow \psi$  \hspace{1cm} (Fixed point rule)

Show the completeness of Kozen’s axiomatization of mu-calculus with respect to modal mu-algebras (Hint: Use the Lindenbaum-Tarski algebra method).