Algebraic modal logic
Summer 2013

Homework 3
(due Friday, 28 June)

1. (BdeRV Exercise 5.3.2) Let Λ be a normal modal logic. Give a detailed proof that the canonical frame $\mathfrak{F}^{\Lambda}$ is isomorphic to the canonical extension of $\mathcal{L}_{\Lambda}$ (the Lindenbaum-Tarski algebra of Λ). [6pts]

2. (BdeRV Exercise 5.3.4) Let W be the set $\mathbb{Z} \cup \{-\infty, \infty\}$ and let S be the successor relation on $\mathbb{Z}$, that is, $S = \{(z, z+1) \mid z \in \mathbb{Z}\}$
(a) Give a BAO whose ultrafilter frame is isomorphic to the frame $\mathfrak{F} = (W, R)$ with $R = S \cup \{(-\infty, \infty), (\infty, \infty)\}$. [4pts]
(b) Give a BAO whose ultrafilter frame is isomorphic to the frame $\mathfrak{F} = (W, R)$ with $R = S \cup (W \times \{(-\infty, \infty), (\infty, \infty)\})$. [4pts]

3. (BdeRV Exercise 5.3.5) An operation on a boolean algebra is called *2-additive* if it satisfies

$$f(x+y+z) = f(x+y) + f(x+z) + f(y+z)$$

Now suppose that $\mathfrak{A} = (A, +, -, 0, f)$ such that $(A, +, -, 0)$ is a boolean algebra on which f is a 2-additive operation. Prove that this algebra can be embedded in a complete and atomic algebra. [10pts]

4. (BdeRV Exercise 5.3.7) Let τ be a similarity type, and let $\mathfrak{F}, \mathfrak{f}$ and $\mathfrak{g}$ be a τ Kripke frame and two general τ-frames, respectively. Prove or disprove the following:
(a)$(\mathfrak{F}^{\sharp})_{\sharp} = \mathfrak{F}$ [4pts]
(b) $\mathfrak{g} \rightarrowtail (\mathfrak{g}_{\sharp})^{\sharp}$ [4pts]
(c) $\mathfrak{g}^* \equiv \mathfrak{f}^*$ only if $\mathfrak{g} \equiv \mathfrak{f}$, [4pts]
(For notations see BdeRV Definition 5.73)

5. *Closure operators and Consequence relations*

Definition 0.1 (Closure operator). Given a set A, a mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is called a *closure operator* on A if, for $X, Y \subseteq A$, it satisfies:
(C1): $X \subseteq C(X)$ (extensive)
(C2): $C^2(X) = C(X)$ (idempotent)
(C3): $X \subseteq Y$ implies $C(X) \subseteq C(Y)$ (isotone)

A subset X of A is called a *closed subset* if $C(X) = X$. The poset of closed subsets of A with set inclusion as the partial ordering is denoted by L_C .

(a) Define an appropriate meet and join of elements of L_C and show that it is a complete lattice. [4pts]
(b) Is the converse of (a) true, that is, is every complete lattice isomorphic to closed subsets of some set with a closure operator. Prove or give a counterexample. [4pts]
(c) Let Frm denote the set of formulas in the modal language and $\Gamma \subseteq \mathcal{P}(\mathsf{Frm})$. Define a mapping $Cn : \mathcal{P}(\mathsf{Frm}) \to \mathcal{P}(\mathsf{Frm})$ as $Cn(\Gamma) = \{\varphi \in \mathsf{Frm} \mid \Gamma \vdash_{\mathbf{K}} \varphi\}$, where $\vdash_{\mathbf{K}}$ is the consequence relation of normal modal logic $\mathbf{K}$.

Show that the mapping Cn is a closure operator. [4pts]

6. *Term algebra and Free algebra*

Definition 0.2 (**Term algebra**)**.** Let X be a set of variables and $\mathcal{F} = \{f_1, \ldots, f_n\}$ an algebraic type. The set $Tm_{\mathcal{F}}(X)$ of terms of type $\mathcal{F}$ over X is the smallest set T such that $X \subseteq T$ and, for every n-ary function symbol $f_i \in \mathcal{F}$ with arity k_i and $t_0, ..., t_{k_i-1} \in T$, $f(t_0, ..., t_{k_i-1}) \in T$.

Definition 0.3 (**Free algebra**)**.** Let K be a class of algebras and X a set. An algebra $\mathbf{F} \in K$ with a map $i : X \to \mathbf{F}$ is called a free K algebra over X, if, for every $\mathbf{A} \in K$ and every map $h : X \to A$, there exists a unique homomorphism $\tilde{h} : \mathbf{F} \to \mathbf{A}$ such that $\tilde{h} \circ i = h$.

(a) Show that, for a type $\mathcal{F}$ and a set X, the term algebra $Tm_{\mathcal{F}}(X)$ is the free algebra over X for the class of all $\mathcal{F}$-algebras. [5pts]
(b) Show that the Lindenbaum-Tarski algebra of the propositional language L is a free Boolean algebra freely generated by the set of all elements $[p]$, where each p is a propositional variable of L. [5pts]