Algebraic modal logic
Summer 2013

Homework 2
(due Friday, 21 June at the beginning of the lecture)

1. Show that the set of filters of Boolean algebra **B** form a distributive lattice (under $\subseteq$) where, for filters F_1, F_2
   (a) $F_1 \wedge F_2 = F_1 \cap F_2$,
   (b) $F_1 \vee F_2 = \{a \in B : a \geq a_1 \wedge a_2 \text{ for some } a_1 \in F_1, a_2 \in F_2\}$. [10 pts]
2. If U is an ultrafilter of a Boolean algebra **B**, show that $\bigwedge U$ exists, and is an atom b or equals 0. In the former case show $U = \{b\}\uparrow$ (prinicipal ultrafilter generated by b). [8 pts]
3. If **B** is the Boolean algebra of finite and cofinite subsets of an infinite set I, show that [8 pts] there is exactly one non-principal ultrafilter of **B**. [8 pts]
4. Show that a finite topological space (X, τ) is a Stone space iff it is *discrete* (i.e., every subset of X is open, or $\tau = \mathcal{P}(X)$). [8 pts]
5. *MacNeille completions and Canonical extensions*

   Definition 0.1 (**MacNeille completion**). A MacNeille completion of a lattice L is any complete lattice C containing L as a sublattice, with L both join-dense and meet-dense in C (that is, each element of C is both a join of elements of L and a meet of elements of L), and such that if C' is another such lattice then there is a unique isomorphism from C to C' fixing L.

   Definition 0.2 (**Canonical extension**). The canonical extension of a lattice $\mathbb{A}$ is a complete lattice $\mathbb{A}^\delta$ containing $\mathbb{A}$ as a sublattice, such that

   (a) Every element of $\mathbb{A}^\delta$ can be expressed both as a join of meets and as a meet of joins of elements from $\mathbb{A}$ (*denseness*);

   (b) For all $S, T \subseteq \mathbb{A}$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{A}^\delta$, there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$. (*compactness*).

   Consider the infinite chains $L = \mathbb{N} \oplus \mathbb{N}^\partial$, $\bar{L}$ and L^δ , in Figure 1 on the next page. Show that
   (a) $\bar{L}$ and L^δ are MacNeille completions of L [4 pts]
   (b) $\bar{L}$ is not a canonical extension of L. [4 pts]
   (c) L^δ is a canonical extension of L (define a dense and compact embedding $\eta : L \hookrightarrow L^\delta$) [4pts]

6. (Exercise 5.2.3 B, deR, V) Let A be a collection of finite and co-finite subsets of $\mathbb{N}$. Define $f : A \to A$ by

$$f(X) = \begin{cases} \{y \in \mathbb{N} \mid y + 1 \in X\} & \text{if X is finite} \\ \mathbb{N} & \text{if X is co-finite} \end{cases}$$

   Prove that $(A, \cup, -, \emptyset, f)$ is a boolean algebra with operators. [10 pts]

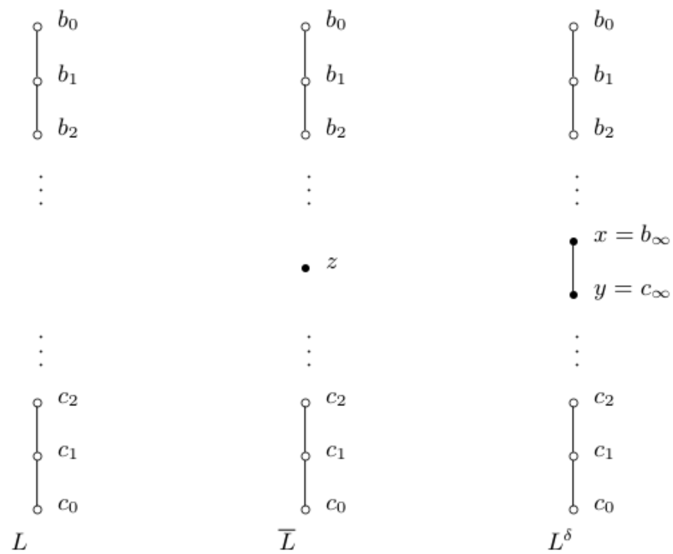

Figure 1: Completion and Canonical extension of a lattice

7. Let $\mathbb{N}_\infty$ be the set of natural numbers with an additional point ∞. Define $\mathcal{T} \subseteq \mathcal{P}(\mathbb{N}_\infty)$ as follows : a subset U of $\mathbb{N}_\infty$ belongs to $\mathcal{T}$ if, either (1) $\infty \notin U$, or (2) $\infty \in U$ and $\mathbb{N}_\infty \setminus U$ is finite.

   (a) Show that $(\mathbb{N}_\infty, \mathcal{T})$ is a topological space. [5 pts]

   (b) Show that $(\mathbb{N}_\infty, \mathcal{T})$ is a Stone space, that is, compact and totally disconnected. (Hint : the clopen subsets of $(\mathbb{N}_\infty, \mathcal{T})$ are finite sets not containing ∞, and their complements.) [5 pts]

8. (Bonus exercise) *Algebraic completeness of modal mu-calculus* [16 pts]

   For a lattice L and a map $f : L \to L$, a element $x \in L$ is a *fixed point* of f if, $f(x) = x$. The *least fixed point* is the least element in the set of fixed points of f. The following theorem gives a method to compute the least fixed point of a monotone map on a complete lattice

   Theorem 0.3 (**Knaster-Tarski Theorem**). *Let $(L, \leq)$ be a comlete lattice and $f : L \to L$ be a monotone map, that is, for each $a, b \in L$, with $a \leq b$ we have $f(a) \leq f(b)$. The Knaster-Tarski thereom states that f has a least fixed point $LFP(f)$, which can be computed as*

   $$LFP(f) = \bigwedge\{a \in L : f(a) \leq a\}$$

   Modal mu-calculus is an extension of basic modal logic with a least fixed point operator, which interprets the least fixed point of a formula, seen as a map on a modal algebra. The syntax of the logic is given as

   $$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mu x \varphi$$

   where $p \in \mathsf{Prop}$ and x occurs positively in φ. The formulas of the modal mu-calculus are interpreted over a modal algebra. The interpretation of $\mu x \varphi$ is given as $LFP(\varphi)$ (read section 3.3 of Yde's lecture notes on algebraic semantics of mu-calculus available at `http://staff.science.uva.nl/~yde/teaching/ml`.

Definition 0.4 (**Modal mu-algebra**). A modal algebra is a modal mu-algebra if the interpretation of $\mu x\varphi$ exists for all formulas φ (where x occurs positively in φ and all algebra assignments.

Definition 0.5. Kozen's axiomatization of mu-calculus consists of the following axiom and rule for the least fixed point operator, in addition to the axioms and rules of modal logic

$\vdash \varphi[\mu x\varphi/x] \to \mu x\varphi$	(Fixed point axiom)
If $\vdash \varphi[\psi/x] \to \psi$, then $\vdash \mu x\varphi \to \psi$	(Fixed point rule)

Show the completeness of Kozen's axiomatization of mu-calculus with respect to modal mu-algebras (Hint: Use the Lindenbaum-Tarski algebra method).