Algorithmic correspondence for intuitionistic modal mu-calculus

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Abstract

In the present paper, the algorithmic correspondence theory developed in (Conradie and Palmigiano, 2012) is extended to mu-calculi with a non-classical base. We focus in particular on the language of bi-intuitionistic modal mu-calculus. We enhance the algorithm ALBA introduced in (Conradie and Palmigiano, 2012) so as to guarantee its success on the class of recursive mu-inequalities, which we introduce in this paper. Key to the soundness of this enhancement are the order-theoretic properties of the algebraic interpretation of the fixed point operators. We show that, when restricted to the Boolean setting, the recursive mu-inequalities coincide with the "Sahlqvist mu-formulas" defined in (van Benthem, Bezhanishvili and Hodkinson, 2012).
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Introduction

Modal mu-calculus [17] is a logical framework combining simple modalities with fixed point operators, enriching the expressivity of modal logic so as to deal with infinite processes like recursion. It has a simple syntax, an easily given semantics, and is decidable. Modal mu-calculus has become a fundamental logical tool in theoretical computer science and has been extensively studied [3], and applied for instance in the context of temporal properties of systems, and of infinite properties of concurrent systems. Many expressive modal and temporal logics such as PDL, CTL, CTL* can be seen as fragments of the modal mu-calculus [3, 15]. It provides a unifying framework connecting modal and temporal logics, automata theory and the theory of games, where fixed point constructions can be used to talk about the long term strategies of players, as discussed in [23].

Correspondence theory studies the relationships between classical first- and second-order logic, and modal logic, interpreted on Kripke frames. A modal and a first-order formula *correspond* if they define the same class of structures. Specifically, Sahlqvist theory is concerned with the identification of syntactically specified classes of modal formulas which correspond to first-order formulas. Sahlqvist-style frame-correspondence theory for modal mu-calculus has recently been developed in [22]. Such analysis strengthens the general mathematical theory of the mu-calculus, facilitates the transfer of results from first-order logic with fixed points, and aids in understanding the meaning of mu-formulas interpreted over frames, which is often difficult to grasp.

The correspondence results in [22] are developed purely model-theoretically. However, they can be naturally encompassed within the existing *algebraic* approach to correspondence theory [5, 9, 10], and generalized to mu-calculi on a weaker-than-classical (and, particularly, *intuitionistic*) base.

There are three types of reasons for studying (bi-)intuitionistic mu-calculi. Firstly, the correspondence results obtained in this setting project onto those obtainable in the classical setting of [22]. Conceptually, this means that the correspondence mechanisms for mu-calculi are intrinsically independent of their being set in classical logic, and hence the non-classical mu-calculi provide clearer insights into their nature, by abstracting from unneeded assumptions.[1] Secondly, these mu-calculi also bring practical advantages, since their greater generality means of course wider applicability. Finally, it can be argued that such a study is now timely, given that closely related areas of logic such as constructive modal logics and type theory are of increasing foundational and practical relevance in such fields as semantics of programming languages [12], and intuitionistic modal mu-calculi can be a valuable tool to these investigations.

Our contribution. As motivated above, we work on a non-classical base, and our setting of choice is bi-intuitionistic modal logic [18, 16]. Besides being interesting in its own right, the bi-intuitionistic modal logic also lends itself to certain types of more systematic analysis. For instance, it provides a methodologically useful setting for the order-theoretic analysis of correspondence theory. Indeed, its array of connectives is representative of the various order-theoretic behaviours one is likely to encounter, and hence provides a blueprint for transferring this analysis to other logics. Moreover, this analysis includes as a special case the correspondence theory for classical modal mu-calculus of [22]. Following the methodology developed in [10], the algebraic and order-theoretic principles underlying these results are isolated. This forms an intermediate level of analysis which is added to the model-theoretic analysis in [22].

In fact, this intermediate level of analysis makes it possible to recognize that even the lattice-distributivity plays no essential role for the crucial order-theoretic preservation properties of fixed points; accordingly, these properties are stated in the vastly more general setting of complete (not necessarily distributive) lattices, paving the way to the development of correspondence theory for substructural mu-calculi.

The fact that the intermediate level of analysis is conducted on algebras makes it possible to develop the crucial part of correspondence theory independently of the specific way the relational semantics is defined, given that different relational semantics can be associated with a given non-classical (fixed point) logic.

In the present paper, we extend the algorithm ALBA of [9] to the language of bi-intuitionistic modal mu-calculus. ALBA is an algorithm, based on a calculus of rewrite rules which, if successful, effectively calculates first-order correspondents for formulas of distributive, intuitionistic and classical modal logics. We define the class of recursive inequalities (see Definition 3.2) for the bi-intuitionistic modal mu-calculus which is the bi-intuitionistic counterpart of the Sahlqvist mu-formulas defined in [22]. We prove that the enhanced ALBA is successful on all recursive mu-inequalities, and hence that each of them has a frame correspondent in first-order logic with least fixed points (FO+LFP) [13] .

It is worth stressing that all the results and in particular all the practical reductions developed for bi-intuitionistic modal mu-calculus are immediately applicable to the classical case.

[1] In particular, the bi-intuitionistic setting accounts for the projection over the classical setting more naturally than the intuitionistic one, for various technical reasons which will be expanded upon in Remark 3.6.

Structure. Within the preliminary section, 1.1 collects some details about the algebraic and relational semantics of bi-intuitionistic modal logic. In 1.2, algebraic-algorithmic correspondence theory is illustrated by means of an extensive example, and in 1.3 the calculus for correspondence for bi-intuitionistic modal logic is introduced and discussed. In Section 2, the stage is set for extending correspondence theory to mu-calculus: in 2.1, the relevant order-theoretic preservation properties of extremal fixed points are stated, in a general setting of complete (not necessarily distributive) lattices. In 2.2, the language and semantics of the bi-intuitionistic modal mu-calculus is introduced, together with the expanded language which facilitates the algebraic correspondence reductions. In 2.3 and 2.4, the formal tools for the enhanced version of ALBA are introduced, in the form of so-called *approximation* and *adjunction rules* for fixed point binders, and the soundness of these rules is proven in terms of the order-theoretic properties of 2.1. In 2.5 the limitations of these rules are discussed, which motivates the developments of Sections 4 and 5. In Section 3, the recursive mu-inequalities are defined in the same uniform style discussed and advocated in [5], which pivots on the order-theoretic properties of the algebraic interpretation of the logical connectives. This class is compared with other Sahlqvist-type classes in the literature. In Section 4, we define certain syntactic shapes of formulas, the (*normal*) *inner formulas*, which guarantee the applicability of the approximation rules as stated in Section 2.3, and of a special rephrasing of the adjunction rules, given in Section 5. In Section 6, we show the execution of the algorithm on two examples. In Section 7, the relation between inner formulas and recursive inequalities is unfolded, which serves as a technical device in the proof that ALBA, augmented with the rules defined in the previous sections, is successful on all recursive inequalities.

1 Preliminaries

In this section we collect some preliminaries on the semantic structures with which we will be working and we illustrate the algebraic-algorithm approach to correspondence theory.

1.1 Perfect modal bi-Heyting algebras

An element $c \neq \bot$ of a complete lattice $\mathbb{C}$ is *completely join-irreducible* iff $c = \bigvee S$ implies $c \in S$ for every $S \subseteq \mathbb{C}$; moreover, c is *completely join-prime* if $c \neq \bot$ and, for every subset S of the lattice, $c \leq \bigvee S$ iff $c \leq s$ for some $s \in S$. *Completely meet-irreducible* and *completely meet-prime* elements re defined order-dually. If c is completely join-prime (resp. meet-prime), then c is completely join-irreducible (resp. meet-irreducible). If $\mathbb{C}$ is *frame distributive* (i.e. finite meets distribute over arbitrary joins) then the completely join-irreducible elements are completely join-prime. The collections of all completely join- and meet-irreducible elements of $\mathbb{C}$ are respectively denoted by $J^\infty(\mathbb{C})$ and $M^\infty(\mathbb{C})$.

A *bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \rightarrow, -, \top, \bot)$ such that both $(A, \wedge, \vee, \rightarrow, \top, \bot)$ and $(A, \wedge, \vee, -, \top, \bot)^\partial$ are Heyting algebras. In particular, the operation $-$ (referred to as 'subtraction', 'exclusion' or 'disimplication') is uniquely identified by the following property holding for every $a, b, c \in A$:

$$a - b \leq c \text{ iff } a \leq c \vee b.$$

In the special case of Boolean algebras, $a - b = a \wedge \neg b$. A *modal bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \rightarrow, -, \top, \bot, \Box, \Diamond)$ such that $(A, \wedge, \vee, \rightarrow, -, \top, \bot)$ is a bi-Heyting algebra and $\Box$ and $\Diamond$ preserve finite meets and joins, respectively.

Definition 1.1. A *perfect* lattice is a complete lattice $\mathbb{C}$ such that $J^\infty(\mathbb{C})$ join-generates $\mathbb{C}$ (i.e. every element of $\mathbb{C}$ is the join of elements in $J^\infty(\mathbb{C})$) and $M^\infty(\mathbb{C})$ meet-generates $\mathbb{C}$ (i.e. every element of $\mathbb{C}$ is the meet of elements in $M^\infty(\mathbb{C})$). A *perfect distributive lattice* is a perfect lattice which is completely distributive, and hence $J^\infty(\mathbb{C})$ coincides with the set of all completely join-prime elements of $\mathbb{C}$ and $M^\infty(\mathbb{C})$ coincides with the set of all completely meet-prime elements of $\mathbb{C}$. A *perfect bi-Heyting algebra* is a bi-Heyting algebra the lattice reduct of which is a perfect distributive lattice. A *perfect modal bi-Heyting algebra* is a modal bi-Heyting algebra the bi-Heyting reduct of which is a perfect bi-Heyting algebra, and moreover such that $\Box$ and $\Diamond$ preserve arbitrary meets and joins, respectively.

A Stone-type duality on objects (extending the finite *Birkhoff* duality) holds between perfect bi-Heyting algebras and *posets*, which is defined as follows: every poset X is associated with the lattice $\mathcal{P}^\uparrow(X)$ of the upward-closed[2] subsets of X, on which the implication and the subtraction are defined as $Y \to Z = (Y^c \cup Z){\downarrow}^c$ and $Y - Z = (Y \cap Z^c){\uparrow}$ for all $Y, Z \in \mathcal{P}^\uparrow(X)$; here $(\cdot)^c$ denotes the complement relative to W; conversely, every perfect bi-Heyting algebra $\mathbb{C}$ is associated with $(J^\infty(\mathbb{C}), \geq)$ where $\geq$ is the reverse lattice order in $\mathbb{C}$, restricted to $J^\infty(\mathbb{C})$.

Just in the same way in which the duality between complete atomic Boolean algebras and sets can be expanded to a duality between complete atomic modal algebras and Kripke frames, the duality between perfect bi-Heyting algebras and posets can be expanded to a duality between perfect modal bi-Heyting algebras and posets endowed with arrays of relations, each of which dualizes one additional operation in the usual way, i.e., n-ary operations give rise to $n+1$-ary relations, and the assignments between operations and relations are defined as in the classical setting. We are not going to report on this duality in full detail (we refer e.g. to [19, 14, 9]), but we limit ourselves to mention that dual frames to perfect modal bi-Heyting algebras can be defined as relational structures $\mathcal{F} = (W, \leq, R_\Diamond, R_\Box)$ such that $(W, \leq)$ is a nonempty poset, $R_\Diamond$ and $R_\Box$ are binary relations on W, and the following inclusions hold:

$$\geq \circ R_\Diamond \circ \geq \ \subseteq \ R_\Diamond \qquad\qquad \leq \circ R_\Box \circ \leq \ \subseteq \ R_\Box.$$

The *complex algebra* of any such relational structure $\mathcal{F}$ (cf. [14, Sec. 2.3]) is

$$\mathcal{F}^+ = (\mathcal{P}^\uparrow(W), \cup, \cap, \varnothing, W, \langle R_\Diamond\rangle, [R_\Box]),$$

where, for every $X \subseteq W$,

$$\begin{array}{lclcl} [R_\Box]X & := & \{w \in W \mid R_\Box[w] \subseteq X\} & = & (R_\Box^{-1}[X^c])^c \\ \langle R_\Diamond\rangle X & := & \{w \in W \mid R_\Diamond[w] \cap X \neq \varnothing\} & = & R_\Diamond^{-1}[X]. \end{array}$$

Here $R[x] = \{w \mid w \in W \text{ and } xRw\}$ and $R^{-1}[x] = \{w \mid w \in W \text{ and } wRx\}$. Moreover, $R[X] = \bigcup\{R[x] \mid x \in X\}$ and $R^{-1}[X] = \bigcup\{R^{-1}[x] \mid x \in X\}$.

1.2 Algebraic-algorithmic correspondence

The contribution of the present paper is set in the context of order-theoretic algorithmic correspondence theory [9, 5]. Correspondence theory originates from the observation that relational structures

[2] A subset Y of a poset X is *upward-closed* if $x \in Y$ and $x \leq y \in X$ implies $y \in Y$. We write $Y{\uparrow} = \{x \in X \mid \exists y(y \in Y \ \& \ y \leq x)\}$. Dually for *downward-closed* subsets and $Y{\downarrow}$.

interpret both classical first- and second-order logic, and modal logic. A modal and a first-order formula *correspond* if they define the same class of structures. Sahlqvist theory aims at characterizing (sub)classes of modal formulas admitting first-order correspondents, and at effectively calculating their correspondents.

This theory goes back to the very well-known Sahlqvist-van Benthem algorithm (see [2]), with the same core motivation and strategy. Namely, the effective computation of first-order frame correspondents for modal formulas through the elimination of monadic second-order universal quantification from the conditions expressing the (local) validity of these formulas. This is typically done by instantiating propositional variables with first-order definable 'minimal valuations'. We refer the reader to [2] for a basic introduction and to [20] for a broader overview.

As mentioned in the introduction, this strategy can be developed in the context of the algebraic semantics of modal logic, and then generalized to various other logics. The algebraic setting helps to distill the essentials of this strategy. We refer the reader to [10] for an in-depth treatment linking the traditional and algebraic approaches, and to [9] for a fully-fledged treatment of the algebraic-algorithmic approach. Before giving a more detailed account of this theory, we will guide the reader through the main principles which make it work, by means of an example.

The algorithm illustrated. Let us start with one of the best known examples in correspondence theory, namely $\Diamond\Box p \to \Box\Diamond p$. It is well known that for every Kripke frame $\mathcal{F} = (W, R)$,

$$\mathcal{F} \Vdash \Diamond\Box p \to \Box\Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall xyz\,(Rxy \wedge Rxz \to \exists u(Ryu \wedge Rzu)).$$

As is discussed at length in [9, 5], every piece of argument used to prove this correspondence on frames can be translated by duality (see Section 1.1) to complex algebras[3]. We will show how this is done in the case of the example above.

As is well known, complex algebras are characterized in purely algebraic terms as complete and atomic BAOs where the modal operations are completely join-preserving. These are also known as *perfect* BAOs [24, Definition 40].

First of all, the condition $\mathcal{F} \Vdash \Diamond\Box p \to \Box\Diamond p$ translates to the complex algebra $\mathbb{A} = \mathcal{F}^+$ of $\mathcal{F}$ as $[\![\Diamond\Box p]\!] \subseteq [\![\Box\Diamond p]\!]$ for every assignment of p into $\mathbb{A}$, so this validity clause can be rephrased as follows:

$$\mathbb{A} \models \forall p[\Diamond\Box p \leq \Box\Diamond p], \tag{1.1}$$

where the order $\leq$ is interpreted as set inclusion in the complex algebra. In perfect BAOs every element is both the join of the completely join-prime elements (the set of which is denoted $J^\infty(\mathbb{A})$) below it and the meet of the completely meet-prime elements (the set of which is denoted $M^\infty(\mathbb{A})$) above it[4]. Hence, taking some liberties in our use of notation, the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \in J^\infty(\mathbb{A}) \mid i \leq \Diamond\Box p\} \leq \bigwedge\{m \in M^\infty(\mathbb{A}) \mid \Box\Diamond p \leq m\}].$$

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [11]), this condition is true if and only if every element in the join is less than or equal to every element in the

[3] cf. [2, Definition 5.21] and also page 4.

[4] In BAOs the completely join-prime elements, the completely join-irreducible elements and the atoms coincide. Moreover, the completely meet-prime elements, the completely meet-irreducible elements and the co-atoms coincide.

meet; thus, condition (1.1) above can be rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\Box p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{1.2}$$

where the variables $\mathbf{i}$ and $\mathbf{m}$ range over $J^\infty(\mathbb{A})$ and $M^\infty(\mathbb{A})$ respectively. Since $\mathbb{A}$ is a perfect BAO, the element of $\mathbb{A}$ interpreting $\Box p$ is the join of the completely join-prime elements below it. Hence, if $i \in J^\infty(\mathbb{A})$ and $i \leq \Diamond\Box p$, because $\Diamond$ is completely join-preserving on $\mathbb{A}$, we have that

$$i \leq \Diamond(\bigvee\{j \in J^\infty(\mathbb{A}) \mid j \leq \Box p\}) = \bigvee\{\Diamond j \mid j \in J^\infty(\mathbb{A}) \text{ and } j \leq \Box p\},$$

which implies that $i \leq \Diamond j_0$ for some $j_0 \in J^\infty(\mathbb{A})$ such that $j_0 \leq \Box p$. Hence, we can equivalently rewrite the validity clause above as follows:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\exists \mathbf{j}(\mathbf{i} \leq \Diamond \mathbf{j} \ \ \& \ \ \mathbf{j} \leq \Box p) \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{1.3}$$

and then use standard manipulations from first-order logic to pull out quantifiers:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \ \& \ \ \mathbf{j} \leq \Box p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{1.4}$$

Now we observe that the operation $\Box$ preserves arbitrary meets in the perfect BAO $\mathbb{A}$. By the general theory of adjunction in complete lattices, this is equivalent to $\Box$ being a right adjoint (cf. [11, Proposition 7.34]). It is also well known that the left or lower adjoint (cf. [11, Definition 7.23]) of $\Box$ is the operation $\blacklozenge$, which can be recognized as the backward-looking diamond P, interpreted with the converse R^{-1} of the accessibility relation R of the frame $\mathcal{F}$ in the context of tense logic (cf. [2, Example 1.25] and [11, Exercise 7.18] modulo translating the notation). Hence the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \ \& \ \ \blacklozenge \mathbf{j} \leq p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{1.5}$$

and then as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \ \& \ \ \exists p(\blacklozenge \mathbf{j} \leq p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m})) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{1.6}$$

At this point we are in a position to eliminate the variable p and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \ \& \ \ \Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{1.7}$$

Let us justify this equivalence: for the direction from top to bottom, fix an interpretation V of the variables $\mathbf{i}, \mathbf{j}$, and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j}$ and $\Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}$. To prove that $\mathbf{i} \leq \mathbf{m}$ holds under V, consider the variant V^* of V such that $V^*(p) = \blacklozenge \mathbf{j}$. Then it can be easily verified that V^* witnesses the antecedent of (1.6) under V; hence $\mathbf{i} \leq \mathbf{m}$ holds under V. Conversely, fix an interpretation V of the variables $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \exists p(\blacklozenge \mathbf{j} \leq p \ \& \ \Box\Diamond p \leq \mathbf{m})$. Then, by monotonicity, the antecedent of (1.7) holds under V, and hence so does $\mathbf{i} \leq \mathbf{m}$, as required. This is an instance of the following result, known as *Ackermann's lemma* ([1], see also [6]):

Lemma 1.2. *Fix an arbitrary propositional language L. Let $\alpha, \beta(p), \gamma(p)$ be L-formulas such that α is p-free, β is positive and γ is negative in p. For any assignment V on an L-algebra $\mathbb{A}$, the following are equivalent:*

1. $\mathbb{A}, V \models \beta(\alpha/p) \leq \gamma(\alpha/p)$ *;*

2. there exists a p-variant V^* *of* V *such that* $\mathbb{A}, V^* \models \alpha \leq p$ *and* $\mathbb{A}, V^* \models \beta(p) \leq \gamma(p)$*,*

where $\beta(\alpha/p)$ *and* $\gamma(\alpha/p)$ *denote the result of uniformly substituting* α *for* p *in* β *and* γ*, respectively.*

The proof is essentially the same as [9, Lemma 4.2]. Whenever, in a reduction, we reach a shape in which the lemma above (or its order-dual) can be applied, we say that the condition is in *Ackermann shape*. If we relax the requirement that p does not occur in α and are willing to admit fixed point operators in our (correspondence) language, we can formulate the following more general version of the Ackermann lemma (see also [7]):

Lemma 1.3. *Let* $\alpha(p)$*,* $\beta(p)$*, and* $\gamma(p)$ *be L-formulas, with* $\alpha(p)$ *and* $\beta(p)$ *positive in p, and* $\gamma(p)$ *negative in p. For any assignment V on a complete L-algebra* $\mathbb{A}$*, the following are equivalent:*

1. $\mathbb{A}, V \models \beta(\mu p.\alpha(p)/p) \leq \gamma(\mu p.\alpha(p))/p)$*;*

2. there exists a p-variant V^* *of* V *such that* $\mathbb{A}, V' \models \alpha(p) \leq p$*, and* $\mathbb{A}, V' \models \beta(p) \leq \gamma(p)$*,*

where $\mu p.\alpha(p)$ *denotes the least fixed point of* $\alpha(p)$*, and need not be an expression in the language L.*

Proof. We begin by noting that, since we are working in a complete lattice, least fixed points of monotone operations exist by the Knaster-Tarski theorem. As regards '1 $\Rightarrow$ 2', let $V'(p) := V(\mu p.\alpha(p))$. As regards '2 $\Rightarrow$ 1', $\mathbb{A}, V' \models \alpha(p) \leq p$ implies that $V'(p)$ is a pre-fixed point of $\alpha(\cdot)$,[5] and hence $\mu p.\alpha(p) \leq V'(p)$. Therefore, $\beta(\mu p.\alpha(p)/p) \leq \beta(V'(p)) \leq \gamma(V'(p)) \leq \gamma(\mu p.\alpha(p)/p)$. □

Taking stock, we note that we have equivalently transformed (1.1) into (1.7), which is a condition in which all propositional variables (corresponding to monadic second-order variables) have been eliminated, and all remaining variables range over completely join- and meet-prime elements. Via the duality, the latter correspond to singletons and complements of singletons, respectively, in Kripke frames. Moreover, $\blacklozenge$ is interpreted on Kripke frames using the converse of the same accessibility relation used to interpret $\Box$. Hence, clause (1.7) translates equivalently into a condition in the first-order correspondence language. To facilitate this translation we first rewrite (1.7) as follows, by reversing the reasoning that brought us from (1.1) to (1.2):

$$\mathbb{A} \models \forall \mathbf{j}[\Diamond \mathbf{j} \leq \Box\Diamond\blacklozenge\mathbf{j}].$$

By again applying the fact that $\Box$ is a right adjoint we obtain

$$\mathbb{A} \models \forall \mathbf{j}[\blacklozenge\Diamond \mathbf{j} \leq \Diamond\blacklozenge\mathbf{j}]. \tag{1.8}$$

Recalling that $\mathbb{A}$ is the complex algebra of $\mathcal{F} = (W, R)$, this gives $\forall w(R[R^{-1}[w]] \subseteq R^{-1}[R[w]]$. Notice that $R[R^{-1}[w]]$ is the set of all states $x \in W$ which have a predecessor z in common with w, while $R^{-1}[R[w]]$ is the set of all states $x \in W$ which have a successor in common with w. This can be spelled out as

$$\forall x \forall w(\exists z(Rzx \wedge Rzw) \rightarrow \exists y(Rxy \wedge Rwy))$$

or, equivalently,

$$\forall z \forall x \forall w((Rzx \wedge Rzw) \rightarrow \exists y(Rxy \wedge Rwy))$$

which is the familiar Church-Rosser condition.

[5]Here $\alpha(\cdot)$ is obtained from the term function α by leaving p free and fixing all other variables to the values prescribed by V.

1.3 The basic calculus for correspondence and recursive Ackermann rules

The example in Section 1.2 illustrated the main strategy for the elimination of second order variables. We transformed the initial validity condition into a shape to which Ackermann's lemma was applicable (i.e., into Ackermann shape). Two order-theoretic ingredients were used to reach Ackermann shape, namely:

(a) The ability to *approximate* elements of the algebra from above or from below using completely join-prime and completely meet-prime elements;

(b) the fact that $\Box$ is a right *adjoint*. More in general, in perfect distributive lattices with operators, all the operations interpreting the logical connectives are either residuals or adjoints.

We can repackage these two observations, together with Ackermann's lemma, in the form of *proof rules*, grouped in the following types: the *approximation rules*, *residuation/adjunction rules* and *Ackermann rules*. These rules, together with the strategy governing the order of their application, as illustrated in Section 1.2, constitute the algorithm ALBA, a rigourous specification if which can be found in [9, Section 6]. ALBA takes an inequality in input, preprocesses it and transforms it into one or more expressions known as quasi-inequalities: given a propositional language $\mathcal{L}$, an *$\mathcal{L}$-quasi-inequality* is an expression of the form $\varphi_1 \leq \psi_1 \,\&\cdots\&\, \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ where the φ_i, ψ_i, φ and ψ are $\mathcal{L}$-formulas.

ALBA's goal is to transform all the obtained quasi-inequalities into (sets of) *pure* quasi-inequalities, i.e., into quasi-inequalities in which no propositional variables occur. If such a state is reached, we say ALBA *succeeds* on the input inequality.

First approximation rule. This rule is applied only once to transform an inequality into a quasi-inequality (as in (1.2)) after some possible preprocessing.

$$\frac{\varphi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \varphi \,\&\, \psi \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]} \text{ (FA)}$$

Approximation rules. Each of the following rules can be proved sound with an argument similar to that used in Section 1.2 to justify the transition from (1.2) to (1.3). For more details, see [9, Lemma 8.4].

$$\frac{\Box\psi \leq \mathbf{m}}{\exists \mathbf{n}(\Box\mathbf{n} \leq \mathbf{m} \,\&\, \psi \leq \mathbf{n})} \text{ ($\Box$Appr)} \qquad \frac{\mathbf{j} \leq \Diamond\psi}{\exists \mathbf{i}(\mathbf{j} \leq \Diamond\mathbf{i} \,\&\, \mathbf{i} \leq \psi)} \text{ ($\Diamond$Appr)}$$

$$\frac{\chi \to \varphi \leq \mathbf{m}}{\exists \mathbf{j}(\mathbf{j} \to \varphi \leq \mathbf{m} \,\&\, \mathbf{j} \leq \chi)} \text{ ($\to$Appr$_1$)} \qquad \frac{\chi \to \varphi \leq \mathbf{m}}{\exists \mathbf{n}(\chi \to \mathbf{n} \leq \mathbf{m} \,\&\, \varphi \leq \mathbf{n})} \text{ ($\to$Appr$_2$)}$$

$$\frac{\mathbf{i} \leq \chi - \varphi}{\exists \mathbf{j}(\mathbf{i} \leq \mathbf{j} - \varphi \,\&\, \mathbf{j} \leq \chi)} \text{ ($-$Appr$_1$)} \qquad \frac{\mathbf{i} \leq \chi - \varphi}{\exists \mathbf{n}(\mathbf{i} \leq \chi - \mathbf{n} \,\&\, \varphi \leq \mathbf{n})} \text{ ($-$Appr$_2$)}$$

Adjunction and residuation rules. Each of the following rules can be proved sound with an argument similar to that used in Section 1.2 to justify the transition from (1.4) to (1.5), cf. [9, Lemma 8.4].

$$\frac{\varphi \vee \chi \leq \psi}{\varphi \leq \psi \quad \chi \leq \psi}\,(\vee\text{LA}) \qquad \frac{\varphi \leq \chi \vee \psi}{\varphi - \chi \leq \psi}\,(\vee\text{RR}) \qquad \frac{\varphi \leq \chi \to \psi}{\varphi \wedge \chi \leq \psi}\,(\to\text{RR})$$

$$\frac{\psi \leq \varphi \wedge \chi}{\psi \leq \varphi \quad \psi \leq \chi}\,(\wedge\text{RA}) \qquad \frac{\chi \wedge \psi \leq \varphi}{\chi \leq \psi \to \varphi}\,(\wedge\text{LR}) \qquad \frac{\chi - \psi \leq \varphi}{\chi \leq \psi \vee \varphi}\,(-\text{LR})$$

Specifically, the rules in the left column above are justified by the fact that $\vee$ and $\wedge$ are respectively the left and the right adjoint of the diagonal map Δ, defined by the assignment $a \mapsto (a, a)$; the ones in the middle and right hand columns above are justified by $\vee$ and $\wedge$ being respectively the right residual of $-$ and the left residual of $\to$. For $\Diamond$ and $\Box$ we have:

$$\frac{\Diamond\varphi \leq \psi}{\varphi \leq \blacksquare\psi}\,(\Diamond\text{LA}) \qquad \frac{\varphi \leq \Box\psi}{\blacklozenge\varphi \leq \psi}\,(\Box\text{RA})$$

Ackermann rules. The soundness of the following rules is justified by Lemma 1.2 and its symmetric version.

$$\frac{\exists p[\&_{i=1}^{n} \alpha_i \leq p \;\&\; \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\bigvee_{i=1}^{n} \alpha_i/p) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i/p)}\,(\text{RA}) \qquad \frac{\exists p[\&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\bot/p) \leq \gamma_j(\bot/p)}\,(\bot)$$

$$\frac{\exists p[\&_{i=1}^{n} p \leq \alpha_i \;\&\; \&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\bigwedge_{i=1}^{n} \alpha_i/p) \leq \beta_j(\bigwedge_{i=1}^{n} \alpha_i/p)}\,(\text{LA}) \qquad \frac{\exists p[\&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\top/p) \leq \beta_j(\top/p)}\,(\top)$$

The rules above are subject to the restrictions that the α_i are p-free, and that the γ_j and the β_j are respectively negative and positive in p. Notice that the rules $(\bot)$ and $(\top)$ can be regarded as the special case of (RA) and (LA) in which $\alpha := \bot$ and $\alpha := \top$, respectively.

Unlike the rules given in the previous paragraphs which apply locally and rewrite individual inequalities, the Ackermann rules involve the set of inequalities in the antecedent of a quasi-inequality as a whole. A quasi-inequality to which one of these rules is applicable is said to be in Ackermann shape. In particular, this requires that either all positive occurrences of p occur *in display* in inequalities of the form $\alpha_i \leq p$ (in the case of (RA)), or that all negative occurrences of p occur in display in inequalities of the form $p \leq \alpha_i$ (in the case of (LA)).

The interested reader may find many examples of correspondence reductions making use of these rules in [9] and [5].

Recursive Ackermann rules. Lemma 1.3 proves the soundness of the following more general *recursive Ackermann rules*, which allow us to eliminate a propositional variable p even if the α_i are not p-free. Note that the recursive Ackermann rule is not part of the original specification of ALBA as given in [9]. However, it can be incorporated into ALBA executions as illustrated in [5, Section 36.8.1] and also in Section 6, below.

$$\frac{\exists p[\&_{i=1}^{n} \alpha_i(p) \leq p \ \& \ \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\mu X.[\bigvee_{i=1}^{n} \alpha_i(X)]/p) \leq \gamma_j(\mu X.[\bigvee_{i=1}^{n} \alpha_i(X)]/p)} \ (\mathrm{RA}_{rec})$$

$$\frac{\exists p[\&_{i=1}^{n} p \leq \alpha_i(p) \ \& \ \&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\nu X.[\bigwedge_{i=1}^{n} \alpha_i(X)]/p) \leq \beta_j(\nu X.[\bigwedge_{i=1}^{n} \alpha_i(X)]/p)} \ (\mathrm{LA}_{rec})$$

The rules are applicable subject to the restrictions that the $\alpha_i(p)$ and β_j are positive in p, that the γ_j are negative in p, and X is a fresh fixed point variable. Notice that these define a generalized Ackermann shape.

2 ALBA for bi-intuitionistic modal mu-calculus: setting the stage

The machinery reviewed above works as it stands also for the reduction of some mu-inequalities. For instance, the inequality $\nu X.\Box(p \wedge X) \leq p$ (cf. [22, Section 5.3]) can be reduced as follows:

$$\begin{array}{lll} & & \forall p[\nu X.\Box(p \wedge X) \leq p] \\ & \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \ \& \ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ (*) & \text{iff} & \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(\mathbf{m} \wedge X) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ & \text{iff} & \forall \mathbf{m}[\nu X.\Box(\mathbf{m} \wedge X) \leq \mathbf{m}]. \end{array}$$

The equivalence marked with $(*)$ is an application of the rule (LA), which can be applied because the term function $\beta(p) = \nu X.\Box(p \wedge X)$ is monotone in p. So all the steps in the previous chain of equivalences can be justified purely in terms of the principles of order-theoretic algorithmic correspondence we have seen above, without the need for any new rule dealing specifically with the fixed point binders. This is possible because the quasi-inequality is already in Ackermann shape after first approximation, and hence we did not need to extract any occurrences of p from under fixed point binders. We will return to this inequality in Example 3.3 and Section 4.2.

Our aim in this paper is to extend these techniques to a larger class of mu-inequalities, including those defined in [22], in which fixed point binders occur in more essential ways. In order to do this we will need to

1. analyze the order-theoretic properties of the term functions associated with mu-calculus formulas, which we do in Section 2.1.

2. define the bi-intuitionistic syntactic and semantic settings for mu-calculus, which we do in Section 2.2.

3. on the basis of the analysis in Section 2.1 formulate approximation and adjunction rules for fixed point binders. This we do in Sections 2.3 and 2.4.

To complete our account it would be sufficient to define a syntactic class of bi-intuitionistic mu-inequalities, as is done in Section 3, and then show that the algorithm enhanced with the rules above succeeds on all its members. However, some non-trivial further justification needs to be given as to how the rules of Sections 2.3 and 2.4 are applicable to the syntactic specifications of Section 3. This is further discussed in Section 2.5.

2.1 Preservation and distribution properties of extremal fixed points

For L and M complete lattices and $G : M \times L \to L$, let $\mu y.G : M \to L$ and $\nu y.G : M \to L$ be the maps respectively given by $b \mapsto LFP(G(b, y))$ and $b \mapsto GFP(G(b, y))$ for each $b \in M$ such that $LFP(G(b, y))$ and $GFP(G(b, y))$ are defined, where $LFP(G(b, y))$ and $GFP(G(b, y))$ denote the least and greatest fixed points of the map $G(b, y) : L \to L$, respectively.

For each such G, and every ordinal κ, let $G^{\kappa\downharpoonleft}(b, y)$ be defined by the following induction: $G^{0\downharpoonleft}(b, y) = y$, $G^{\kappa+1\downharpoonleft}(b, y) = G(b, G^{\kappa\downharpoonleft}(b, y))$ and for λ a limit ordinal, $G^{\lambda\downharpoonleft}(b, y) = \bigwedge_{\kappa<\lambda} G^{\kappa\downharpoonleft}(b, y)$. Also, for a map $F : L \to L$ we define $F^{\kappa\upharpoonleft}(x)$ for all ordinals κ by induction as follows: $F^{0\upharpoonleft}(x) = x$, $F^{\kappa+1\upharpoonleft}(x) = F(F^{\kappa\upharpoonleft}(x))$ and $F^{\lambda\upharpoonleft}(y) = \bigvee_{\kappa<\lambda} F^{\kappa\upharpoonleft}(x)$.

Lemma 2.1. *Let L, M and G be as above.*

1. *If $G : M \times L \to L$ is completely meet-preserving, then the map $g^\kappa : M \to L$ defined by the assignment $b \mapsto G^{\kappa\downharpoonleft}(b, \top)$ is completely meet-preserving for every ordinal κ.*

2. *If $F : L \to M \times L$ is the left adjoint of G, and $F_1 : L \to M$ and $F_2 : L \to L$ are such that $F = (F_1, F_2)$, then F_1 and F_2 are completely join-preserving.*

3. *If F, F_1 and F_2 are as in the previous item, then for every ordinal κ, the left adjoint of g^κ is the map defined by the assignment $a \mapsto F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\upharpoonleft}(a))$.*

4. *If G is completely meet-preserving, then $\nu y.G : M \to L$ is defined everywhere on M, and is completely meet-preserving.*

5. *If G is completely meet-preserving, then the left adjoint of $\nu y.G$ is the map defined by the assignment $a \mapsto F_1(a \vee \mu y.F_2(a \vee y))$.*

Proof. 1. Let $S \subseteq M$. We proceed by induction on κ: we have $G^{1\downharpoonleft}(\bigwedge S, \top) = G(\bigwedge S, \top) = G(\bigwedge\{(s, \top) \mid s \in S\}) = \bigwedge\{G(s, \top) \mid s \in S\} = \bigwedge\{G^{1\downharpoonleft}(s, \top) \mid s \in S\}$, where the penultimate equality holds by the assumption that $G : M \times L \to L$ is completely meet-preserving and the fact that the second coordinate is $\top$.

Assume the claim holds for κ and consider the case for $\kappa + 1$:

$$\begin{array}{rcll} G^{(\kappa+1)\downharpoonleft}(\bigwedge S, \top) & = & G(\bigwedge S, G^{\kappa\downharpoonleft}(\bigwedge S, \top)) & \\ & = & G(\bigwedge S, \bigwedge\{G^{\kappa\downharpoonleft}(s, \top) \mid s \in S\}) & \text{(Induction hypothesis)} \\ & = & G(\bigwedge\{(s, G^{\kappa\downharpoonleft}(s, \top)) \mid s \in S\}) & \\ & = & \bigwedge\{G(s, G^{\kappa\downharpoonleft}(s, \top)) \mid s \in S\} & (G \text{ completely meet-preserving}) \\ & = & \bigwedge\{G^{(\kappa+1)\downharpoonleft}(s, \top) \mid s \in S\}. & \end{array}$$

If λ is a limit ordinal, then

$$
\begin{aligned}
G^{\lambda\downarrow}(\bigwedge S, \top) &= \bigwedge_{\kappa<\lambda} G^{\kappa\downarrow}(\bigwedge S, \top) \\
&= \bigwedge_{\kappa<\lambda} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid s \in S\} \quad \text{(Induction hypothesis)} \\
&= \bigwedge_{s\in S} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid \kappa < \lambda\} \quad \text{(Associativity and commutativity)} \\
&= \bigwedge_{s\in S} G^{\lambda\downarrow}(s, \top).
\end{aligned}
$$

2. Let $S \subseteq L$. The inequality $\bigvee\{F_1(s) \mid s \in S\} \leq F_1(\bigvee S)$ follows immediately from the fact that F is order-preserving and hence F_1 and F_2 are. Conversely, fix $b \in M$ arbitrarily, suppose that $\bigvee\{F_1(s) \mid s \in S\} \leq b$ and let us show that $F_1(\bigvee S) \leq b$:

$$
\begin{aligned}
\bigvee\{F_1(s) \mid s \in S\} \leq b \quad &\text{iff} \quad F_1(s) \leq b \text{ for each } s \in S \\
&\text{iff} \quad F(s) \leq (b, \top) \text{ for each } s \in S \\
&\text{iff} \quad s \leq G(b, \top) \text{ for each } s \in S \\
&\text{iff} \quad \bigvee S \leq G(b, \top) \\
&\text{iff} \quad F(\bigvee S) \leq (b, \top) \\
&\text{iff} \quad F_1(\bigvee S) \leq b.
\end{aligned}
$$

The case for F_2 can be proved similarly.

3. We proceed by induction on κ. If $\kappa = 1$, then for every $a \in L$ and $b \in M$, we have that $a \leq G(b, \top)$ iff $F(a) \leq (b, \top)$ iff $F_1(a) \leq b$ and $F_2(a) \leq \top$ iff $F_1(a) \leq b$, which proves the base case.

Assume the claim holds for κ and consider the case for $\kappa + 1$:

$$
\begin{aligned}
a \leq G^{(\kappa+1)\downarrow}(b, \top) \quad &\text{iff} \quad a \leq G(b, G^{\kappa\downarrow}(b, \top)) \\
&\text{iff} \quad F(a) \leq (b, G^{\kappa\downarrow}(b, \top)) \\
&\text{iff} \quad F_1(a) \leq b \text{ and } F_2(a) \leq G^{\kappa\downarrow}(b, \top) \\
&\text{iff} \quad F_1(a) \leq b \text{ and } F_1(F_2(a) \vee \textstyle\bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(F_2(a))) \leq b \\
&\text{iff} \quad F_1(a) \vee F_1(F_2(a) \vee \textstyle\bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(F_2(a))) \leq b \\
&\text{iff} \quad F_1(a) \vee F_1(F_2(a) \vee \textstyle\bigvee_{2\leq\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b \\
&\text{iff} \quad F_1(a \vee F_2(a) \vee \textstyle\bigvee_{2\leq\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b \\
&\text{iff} \quad F_1(a \vee \textstyle\bigvee_{\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b.
\end{aligned}
$$

Let λ be a limit ordinal and assume that the claim holds for all $\kappa < \lambda$:

$$
\begin{aligned}
a \leq G^{\lambda\downarrow}(b, \top) \quad &\text{iff} \quad a \leq \textstyle\bigwedge_{\kappa<\lambda} G^{\kappa\downarrow}(b, \top) \\
&\text{iff} \quad a \leq G^{\kappa\downarrow}(b, \top) \text{ for every } \kappa < \lambda \\
&\text{iff} \quad F_1(a \vee \textstyle\bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \text{ for every } \kappa < \lambda \\
&\text{iff} \quad \textstyle\bigvee_{\kappa<\lambda} F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \\
&\text{iff} \quad F_1(a \vee \textstyle\bigvee_{\kappa<\lambda} \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \quad \text{(by item 2 above)} \\
&\text{iff} \quad F_1(a \vee \textstyle\bigvee_{\kappa<\lambda} F_2^{\kappa\uparrow}(a)) \leq b.
\end{aligned}
$$

4. Since G is completely meet-preserving, G is monotone in each coordinate. Hence, by the Knaster-Tarski theorem, $\nu y.G$ is everywhere defined. By the general theory of fixed points (see [13]), for all $b \in M$, we have $\nu y.G(b, y) = \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(b, \top)$. Hence,

$$
\begin{aligned}
\nu y.G(\bigwedge S, y) &= \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(\bigwedge S, \top)\\
&= \bigwedge_{\kappa\geq 1} \bigwedge\{G^{\kappa\downarrow}(s,\top) \mid s\in S\} \quad \text{(item 1 above)}\\
&= \bigwedge_{s\in S} \bigwedge\{G^{\kappa\downarrow}(s,\top) \mid \kappa\geq 1\} \quad \text{(Associativity and commutativity)}\\
&= \bigwedge_{s\in S} \nu y.G(s,y).
\end{aligned}
$$

5. For all $a \in L$ and $b \in M$,

$$
\begin{aligned}
a \leq \nu y.G(b,y) \quad &\text{iff} \quad a \leq \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(b,\top)\\
&\text{iff} \quad a \leq G^{\kappa\downarrow}(b,\top) \text{ for every } \kappa\geq 1\\
&\text{iff} \quad F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \text{ for every } \kappa \geq 1 \quad \text{(by item 3 above)}\\
&\text{iff} \quad \bigvee_{\kappa\geq 1} F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b\\
&\text{iff} \quad F_1(a \vee \bigvee_{\kappa\geq 1}\bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \quad \text{(by item 2 above)}\\
&\text{iff} \quad F_1(a \vee \bigvee_{\kappa\geq 1} F_2^{\kappa\uparrow}(a)) \leq b\\
&\text{iff} \quad F_1(a \vee \mu y.F_2(a\vee y)) \leq b.
\end{aligned}
$$

□

Remark 2.2. In the following sections we will use the lemma above with $M = L^\epsilon$ for some order type ϵ over n. In such a setting the map $F_1 : L \to L^\epsilon$ takes the form $(F_{1,1}, \ldots, F_{1,n})$ where $F_{1,i} : L \to L^{\epsilon_i}$ for each $1 \leq i \leq n$. Hence the left adjoint of $\nu y.G(\overline{x}, y) : L^\epsilon \to L$ is the map defined by the assignment $a \mapsto (F_{1,1}(a \vee \mu y.F_2(a \vee y)), \ldots, F_{1,n}(a \vee \mu y.F_2(a \vee y)))$, i.e., for all $a \in L$ and $\overline{b} \in L^\epsilon$,

$$
a \leq \nu y.G(\overline{b}, y) \quad \text{iff} \quad \mathop{\&}_{1\leq i\leq n} F_{1,i}(a \vee \mu y.F_2(a\vee y)) \leq^{\epsilon_i} b_i.
$$

Lemma 2.3. *Let L, M_1 and M_2 be complete lattices.*

1. *If $f : L \to L$ preserves all finite non-empty joins and $g_i : M_i \to L$, $i \in \{1,2\}$, then*

$$
\mu x.[f(x) \vee (g_1(x_1) \vee g_2(x_2))] = \mu x.[f(x) \vee g_1(x_1)] \vee \mu x.[f(x) \vee g_2(x_2)].
$$

2. *If $f : L \to L$ preserves all finite non-empty meets and $g_i : M_i \to L$, $i \in \{1,2\}$, then*

$$
\nu x.[f(x) \wedge (g_1(x_1) \wedge g_2(x_2))] = \nu x.[f(x) \wedge g_1(x_1)] \wedge \nu x.[f(x) \wedge g_2(x_2)].
$$

Proof. We only prove item 1, item 2 being order dual.

$$
\begin{aligned}
&\mu x.[f(x) \vee (g_1(x_1) \vee g_2(x_2))]\\
= \; &\bigvee_{\kappa\geq 0}\left(f^{\kappa+1}(\bot) \vee f^{\kappa}(g_1(x_1) \vee g_2(x_2))\right)\\
= \; &\bigvee_{\kappa\geq 0}\left(f^{\kappa+1}(\bot) \vee f^{\kappa}(g_1(x_1))\right) \vee \bigvee_{\kappa\geq 0}\left(f^{\kappa+1}(\bot) \vee f^{\kappa}(g_2(x_2))\right)\\
= \; &\mu x.[f(x) \vee g_1(x_1)] \vee \mu x.[f(x) \vee g_2(x_2)].
\end{aligned}
$$

□

In applying the lemma above, M_1 and M_2 will typically be powers of L. Accordingly, x_1 and x_2 will tuples of variables which we will write as $\overline{x}_1$ and $\overline{x}_2$.

2.2 The bi-intuitionistic modal mu-language and its semantics

Syntax. Let *AtProp* and *FVar* be disjoint sets of propositional variables and of fixed point variables (the elements of which are respectively denoted by p, q, r and by X, Y, Z). Let x, y, z be general purpose variables, which can be either used as place-holder variables, or as generic variables ranging in $AtProp \cup FVar$. Let us define, by simultaneous recursion,
(a) the set $\mathcal{L}$ of bi-intuitionistic modal mu-formulas[6] over *AtProp* and *FVar*,
(b) their signed (positive or negative) generation trees, and
(c) the set $FV(\varphi)$ of their free variables,
as follows: $\top$ and $\bot$ are bi-intuitionistic modal mu-formulas; their $*$-signed generation trees (for $* \in \{+, -\}$) consist of the single nodes $*\top$ and $*\bot$, respectively, and $FV(\top) = FV(\bot) = \varnothing$. Any $x \in AtProp \cup FVar$ is a bi-intuitionistic modal mu-formula; its $*$-signed generation tree (for $* \in \{+, -\}$) consists of one node, labelled by $*x$, and $FV(x) = \{x\}$. If φ and ψ are modal mu-formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \varphi - \psi, \Box\varphi, \Diamond\varphi$; for $\odot \in \{\wedge, \vee, \Box, \Diamond\}$, their $*$-signed generation tree consists of a root node, labelled by $*\odot$, whose only child (children) is (are) the root(s) of the $*$-signed generation tree(s) of the immediate subformula(s); the $*$-signed generation tree of $\varphi \to \psi$ consists of a root node, labelled by $* \to$, whose only children are the roots of the $*^\partial$-signed generation tree of φ and of the $*$-signed generation tree of ψ (where $*^\partial = +$ if $* = -$, and $*^\partial = -$ if $* = +$); the $*$-signed generation tree of $\varphi - \psi$ consists of a root node, labelled by $*-$, whose only children are the roots of the $*$-signed generation tree of φ and of the $*^\partial$-signed generation tree of ψ; for $\odot \in \{\Box, \Diamond\}$, we let $FV(\odot\varphi) = FV(\varphi)$, and for $\odot \in \{\wedge, \vee, \to, -\}$, we let $FV(\varphi \odot \psi) = FV(\varphi) \cup FV(\varphi)$. If every free occurrence of X in the positive generation tree of φ is labelled positively, then $\mu X.\varphi$ and $\nu X.\varphi$ are modal mu-formulas; for $\odot \in \{\mu X, \nu X\}$, the $*$-signed generation tree of $\odot.\varphi$ consists of a root node, labelled by $*\odot$, whose only child is the root of the $*$-signed generation tree of φ; we let $FV(\odot.\varphi) = FV(\varphi) \setminus \{X\}$. An occurrence of X in φ is *bound* if $X \notin FV(\varphi)$. A *sentence* is a modal mu-formula with no free fixed point variables. The symbol $\varphi(p_1, ..., p_n, X_1, ..., X_m)$ indicates that the propositional variables and free fixed point variables in φ are among $p_1, ..., p_n$ and $X_1, ..., X_m$ respectively; in $\varphi(p_1, ..., p_n, X_1, ..., X_m)$, which we will typically abbreviate as $\varphi(\overline{p}, \overline{X})$, the variables $p_1, ..., p_n, X_1, ..., X_m$ will be understood as pairwise distinct. For modal mu-formulas φ and ψ and $x \in AtProp \cup FVar$, the symbol $\varphi(\psi/x)$ denotes the mu-formula obtained by replacing all free occurrences of x in φ by ψ.

Semantics and the expanded language $\mathcal{L}^+$. The non-fixed point fragment of this language can be interpreted on several types of relational structures such as those described in the Section 1.1; each interpretation yields a different corresponding definition of complex algebra. Irrespective of these differences, the complex algebras of these relational structures are always *perfect modal bi-Heyting algebras* (see Definition 1.1 in Section 1.1). Each operation in such a perfect algebra is either a *residual* or an *adjoint* (see e.g. [11]). The core of the theory presented in this paper can (and will) be developed only on the basis of these properties, hence independently of any particular choice of relational dual semantics.

Term functions are associated with $\mathcal{L}$-formulas in the usual way, see e.g., [4, Definition 10.2]. In particular, as to the interpretation of fixed point binders, if $\varphi(\overline{p}, Y, \overline{X})$ is positive in Y, then its associated term function is monotone in Y and hence, by the Knaster-Tarski theorem, for every given assignment

[6]Henceforth we will sometimes refer to bi-intuitionistic modal mu-formulas as modal mu-formulas, mu-formulas, or simply formulas.

of elements to $\overline{p}$ and $\overline{X}$, the resulting function in Y has a greatest and a least fixed point, which are, respectively, the values for $\nu Y.\varphi(\overline{p}, Y, \overline{X})$ and $\mu Y.\varphi(\overline{p}, Y, \overline{X})$ under the given assignment.

As discussed in [9] for the case of distributive and intuitionistic modal logic, the special properties of perfect (distributive) lattices make it possible to define an interpretation for the following *expanded modal mu-language* $\mathcal{L}^+$, which is built over $AtProp \cup FVar \cup \mathsf{NOM} \cup \mathsf{CNOM}$, where the variables $\mathbf{i}, \mathbf{j} \in \mathsf{NOM}$ (called *nominals*) and $\mathbf{m}, \mathbf{n} \in \mathsf{CNOM}$ (called *co-nominals*) are respectively interpreted in any perfect bi-Heyting algebra $\mathbb{C}$ as elements of $J^\infty(\mathbb{C})$ and of $M^\infty(\mathbb{C})$ (see Section 1.1), additionally closing under the modal operators $\blacklozenge$ and $\blacksquare$ (respectively interpreted in $\mathbb{C}$ as the left adjoint of $\Box^{\mathbb{C}}$ and as the right adjoint of $\Diamond^{\mathbb{C}}$). The $*$-signed generation tree and set $FV(\varphi)$ of any $\mathcal{L}^+$-formula φ are defined as in the case of $\mathcal{L}$-formulas. A formula of $\mathcal{L}^+$ is *pure* if it contains no $p \in AtProp$.

Notational conventions. For every formula φ, let $\neg\varphi$ and $\sim\varphi$ abbreviate $\varphi \to \bot$ and $\top - \varphi$ respectively. An *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order-type ϵ, let ϵ^∂ be its *opposite* order-type, i.e., $\epsilon_i^\partial = 1$ iff $\epsilon_i = \partial$ for every $1 \leq i \leq n$. In what follows we will find it convenient to use the following conventions: we write $\top^1$ and $\top^\partial$ for $\top$ and $\bot$ respectively; likewise, we write $\bot^1$ and $\bot^\partial$ for $\bot$ and $\top$ respectively. Analogous conventions will hold for $\wedge, \vee, \mu, \nu, \leq$; in particular, $\wedge^\partial, \vee^\partial, \mu^\partial, \nu^\partial, \leq^\partial$ will respectively denote $\vee, \wedge, \nu, \mu, \geq$. The exponent in these conventions will typically be a generic ϵ_i for some order-type ϵ. Hence, for instance, $\bot^{\epsilon_i}$ will denote $\bot$ if $\epsilon_i = 1$ and $\top$ if $\epsilon_i = \partial$. Similarly, $\mathbf{j}^{\epsilon_i}$ denotes a nominal if $\epsilon_i = 1$ and a conominal if $\epsilon_i = \partial$, and dually, $\mathbf{n}^{\epsilon_i}$ denotes a conominal if $\epsilon_i = 1$ and a nominal if $\epsilon_i = \partial$. We will use the symbols $\&$, $\parr$, and $\Rightarrow$, interpreted as conjunction, disjunction, and implication, respectively, to combine $\mathcal{L}^+$-inequalities into quasi-inequalities. Given two tuple of variables $\overline{x}$ and $\overline{y}$, denote by $\overline{x} \oplus \overline{y}$ their concatenation.

A glimpse at the first-order correspondence language. Pure formulas can be equivalently translated over the relational semantics (see Section 1.1) via a well known standard translation process, similar to the one defined in [7], see also [9] and [5]. This translation targets the associated first-order correspondence language augmented with least fixed points (see [13]). Since there are many options when it comes to relational dual semantics for non-classical logics of this type, we have chosen not to commit to a specific translation, but to focus only on the reduction process up to the elimination of propositional variables, as this remains invariant, irrespective of the choice of the specific relational semantics. Depending on this choice, the final propositional variable-free clause above will then receive different translations.

2.3 Approximation rules and their soundness

Let ϵ be an order-type on an n-tuple $\overline{x}$. Recall that $\mathbf{j}^{\epsilon_i}$ denotes a nominal if $\epsilon_i = 1$ and a conominal if $\epsilon_i = \partial$. Dually, $\mathbf{n}^{\epsilon_i}$ denotes a conominal if $\epsilon_i = 1$ and a nominal if $\epsilon_i = \partial$. We let $\overline{\mathbf{i}}_i^{\,\epsilon}$ be the n-tuple whose i-th coordinate is $\mathbf{i}^{\epsilon_i}$ and whose j-th coordinate is $\bot^{\epsilon_j}$ for all $j \neq i$. Dually, we let $\overline{\mathbf{n}}_i^{\,\epsilon}$ be the n-tuple whose i-th coordinate is $\mathbf{n}^{\epsilon_i}$ and whose j-th coordinate is $\top^{\epsilon_j}$ for all $j \neq i$.[7]

$$\frac{\mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})}{\parr_{i=1}^{n}(\exists \mathbf{j}^{\epsilon_i}[\mathbf{i} \leq \mu X.\psi(\overline{\mathbf{j}}_i^{\,\epsilon}/\overline{x}, X, \overline{z}) \;\&\; \mathbf{j}^{\epsilon_i} \leq^{\epsilon_i} \varphi_i])}\ (\mu^\epsilon\text{-A})$$

[7] Of course, if we fix a value for $\mathbf{i}^{\epsilon_i}$, then $\overline{\mathbf{i}}_i^{\,\epsilon}$ denotes the element in $J^\infty(\mathbb{A}^\epsilon)$ corresponding to $\mathbf{i}^{\epsilon_i}$ in the i-th coordinate. Dually, $\overline{\mathbf{n}}_i^{\,\epsilon}$ ranges in $M^\infty(\mathbb{A}^\epsilon)$ in an analogous way.

$$\frac{\nu X.\varphi(\overline{\psi}/\overline{x}, X, \overline{z}) \leq \mathbf{m}}{\bigparr_{i=1}^{n}(\exists \mathbf{n}^{\epsilon_i}[\nu X.\varphi(\overline{\mathbf{n}}_i^{\epsilon}/\overline{x}, X, \overline{z}) \leq \mathbf{m} \;\&\; \psi_i \leq^{\epsilon_i} \mathbf{n}^{\epsilon_i}])} \;(\nu^{\epsilon}\text{-A})$$

where

1. in each rule, the tuples $\overline{x}$ and $\overline{z}$ are disjoint, and the variables $\overline{x} \in Var$ do not occur in any formula in $\overline{\psi}$ or in $\overline{\varphi}$;
2. in (μ^{ϵ}-A) the associated term function of $\psi(\overline{x}, X, \overline{z})$ is completely $\bigvee$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\epsilon} \times \mathbb{C}$, for any perfect modal bi-Heyting algebra $\mathbb{C}$;
3. in (ν^{ϵ}-A) the associated term function of $\varphi(\overline{x}, X, \overline{z})$ is completely $\bigwedge$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\epsilon} \times \mathbb{C}$, for any perfect modal bi-Heyting algebra $\mathbb{C}$.

The soundness of (μ^{ϵ}-A) is proven in the following proposition and that of (ν^{ϵ}-A) is dual.

Proposition 2.4. *Let $\psi(\overline{x}, X, \overline{z}), \overline{\varphi} \in \mathcal{L}^{+}$ such that $\overline{x}$ and $\overline{z}$ are disjoint and the term function associated with $\psi(\overline{x}, X, \overline{z})$ is completely $\bigvee$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\epsilon} \times \mathbb{C}$. Let V be an assignment on $\mathbb{C}$. Then the following are equivalent:*

1. *$\mathbb{C}, V \models \mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})$,*
2. *$\mathbb{C}, V' \models \mathbf{i} \leq \mu X.\psi(\overline{\mathbf{j}}_i^{\epsilon}/\overline{x}, X, \overline{z})$ and $\mathbb{C}, V' \models \mathbf{j}^{\epsilon_i} \leq^{\epsilon_i} \varphi_i$ for some $\mathbf{j}^{\epsilon_i}$-variant V' of V, and some $1 \leq i \leq n$.*

Proof. (2) $\Rightarrow$ (1) follows by ϵ-monotonicity. Conversely, assume that $\mathbb{C}, V \models \mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})$. The assumption implies, by the order dual of Lemma 2.1.1 with $M = \mathbb{C}^{\epsilon}$ and $L = \mathbb{C}$, that the term function associated with $\mu X.\psi(\overline{x}, X, \overline{z})$ obtained by fixing $\overline{z}$ according to V is completely join-preserving in $\mathbb{C}^{\epsilon}$. Since $\mathbb{C}$, and hence $\mathbb{C}^{\epsilon}$, is a perfect modal bi-Heyting algebra, we have:

$$\mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z}) = \bigvee\{\mu X.\psi(j, X, \overline{z}) \mid j \in J^{\infty}(\mathbb{C}^{\epsilon}) \;\&\; j \leq \overline{\varphi}\}.$$

Since $V(\mathbf{i}) \in J^{\infty}(\mathbb{C})$, this implies that $V(\mathbf{i}) \leq \mu X.\psi(j_0, X, \overline{z})$ for some $j_0 \in J^{\infty}(\mathbb{C}^{\epsilon})$ such that $j_0 \leq \overline{\varphi}$. Notice that j_0 is an n-tuple which is equal to $\bot^{\mathbb{C}^{\epsilon}}$ except for exactly one coordinate, the i-th say, which is equal to some $j_{0_i} \in J^{\infty}(\mathbb{C}^{\epsilon_i})$. Let V' be the $\mathbf{j}^{\epsilon_i}$-variant of V which sends $\mathbf{j}^{\epsilon_i}$ to $j_{0_i} \in J^{\infty}(\mathbb{C}^{\epsilon_i})$. Then (2) holds under this choice of i and V'. □

Example 2.5. Consider the inequality $\mu X.[{\sim}\nu Y.[(Y \wedge q) \wedge \neg(X \vee {\sim}p)] \wedge \Box\Diamond p] \leq \Diamond(p \wedge \Box\neg q)$. After first approximation we get:

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \mu X.[{\sim}\nu Y.[(Y \wedge q) \wedge \neg(X \vee {\sim}p)] \wedge \Box\Diamond p] \;\&\; \Diamond(p \wedge \Box\neg q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \quad (2.1)$$

The formula on the left-hand side of the inequality is a substitution instance of $\mu X.\psi(p/x, q/y, X, \Box\Diamond p/z)$, where $\psi = {\sim}\nu Y.[(Y \wedge y) \wedge \neg(X \vee {\sim}x)] \wedge z$. The term function associated with ψ is completely join-preserving in (x, y, X) for the order-type $\epsilon = (\partial, 1)$ on (x, y). Hence, we can apply (μ^{ϵ}-A) to the first inequality in the antecedent of the quasi-inequality above, which transforms it into:

$$\begin{array}{rl} & \exists \mathbf{j}^{\epsilon_x}(\mathbf{i} \leq \mu X.[{\sim}\nu Y.[(Y \wedge \bot^{\epsilon_y}) \wedge \neg(X \vee {\sim}\mathbf{j}^{\epsilon_x})] \wedge \Box\Diamond p] \;\&\; \mathbf{j}^{\epsilon_x} \leq^{\epsilon_x} p) \\ \bigparr & \exists \mathbf{j}^{\epsilon_y}(\mathbf{i} \leq \mu X.[{\sim}\nu Y.[(Y \wedge \mathbf{j}^{\epsilon_y}) \wedge \neg(X \vee {\sim}\bot^{\epsilon_x})] \wedge \Box\Diamond p] \;\&\; \mathbf{j}^{\epsilon_y} \leq^{\epsilon_y} q), \end{array}$$

which, recalling that $\epsilon_x = \partial$ and $\epsilon_y = 1$, becomes

$$\begin{array}{rl} & \exists \mathbf{n}_x(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \bot) \wedge \neg(X \vee \sim \mathbf{n}_x)] \wedge \Box\Diamond p] \ \& \ p \leq \mathbf{n}_x) \\ \parr & \exists \mathbf{j}_y(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg(X \vee \sim \top)] \wedge \Box\Diamond p] \ \& \ \mathbf{j}_y \leq q), \end{array}$$

which can be further simplified to

$$\begin{array}{rl} & \exists \mathbf{n}_x(\mathbf{i} \leq \Box\Diamond p \ \& \ p \leq \mathbf{n}_x) \\ \parr & \exists \mathbf{j}_y(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg X] \wedge \Box\Diamond p] \ \& \ \mathbf{j}_y \leq q). \end{array}$$

Substituting the clause above into (2.1), and distributing first & and then ⇒ over $\parr$, we obtain the conjunction of the following two quasi-inequalities, on each of which we can proceed separately:

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}_x[(\mathbf{i} \leq \Box\Diamond p \ \& \ p \leq \mathbf{n}_x \ \& \ \Diamond(p \wedge \Box\neg q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.2}$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_y[(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg X] \wedge \Box\Diamond p] \ \& \ \mathbf{j}_y \leq q \ \& \ \Diamond(p \wedge \Box\neg q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.3}$$

In the quasi-inequality (2.2), the variable q can be eliminated by an application of ($\top$), which transforms $\Diamond(p \wedge \Box\neg q) \leq \mathbf{m}$ into $\Diamond p \leq \mathbf{m}$; now, after applying the adjunction rule ($\Diamond$LA) to the latter inequality, we get

$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}_x[(\mathbf{i} \leq \Box\Diamond p \ \& \ p \leq \mathbf{n}_x \ \& \ p \leq \blacksquare \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape for (LA), applying which yields

$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}_x[\mathbf{i} \leq \Box\Diamond(\mathbf{n}_x \wedge \blacksquare \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

from which all remaining occurrences of propositional variables have been eliminated. The quasi-inequality (2.3) is in Ackermann shape with respect to q; applying (RA) yields:

$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_y[(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg X] \wedge \Box\Diamond p] \ \& \ \Diamond(p \wedge \Box\neg \mathbf{j}_y) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

Applying ($\Diamond$LA) and then ($\wedge$LR) to $\Diamond(p \wedge \Box\neg \mathbf{j}_y) \leq \mathbf{m}$ yields

$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_y[(\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg X] \wedge \Box\Diamond p] \ \& \ p \leq \Box\neg \mathbf{j}_y \rightarrow \blacksquare \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape; applying (LA) yields

$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}_y[\mathbf{i} \leq \mu X.[\sim \nu Y.[(Y \wedge \mathbf{j}_y) \wedge \neg X] \wedge \Box\Diamond(\Box\neg \mathbf{j}_y \rightarrow \blacksquare \mathbf{m})] \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

from which all remaining occurrences of propositional variables have been eliminated.

2.4 Adjunction rules and their soundness

$$\frac{\chi \leq \nu X.\varphi(\overline{\varphi}/\overline{x}, X, \overline{z})}{\&_{i=1}^{n} F_{1,i}(\chi \vee \mu Y.F_2(\chi \vee Y, \overline{z}), \overline{z}) \leq^{\epsilon_i} \varphi_i} \ (\nu^{\epsilon}\text{-Adj})$$

where $\varphi, \overline{\varphi}, \chi \in \mathcal{L}^+$, the arrays of variables $\overline{x}$ and $\overline{z}$ are disjoint, $\overline{x}$ has arity n, the term function associated with $\varphi(\overline{x}, X, \overline{z})$ is a right adjoint in $(\overline{x}, X) \in \mathbb{C}^{\epsilon} \times \mathbb{C}$ for any perfect modal bi-Heyting algebra $\mathbb{C}$, and $F = ((F_{1,i}(y, \overline{z}))_{i=1}^{n}, F_2(y, \overline{z})) : \mathbb{C} \to \mathbb{C}^{\epsilon} \times \mathbb{C}$ is its left adjoint.

$$\frac{\mu X.\psi(\overline{\psi}/\overline{x}, X, \overline{z}) \leq \chi}{\&_{i=1}^{n} \psi_i \leq^{\epsilon_i} G_{1,i}(\chi \wedge \nu Y.G_2(\chi \wedge Y, \overline{z}), \overline{z})} \ (\mu^\epsilon\text{-Adj})$$

where $\psi, \overline{\psi}, \chi \in \mathcal{L}^+$, the arrays of variables $\overline{x}$ and $\overline{z}$ are disjoint, $\overline{x}$ has arity n, the term function associated with $\psi(\overline{x}, X, \overline{z})$ is a left adjoint in $(\overline{x}, X) \in \mathbb{C}^\epsilon \times \mathbb{C}$ for any perfect modal bi-Heyting algebra $\mathbb{C}$, and $G = ((G_{1,i}(y, \overline{z}))_{i=1}^n, G_2(y, \overline{z})) : \mathbb{C} \to \mathbb{C}^\epsilon \times \mathbb{C}$ is its right adjoint.

The next proposition formally states and proves the soundness of (ν^ϵ-Adj). The soundness of the rule (μ^ϵ-Adj) can be proven similarly using an order-dual version of Lemma 2.1.

Proposition 2.6. *Let $\varphi(\overline{x}, X, \overline{z})$, χ, and F be as in (ν^ϵ-Adj). Let $\mathbb{C}$ be a complete modal bi-Heyting algebra and let V be an assignment on $\mathbb{C}$. Then the following are equivalent:*

1. $\mathbb{C}, V \models \chi \leq \nu X.\varphi(\overline{\varphi}/\overline{x}, X, \overline{z})$,
2. $\mathbb{C}, V \models \&_{i=1}^{n} F_{1,i}(\chi \vee \mu Y.F_2(\chi \vee Y, \overline{z}), \overline{z}) \leq^{\epsilon_i} \varphi_i$.

Proof. The statement immediately follows from Lemma 2.1.5 with $M = \mathbb{C}^\epsilon$, $L = \mathbb{C}$, and G the term function $\varphi(\overline{x}, X, V(\overline{z}))$, cf. Remark 2.2. □

2.5 From sematic to syntactic rules

The conditions of applicability of the rules defined in Sections 2.3 and 2.4 are given in terms of the order-theoretic properties of the term functions associated with the argument of the fixed point binder. This makes the present formulation of these rules unsuitable for inclusion in an extended calculus for correspondence, which is supposed to be a purely syntactic tool. This also makes the practical application of these rules very inconvenient, since the order-theoretic properties have to be verified each time. These difficulties are further compounded by the fact that, unlike other approximation and adjunction rules that apply to a single connective at a time, here we need to consider an entire subformula as a whole. Another serious difficulty is posed by the conclusions of the adjunction rules, which give no information as to how the F_i and G_i are to be computed, or whether they are expressible as $\mathcal{L}^+$-term functions at all. It is therefore highly desirable to have syntactic versions of these rules.

In Section 4, a syntactic class of formulas, called the inner formulas, is defined which is shown to verify the assumptions for the applicability of the approximation and adjunction rules. In Section 5, an effective procedure is given for computing the corresponding F_i and G_i as $\mathcal{L}^+$-term functions when the adjunction rules are applied to inner formulas. In Section 7, the inner formulas are used to show that the extended ALBA successfully computes FO+LFP correspondents for all the recursive mu-inequalities as defined Section 3.

3 Recursive mu-inequalities and Sahlqvist mu-formulas

In the present section, the definition of recursive inequalities for the signature of bi-intuitionistic modal mu-calculus is introduced. The style of this definition closely follows that of [9], in that is grounded on a certain classification of the nodes in the signed generation trees of formulas (cf. Table 1). However, one major difference with [9] is that the classification of nodes adopted in the present paper is based on the order-theoretic properties which the operations interpreting the logical connectives enjoy, rather than on those they lack. This is reflected in the names of the groupings in Table 1: SLA, SRA, SLR and SRR stand for syntactically left adjoint, syntactically right adjoint, syntactically left residual and syntactically right residual, respectively. For further discussion see [5, Section 36.7]. In order to establish connections with the model-theoretic analysis conducted in [22], nodes are firstly classified as *inner* and *outer skeleton nodes* and *PIA nodes*, cf. Table 1. This order-theoretic classification is then applied within these categories.

Note that in Table 1 an array of signed connectives wider than that of the language of bi-intuitionistic modal mu-calculus is classified. This serves as a template for extending the definition of ϵ-recursive inequalities to different languages. Specifically, the extra connectives $\circ$, $\star$, $\lhd$, and $\rhd$ serve as generic connectives which respectively are (completely) join-preserving in each coordinate, (completely) meet-preserving in each coordinate, (completely) meet-reversing[8], and (completely) join-reversing. Notice, in particular, that the order-theoretic behaviour of the defined connectives $\sim$ and $\neg$ matches that of $\lhd$ and $\rhd$, respectively, and hence they will be classified in the same way as $\lhd$ and $\rhd$.

3.1 Recursive mu-inequalities

Recall that an order-type over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order-type ϵ, let ϵ^∂ be the *opposite* order-type, i.e., $\epsilon^\partial_i = 1$ iff $\epsilon_i = \partial$ for every $1 \leq i \leq n$.

For any $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$, any order-type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in the signed generation tree of φ is a (leaf) node $+p_i$ with $\epsilon_i = 1$, or $-p_i$ with $\epsilon_i = \partial$. An *ϵ-critical branch* in the tree is a branch terminating in an ϵ-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ϵ-critical nodes are *to be solved for, according to ϵ*. Sometimes, abusing terminology, we will talk about order-types ϵ over arrays $\overline{x}$ of variables, meaning that for every x_i in $\overline{x}$ there is an ϵ_i in ϵ which applies to it.

For every $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$, and every order-type ϵ, we say that $+\varphi$ (resp. $-\varphi$) *agrees with* ϵ, and write $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$), if every leaf node in the signed generation tree of $+\varphi$ (resp. $-\varphi$) which is labelled with a propositional variable is ϵ-critical. In other words, $\epsilon(+\varphi)$ (resp. $\epsilon(-\varphi)$) means that all propositional variable occurrences corresponding to leaves of $+\varphi$ (resp. $-\varphi$) are to be solved for according to ϵ. We will also make use of the *sub-tree relation* $\gamma \prec \varphi$, which extends to signed generation trees, and we will write $\epsilon(\gamma) \prec *\varphi$ to indicate that γ, regarded as a sub- (signed generation) tree of $*\varphi$, agrees with ϵ.

Definition 3.1. Nodes in signed generation trees will be called *skeleton nodes* and *PIA nodes* according to the specification given in Table 1. A branch in a signed generation tree $*\varphi$, for $* \in \{+, -\}$, ending in a propositional variable is an *ϵ-good branch* if, apart from the leaf, it is the concatenation of three paths P_1, P_2, and P_3, each of which may possibly be of length 0, such that P_1 is a path from the leaf consisting only of PIA-nodes, P_2 consists only of inner skeleton-nodes, and P_3 consists only of outer skeleton-nodes. Moreover,

1. The formula corresponding to the uppermost node on P_1 is a mu-sentence.
2. On any SRR-node in P_1 of the form $\gamma \odot \beta$, where β is the side where the branch lies, γ is a mu-sentence and $\epsilon^\partial(\gamma) \prec *\varphi$ (see above for this notation).

[8]For any complete lattices P, Q, a map $f : P \to Q$ is completely join-reversing if $f(\bigvee S) = \bigwedge\{f(s) \mid s \in S\}$ for any $S \subseteq P$, and completely meet-reversing if $f(\bigwedge S) = \bigvee\{f(s) \mid s \in S\}$ for any $S \subseteq P$.

Outer Skeleton	Inner Skeleton	PIA
Δ-adjoints	Binders	Binders
$+\ \vee\ \wedge$	$+\ \mu$	$+\ \nu$
$-\ \wedge\ \vee$	$-\ \nu$	$-\ \mu$
SLR	SLA	SRA
$+\ \Diamond\ \lhd\ \circ\ -$	$+\ \Diamond\ \lhd\ \vee$	$+\ \Box\ \rhd\ \wedge$
$-\ \Box\ \rhd\ \star\ \to$	$-\ \Box\ \rhd\ \wedge$	$-\ \Diamond\ \lhd\ \vee$
	SLR	SRR
	$+\ \wedge\ \circ\ -$	$+\ \vee\ \star\ \to$
	$-\ \vee\ \star\ \to$	$-\ \wedge\ \circ\ -$

Table 1: Skeleton and PIA nodes.

Unravelling the condition $\epsilon^\partial(\gamma) \prec *\varphi$ specifically to the $\mathcal{L}$-signature, we obtain:
a) if $\gamma \odot \beta$ is $+(\gamma \star \beta)$, $+(\gamma \vee \beta)$, $+(\beta \to \gamma)$, or $-(\beta - \gamma)$, then $\epsilon^\partial(+\gamma)$;
b) if $\gamma \odot \beta$ is $+(\gamma \to \beta)$, $-(\gamma \wedge \beta)$, $-(\gamma \circ \beta)$, or $-(\gamma - \beta)$, then $\epsilon(+\gamma)$, i.e., $\epsilon^\partial(-\gamma)$.

3. On any SLR-node in P_2 of the form $\gamma \odot \beta$, where β is the side where the branch lies, γ is a mu-sentence and $\epsilon^\partial(\gamma) \prec *\varphi$ (see above for this notation).
Unravelling the condition $\epsilon^\partial(\gamma) \prec *\varphi$ specifically to the $\mathcal{L}$-signature, we obtain:
a) if $\gamma \odot \beta$ is $-(\gamma \star \beta)$,$-(\gamma \vee \beta)$, $-(\beta \to \gamma)$, or $+(\beta - \gamma)$, then $\epsilon(+\gamma)$, i.e., $\epsilon^\partial(-\gamma)$;
b) if $\gamma \odot \beta$ is $-(\gamma \to \beta)$, $+(\gamma \wedge \beta)$, $+(\gamma \circ \beta)$, or $+(\gamma - \beta)$, then $\epsilon^\partial(+\gamma)$.

Definition 3.2. Given an order-type ϵ, the signed generation tree $*\varphi$, with $* \in \{-, +\}$, of an $\mathcal{L}$-sentence $\varphi(p_1, \ldots p_n)$ is *ϵ-recursive* if every ϵ-critical branch is ϵ-good. Such a signed generation is *non-trivially ϵ-recursive* if contains at least one ϵ-critical branch.

An $\mathcal{L}$-inequality $\varphi \leq \psi$ is *ϵ-recursive* if the signed generation trees $+\varphi$ and $-\psi$ are both ϵ-recursive. An $\mathcal{L}$-inequality $\varphi \leq \psi$ is *recursive* if it is ϵ-recursive for some order-type ϵ.

The signed generation tree $*\varphi$, with $* \in \{-, +\}$, is *ϵ-PIA* if it is ϵ-recursive and all ϵ-critical branches consist only of PIA-nodes. Such a signed generation is *non-trivially ϵ-PIA* if contains at least one ϵ-critical branch.

Example 3.3. The inequality $\nu X.\Box(p \wedge X) \leq p$, corresponding to the formula $\nu X.\Box(p \wedge X) \to p$ from [22, Section 5.3] was discussed at the beginning of Section 2. This inequality is ϵ-recursive for $\epsilon = (1)$ and $\epsilon = (\partial)$. In Section 2 we gave the ALBA-reduction according to $\epsilon = (\partial)$. In Section 4.2 we discuss how to do a reduction according to $\epsilon = (1)$.

Example 3.4. The inequality $\nu X.\neg(p \wedge \neg X) \leq \Diamond\Box p$ is not ϵ-recursive for any order-type ϵ. Indeed, if $\epsilon_p = \partial$ then, on the critical branch in $+\nu X.\neg(p \wedge \neg X)$, the $-\wedge$ is an SRR node which separates the p and the fixed point variable X. If $\epsilon_p = 1$ then the critical branch in $-\Diamond\Box p$ is clearly not good. On the other hand, the unfolding of the fixed point stabilizes after the first step as $\top$, hence the inequality is equivalent to $\top \leq \Diamond\Box p$ which is ϵ-recursive for $\epsilon_p = \partial$. In fact, the first-order definability of $\top \leq \Diamond\Box p$ already follows from the fact that it is monotone in p.

Example 3.5. Consider the inequality

$$\Diamond \mu X.[(p \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge \mu Z.{\sim}(\Box p \wedge \neg Z))) \to \Diamond\Box p]] \leq \Diamond\Box p.$$

This is ϵ-recursive with $\epsilon_p = 1$. Indeed, in the positive generation tree of the left-hand side, there are two critical branches, respectively corresponding to the first and third occurrences of p in the formula, counting from the left. The branch leading from the first is $+p, +\vee, +\vee, +\mu X, +\Diamond$, and partitioning this as $P_1 = \varnothing$, $P_2 = +\vee, +\vee, +\mu X$, and $P_3 = +\Diamond$ satisfies the requirements of Definition 3.2. The branch leading from the third occurrence of p is

$$+p, \underbrace{+\Box, +\wedge, -{\sim}, -\mu Z}_{P_1}, \underbrace{-\wedge, +{\sim}, +\vee, +\Diamond, - \to, -\nu Y, +{\sim}, +\vee, +\mu X}_{P_2}, \underbrace{+\Diamond}_{P_3},$$

and partitioning it as indicated satisfies the requirements of Definition 3.2. In particular, there are no SRR nodes, and the only occurring SLR node is $- \to$, which satisfies condition 3(a) of the definition since $\Diamond\Box p$ is a sentence and $\epsilon^\partial(-\Diamond\Box p)$.

Remark 3.6. Definition 3.2 implies that on a good branch, within P_2 and within P_1, occurrences of nodes $\to$ and $-$ where the branch goes through the child corresponding to the antitone coordinate need to be in strict alternation. This can be seen, e.g., in Example 3.5 in the P_2-part of the displayed branch. This implies that, if we restrict to the signature of intuitionistic modal logic by removing $-$, we would be able to change polarity at most once within the P_2 and P_1 parts of a good branch. Given the further restrictions imposed by Definition 3.1.2 and 3.1.3, this would imply that no good branch could go through the antitone coordinate of $\to$ within the scope of a fixed point binder, thus severely restricting the diversity of order-theoretic behaviour within the resulting class of recursive mu-inequalities. This brings with it the added inconvenience that, when projecting onto the classical setting (see Section 3.2.2 below) we would have to restrict the range to formulas in negation normal form.

3.2 General syntactic shapes and a comparison with existing Sahlqvist-type classes

The aim of the present subsection is to position the ϵ-recursive mu-inequalities with respect to the general syntactic shape of Sahlqvist/Inductive/Recursive inequalities discussed in [5, Subsections 36.6.1 and 36.7.2], and to compare them with the Sahlqvist mu-formulas defined in [22, Definition 3.4].

3.2.1 Recursive mu-inequalities and the general Sahlqvist/Inductive/Recursive shape

In a series of papers including [9], [5], [8] and [10], we have been developing a general, unified theory of correspondence, which is designed to be uniformly applicable across languages. Functional to this purpose, general shapes of Sahlqvist-type conditions on inequalities have been developed, which are based on the order-theoretic properties of the algebraic interpretations of the logical connectives. These shapes are discussed in an intuitive an non-technical way in the survey paper [5, Section 36.7, 36.8.2]. In the discussion below we will therefore make reference to that paper.

It can be straightforwardly checked that the outer-skeleton nodes (see Table 1) of an ϵ-recursive mu-inequality satisfy the same order-theoretic requirements of the nodes of an ϵ-Sahlqvist inequality [5, definitions 36.6.2 and 36.6.3] in which the length of the P_1 paths of ϵ-critical branches is 0. It is also straightforward to see that, in any ϵ-recursive mu-inequality, the ϵ-PIA subtrees are defined in such a way that at most one ϵ-critical branch may pass through any given SRR-node; as discussed in [5, Subsection 36.7.2], this is the defining feature of ϵ-Recursive inequalities across languages. The specific definition of the PIA-subtrees for mu-languages incorporates extra conditions regulating the relative positions of free fixed point variables and ϵ-critical variables in each subtree; as we will see further, these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties, ultimately guaranteeing the applicability of the μ- and ν-adjunction rules.

The inner skeleton essentially arises by the addition of fixed point binders, in the appropriate polarity, to the 'outer skeleton' shape. This introduction blocks the application of the Δ-rules ($\wedge$ RA) and ($\vee$ LA) (and, more generally, also the possibility of applying rules to single connectives), leaving us with only μ- and ν-approximation rules. Hence, in inner skeletons, all the nodes are reclassified according to the properties which they enjoy and which are now relevant. Similar to the PIA-subtrees, the inner-skeleton shape incorporates extra conditions regulating the relative positions of free fixed point variables and ϵ-critical variables; as we will see in the remainder of the paper, these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties guaranteeing the applicability of the μ- and ν-approximation rules.

The shape of ϵ-recursive mu-inequalities provides a uniform 'winning strategy' for the success of ALBA, analogous to the one described for ϵ-inductive and ϵ-Sahlqvist inequalities in [9, Section 10] and [5, Subsection 36.6.1]. Indeed, as we will show in Section 7, the order-type ϵ tells us which occurrences of a given variable we need to solve for so as to reach Ackermann shape, and the ϵ-recursive shape guarantees that this is always possible. Specifically, going down a critical branch, we can surface the PIA-subtree, containing the ϵ-critical occurrences of propositional variables, by means of applications of approximation rules to the skeleton nodes. Then adjunction/residuation rules such as (μ-Adj) and (ν-A) are applied to the PIA-subtrees so as to display the ϵ-critical occurrences, and to simultaneously calculate the minimal valuation for them. Finally, notice that the remaining occurrences of variables are of the opposite order-type: this guarantees that they have the right polarity to receive the calculated minimal valuations, as prescribed by (LA), (RA) or their recursive counterparts. An exhaustive and formal account of this procedure will be given in Section 7.

Finally, as hinted above, notice that the winning strategy outlined so far does not provide information about which version of the Ackermann rule will actually be applied in the reduction procedure. Should we want to guarantee that either (LA) or (RA) will be applied, and not their recursive counterparts, we need to strengthen Definition 3.2 so as to guarantee that, when displaying the critical occurrences in inequalities of the form $\alpha \leq p$ or $p \leq \alpha$, the formula α is p-free. This requirement can be enforced by introducing the (Ω, ϵ)-*inductive mu-inequalities* along the lines of the (Ω, ϵ)-inductive DML/IML inequalities of [9, Definition 3.1]: namely, by imposing a partial ordering Ω upon the variables in Recursive inequalities, and demanding not only that at most one ϵ-critical branch pass through any given SRR-node, but also that if an ϵ-critical branch passes through an SRR-node, all variables occurring on other branches passing through it have to be strictly Ω-smaller than the variable on the critical branch.

3.2.2 Recursive mu-inequalities and Sahlqvist mu-formulas

In [22], the following notions are introduced in the language of classical modal mu-calculus:

Definition 3.7. The class of *PIA formulas* is recursively defined as follows:

$$\varphi ::= p \mid X \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi \mid \nu X.\varphi \mid \neg\pi \vee \varphi,$$

where $p \in AtProp$, $X \in FVar$, and π is a positive sentence. The class of *Sahlqvist mu-formulas* is recursively defined as follows:

$$\chi ::= X \mid \pi \mid \neg\varphi \mid \chi_1 \wedge \chi_2 \mid \Box\chi \mid \nu X.\chi \mid \pi \vee \chi \mid \sigma_1 \vee \sigma_2,$$

where $p \in AtProp$, $X \in FVar$, φ is a PIA sentence, π is a positive sentence, and σ_1 and σ_2 are Sahlqvist mu-formulas which are sentences.

Let us consider the mapping τ from the language of classical modal mu-calculus to the language of bi-intuitionistic modal mu-calculus recursively defined as expected (in particular, $\tau(\neg\xi) := \tau(\xi) \to \bot$); let us also translate formulas as inequalities by the mapping $\tau'(\xi) := \top \leq \tau(\xi)$. Conversely, consider the mapping λ recursively defined as expected on the connectives which have a primitive classical counterpart, and such that:

$$\begin{array}{rcl} \lambda(\xi_1 \to \xi_2) & = & \neg\lambda(\xi_1) \vee \lambda(\xi_2) \\ \lambda(\xi_1 - \xi_2) & = & \lambda(\xi_1) \wedge \neg\lambda(\xi_2) \\ \lambda(\nu X.\xi(X)) & = & \neg\mu X.\neg\xi(\neg X/X). \end{array}$$

Let us also translate bi-intuitionistic inequalities into classical formulas by the mapping $\lambda'(\xi_1 \leq \xi_2) = \neg\lambda(\xi_1) \vee \lambda(\xi_2)$. We omit the proof of the following proposition, which is straightforward but tedious.

Proposition 3.8. *1. Every formula ξ of modal mu-calculus is logically equivalent to $\lambda'(\tau'(\xi))$;*

2. *for every Sahlqvist mu-formula χ, the inequality $\top \leq \tau(\chi)$ is an ϵ-recursive inequality with $\epsilon = \overline{1}$;*

3. *for every $\overline{1}$-recursive inequality $\xi_1 \leq \xi_2$, the formula $\neg\lambda(\xi_1) \vee \lambda(\xi_2)$ is a Sahlqvist mu-formula.*

The analysis of PIA-formulas conducted in [22] can be summarized in the slogan "PIA formulas provide minimal valuations". In this respect, the crucial model-theoretic property possessed by PIA-formulas is the *intersection property*, isolated by van Benthem in [21], which means that a formula, seen as an operation on the complex algebra of a frame, commutes with arbitrary intersections of subsets. The order-theoretic import of this property is clear: a formula has the intersection property iff the term function associated with is completely meet-preserving. In the complete lattice setting in which we find ourselves, this is equivalent to it being a right adjoint; this is exactly the order-theoretic property guaranteeing the soundness of adjunction/residuation rules like (μ-Adj) and (ν-Adj).

4 Inner formulas and their normal forms

As discussed in Section 2.5, the aim of the present section is to introduce and study a class of mu-formulas, the *inner formulas*, the syntactic shape of which guarantees that their associated term functions enjoy the order-theoretic properties which in turn guarantee that the approximation and adjunction rules are systematically applicable to them.

This is the most technically involved section of the paper; for the sake of clarity, it is organized as follows: in Subsection 4.1, inner formulas are defined, and is shown that their associated term functions indeed satisfy the mentioned order-theoretic requirements; in Subsection 4.2, two case studies are discussed, which focus in particular on how to effectively calculate the adjoints of inner formulas; this discussion motivates the introduction, in Subsection 4.3, of the notion of inner formulas in normal form, and its ensuing normalization proposition; finally, in Subsection 4.4, a lemma is proven which provides the effective computation of the adjoints of inner formulas in normal form.

4.1 Inner formulas

Definition 4.1. Let $\overline{y}, \overline{z} \subseteq Var$ and $\overline{X} \subseteq FVar$ be tuples, each consisting of pairwise different variables, such that $\overline{y}$ and $\overline{z}$ are disjoint. Let $\overline{x} = \overline{y} \oplus \overline{X}$ and let δ be an order-type on $\overline{x} = (x_i)_{i=1}^n$. The δ-$\Box$ *and* δ-$\Diamond$ $(\overline{x}, \overline{z})$-*inner formulas* ($(\overline{x}, \overline{z})$-$IF_\delta^\Box$ and $(\overline{x}, \overline{z})$-$IF_\delta^\Diamond$), the free variables of which are contained in $(\overline{x}, \overline{z})$, are given by the following simultaneous recursion (for the sake of readability, the parameters $\overline{x}$ and $\overline{z}$ are omitted):

$$\begin{array}{lcl} \mathrm{IF}_\delta^\Box \ni \varphi & ::= & \top \mid x_i \mid \Box\varphi \mid \varphi_1 \wedge \varphi_2 \mid \nu Y.\varphi' \mid \pi \to \varphi \mid \pi \vee \varphi \mid \psi^c \to \pi \\ \mathrm{IF}_\delta^\Diamond \ni \psi & ::= & \bot \mid x_i \mid \Diamond\psi \mid \psi_1 \vee \psi_2 \mid \mu Y.\psi' \mid \psi - \pi \mid \pi \wedge \psi \mid \pi - \varphi^c \end{array}$$

where

1. $\delta_i = 1$ in the base of the recursion, for $1 \leq i \leq n$,

2. π is $\pi(\overline{z}) \in \mathcal{L}^+$,

3. $\varphi' = \varphi'(\overline{y} \oplus \overline{X}', \overline{z})$ and $\psi' = \psi'(\overline{y} \oplus \overline{X}', \overline{z})$ are $\mathrm{IF}_{\delta'}^\Box$ and $\mathrm{IF}_{\delta'}^\Diamond$, respectively, with $\overline{X}' = \overline{X} \oplus Y$ and $\delta' = \delta \oplus 1$,

4. $\psi^c \in (\overline{x}, \overline{z})$-$\mathrm{IF}_{\delta^\partial}^\Diamond$ and $\varphi^c \in (\overline{x}, \overline{z})$-$\mathrm{IF}_{\delta^\partial}^\Box$.

5. All other formulas have their free variables among $(\overline{x}, \overline{z})$.

With similar side conditions we can define the δ-$\blacksquare$ *and* δ-$\Diamond\blacklozenge$ $(\overline{x}, \overline{z})$-*inner formulas* ($(\overline{x}, \overline{z})$-$IF_\delta^{\Box\blacksquare}$ and $(\overline{x}, \overline{z})$-$IF_\delta^{\Diamond\blacklozenge}$) by the following simultaneous recursion:

$$\begin{array}{lcl} \mathrm{IF}_\delta^{\Box\blacksquare} \ni \varphi & ::= & \top \mid x_i \mid \Box\varphi \mid \blacksquare\varphi \mid \varphi_1 \wedge \varphi_2 \mid \nu Y.\varphi' \mid \pi \to \varphi \mid \pi \vee \varphi \mid \psi^c \to \pi \\ \mathrm{IF}_\delta^{\Diamond\blacklozenge} \ni \psi & ::= & \bot \mid x_i \mid \Diamond\psi \mid \blacklozenge\psi \mid \psi_1 \vee \psi_2 \mid \mu Y.\psi' \mid \psi - \pi \mid \pi \wedge \psi \mid \pi - \varphi^c \end{array}$$

In what follows, the letter φ and ψ (possibly with superscripts or indexes) will denote $\mathrm{IF}^\Box$- and $\mathrm{IF}^\Diamond$-formulas, respectively.

Note that every $\mathrm{IF}^\Box$-formula is an $\mathrm{IF}^{\Box\blacksquare}$-formula and that every $\mathrm{IF}^\Diamond$-formula is a $\mathrm{IF}^{\Diamond\blacklozenge}$-formula.

Remark 4.2. The above definition is tailored to ensure that for any perfect modal bi-Heyting algebra L (cf. Definition 1.1), the term function associated with a $\mathrm{IF}_\delta^{\Box\blacksquare}$ (respectively, $\mathrm{IF}_\delta^{\Diamond\blacklozenge}$) formula is a right (respectively, left) adjoint from $L^\delta \to L$ fixing the variables $\overline{z}$ as parameters (see lemma below).

In particular this requires that in the associated generation tree, on each branch ending in an x_i the nodes corresponding to the negative sides of $\to$ and $-$ are in strict alternation. Moreover, any alternation between $\mathrm{IF}^{\boxslash}$ and $\mathrm{IF}^{\diamondminus}$ is accompanied by a change of polarity. Finally, these considerations imply that, in the signature of intuitionistic modal logic, where the subtraction symbol is removed, change of polarity on these ‘critical’ branches can occur at most once.

Lemma 4.3. *For any perfect modal bi-Heyting algebra $\mathbb{C}$,*

1. *the term function associated with any $IF^{\boxslash}_{\delta}$-formula $\varphi(\overline{x}, \overline{z})$ is completely meet-preserving as a map $\mathbb{C}^{\delta} \to \mathbb{C}$, fixing the variables $\overline{z}$, and*
2. *the term function associated with any $IF^{\diamondminus}_{\delta}$-formula $\psi(\overline{x}, \overline{z})$ is completely join-preserving as a map $\mathbb{C}^{\delta} \to \mathbb{C}$, fixing the variables $\overline{z}$.*

Proof. By simultaneous induction on φ and ψ. The base cases are clear, as are the cases corresponding to the third, fourth and fifth columns in the recursive definition above. The case for φ of the form $\nu Y.\varphi'(\overline{y} \oplus \overline{X}', \overline{z})$ follows by the induction hypothesis and Lemma 2.1.4. Analogously the case for ψ of the form $\mu Y.\psi'(\overline{y} \oplus \overline{X}', \overline{z})$ follows by the induction hypothesis and the order-dual of Lemma 2.1.4. The cases corresponding to the fifth and sixth columns in the recursive definition follow from the induction hypothesis, the fact that $\to$ and $\vee$ are completely meet-preserving in their positive coordinates, while $-$ and $\wedge$ are completely join-preserving in their positive coordinates, and the fact that variables from $\overline{x}$ appear in at most one coordinate of each, which are moreover positive. Similarly, the cases corresponding to the last column follow from the fact that $-$ and $\to$ are respectively completely meet and join-reversing in their negative coordinates, and the fact that variables from $\overline{x}$ appear in at most their negative coordinates. $\square$

4.2 Towards syntactic adjunction rules

The lemma above guarantees that the approximation rules (μ^{δ}-A) and (ν^{δ}-A) can be respectively applied in particular to inequalities of the form $\mathbf{i} \leq \mu X.\psi(\overline{y}, X, \overline{z})$ and $\nu X.\varphi(\overline{y}, X, \overline{z}) \leq \mathbf{m}$, such that $\mu X.\psi$ and $\nu X.\varphi$ are $(\overline{y}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$- and $(\overline{y}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$-*sentences* respectively. For the same reasons, also the general adjunction rules can be applied to inequalities featuring δ-$\Box$ and δ-$\Diamond$ $(\overline{y}, \overline{z})$-inner sentences as main formulas on the appropriate sides. However, the general adjunction rules do not provide any information as to how the adjoint map can be effectively computed as term functions. Indeed, in what follows, we will work towards new adjunction rules which explicitly incorporate such computations. These new rules will be given in terms of a syntactic refinement of inner formulas, introduced in the next subsection. In order to motivate this refinement, it will be useful to consider the following pair of examples.

Consider the inequality $\nu X.\Box(p \wedge X) \leq p$, which we already solved towards the end of Section 1.2. Notice that $\nu X.\Box(x \wedge X)$ is an $(x, \varnothing)$-$\mathrm{IF}^{\Box}_{\delta}$ formula, with $\delta = (1)$. An alternative and more instructive reduction proceeds as follows: after first approximation we get

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \;\&\; p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

Trying to solve for the occurrence of p in $\mathbf{i} \leq \nu X.\Box(p \wedge X)$, we *unfold* the fixed point (see, e.g., [13]) and obtain $\mathbf{i} \leq \bigwedge_{\kappa \geq 1} \Box^{\kappa} p$. This is equivalent to $\mathbf{i} \leq \Box^{\kappa} p$ for every $\kappa \geq 1$. By general adjunction, each such inequality is equivalent to $\blacklozenge^{\kappa}\mathbf{i} \leq p$. Hence we have:

$$\mathbf{i} \leq \bigwedge_{\kappa \geq 1} \Box^{\kappa} p \quad \text{iff} \quad \bigvee_{\kappa \geq 1} \blacklozenge^{\kappa}\mathbf{i} \leq p.$$

Noticing that $\bigvee_{\kappa\geq 1} \blacklozenge^\kappa \mathbf{i}$ is the unfolding of $\mu X.\blacklozenge(X \vee \mathbf{i})$, the quasi-inequality displayed above is equivalent to

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mu X.\blacklozenge(X \vee \mathbf{i}) \leq p) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape and yields

$$\forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(X \vee \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

This example illustrates an effective computation of the left adjoint of an $\mathrm{IF}^{\Box}_{\delta}$-formula, with δ constantly 1. Consider now the analogous computation of the adjoint of an $\mathrm{IF}^{\Box}_{\delta}$-formula, where δ is not constantly 1; for instance the left adjoint of $\nu X.\neg\Diamond(p \vee {\sim} X)$. It is easy to see that, unfolding this fixed point, one gets to a conjunction $\bigwedge_{\kappa\geq 0} \rhd_\kappa(p)$, where for every ordinal κ, the symbol $\rhd_\kappa(p)$ denotes an $\mathcal{L}^+$-term which is (completely) join-reversing in p. Hence, proceeding as we did in the previous computation we obtain:

$$\mathbf{i} \leq \bigwedge_{\kappa\geq 0} \rhd_\kappa p \quad \text{iff} \quad p \leq \bigwedge_{\kappa\geq 0} \blacktriangleright_\kappa \mathbf{i}.$$

The main difference between this clause and the analogous clause displayed in the previous computation is that we are not yet in a position to recognize $\bigwedge_{\kappa\geq 1} \blacktriangleright_\kappa \mathbf{i}$ as the unfolding of some fixed point. In particular, for this, we would need to see the parameter κ explicitly as the exponential $()^\kappa$ applied to some term. This term can be calculated either inductively for each κ, or observing that $\nu X.\neg\Diamond(p \vee {\sim} X) = \nu X.[\neg\Diamond p \wedge \neg\Diamond{\sim} X]$, unfolding which yields $\bigwedge_{\kappa\geq 0}(\neg\Diamond{\sim})^\kappa(\neg\Diamond p)$. Now the displayed clause above becomes:

$$\mathbf{i} \leq \bigwedge_{\kappa\geq 0}(\neg\Diamond{\sim})^\kappa(\neg\Diamond p) \quad \text{iff} \quad \bigvee_{\kappa\geq 0}({\sim}\blacksquare\neg)^\kappa(\mathbf{i}) \leq \neg\Diamond p \quad \text{iff} \quad p \leq \blacksquare\neg \bigvee_{\kappa\geq 0}({\sim}\blacksquare\neg)^\kappa(\mathbf{i}).$$

Notice that the term $\nu X.[\neg\Diamond p \wedge \neg\Diamond{\sim} X]$ which was obtained by distributing $\neg\Diamond$ over $\vee$ can be seen as the result of substituting $\neg\Diamond p$ for x in $\nu X.[x \wedge \neg\Diamond{\sim} X]$, and that the latter is an $\mathrm{IF}^{\Box}_{\delta'}$-formula with δ' constantly 1. This neatly breaks the computation of the adjoint into two steps, the first of which calculates the adjoint of the 'right-side-up' fixed point, and the second composes it with the adjoint of the negative term $\neg\Diamond p$. This is the basic idea underlying the notion of normal forms in the following subsection.

4.3 Normal forms and normalization

Definition 4.4. The *normal* $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$- and $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formulas are given by the same simultaneous recursion as in Definition 4.1, subject to the following additional constraints:

1. if φ is of the form $\nu Y.\varphi'(\overline{x}', \overline{z})$, where $\overline{x}' = \overline{y} \oplus \overline{X}'$ and $\overline{X}' = \overline{X} \oplus Y$, then there exists an $(\overline{y}' \oplus \overline{X}', \overline{z})$-$\mathrm{IF}^{\Box}_{\delta'}$-formula φ'', where δ' is the order-type over $\overline{y}' \oplus \overline{X}'$ which is constantly 1 over $\overline{y}'$ and restricts to δ over $\overline{X}'$, such that $\varphi'(\overline{x}', \overline{z}) = \varphi''(\overline{\varphi}/\overline{y}', \overline{X}', \overline{z})$ where the $\overline{\varphi}$ are normal $(\overline{y}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta''}$-sentences, where δ'' is the restriction of δ to $\overline{y}$.
2. if ψ is of the form $\mu Y.\psi'(\overline{x}', \overline{z})$, where $\overline{x}' = \overline{y} \oplus \overline{X}'$ and $\overline{X}' = \overline{X} \oplus Y$, then there exists an $(\overline{y}' \oplus \overline{X}', \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta'}$-formula ψ'', where δ' is the order-type over $\overline{y}' \oplus \overline{X}'$ which is constantly 1 over $\overline{y}'$ and restricts to δ over $\overline{X}'$, such that $\psi'(\overline{x}', \overline{z}) = \varphi''(\overline{\psi}/\overline{y}', \overline{X}', \overline{z})$ where the $\overline{\psi}$ are normal $(\overline{y}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta''}$-sentences, where δ'' is the restriction of δ to $\overline{y}$.

Lemma 4.5. *1. Every $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-formula φ with $x_1, x_2 \in \overline{x}$ is equivalent to an $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-formula of the form $\varphi_1(\overline{x}_1,\overline{z}) \wedge \varphi_2(\overline{x}_2,\overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$.*

2. *Every $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-formula ψ with $x_1, x_2 \in \overline{x}$ is equivalent to an $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-formula of the form $\psi_1(\overline{x}_1,\overline{z}) \vee \psi_2(\overline{x}_2,\overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$.*

Proof. By simultaneous induction on φ and ψ. The base cases when $\varphi \in \{x, \top\}$ and $\psi \in \{x, \bot\}$ follow by noting that $\varphi \equiv \varphi \wedge \top$ and $\psi \equiv \psi \vee \bot$, respectively. The case $\varphi = \Box\varphi'$ follows by the induction hypothesis and the distributivity of $\Box$ over $\wedge$. The case $\varphi = \varphi_1 \wedge \varphi_2$ follows by the induction hypothesis and associativity and commutativity of $\wedge$. The case $\varphi = \psi^c \to \pi$ follows by the induction hypothesis on the IF$^{\Diamond}_{\delta^\partial}$-formula ψ^c and the fact that $\to$ turns $\vee$ into $\wedge$ in its first coordinate. Consider the case $\varphi = \nu Y.\varphi'(\overline{x}',\overline{z})$ with $\overline{x}' = \overline{x} \oplus Y$ and $\delta' = \delta \oplus 1$. By induction hypothesis $\varphi'(\overline{x}',\overline{z}) \equiv \varphi_1(\overline{x}_1 \oplus Y,\overline{z}) \wedge \varphi_2(\overline{x}_2 \oplus Y,\overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$. By applying the induction hypothesis again to $\varphi_1(\overline{x}_1 \oplus Y,\overline{z})$ (w.r.t. x_1 and Y) and $\varphi_2(\overline{x}_2 \oplus Y,\overline{z})$ (w.r.t. x_2 and Y) we obtain

$$
\begin{aligned}
& \varphi'(\overline{x}',\overline{z}) \\
\equiv\ & [\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_1''(\overline{x}_1,\overline{z})] \wedge [\varphi_2'(\overline{x}_2' \oplus Y,\overline{z}) \wedge \varphi_2''(\overline{x}_2,\overline{z})] \\
\equiv\ & [\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z})] \wedge [\varphi_1''(\overline{x}_1,\overline{z}) \wedge \varphi_2''(\overline{x}_2,\overline{z})],
\end{aligned}
$$

where $\overline{x}_1' = \overline{x}_1 \setminus \{x_1\}$ and $\overline{x}_2' = \overline{x}_2 \setminus \{x_2\}$. Note that $\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z})$ is an $(\overline{x} \oplus Y,\overline{z})$-IF$^{\Box}_{\delta'}$-formula, and hence, by Lemma 4.3, it is a right adjoint in $\overline{x} \oplus Y$. Therefore, it preserves non-empty joins in Y. Hence, by Lemma 2.3, applied to $f(Y) = \varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z})$, $g_1(\overline{x}_1) = \varphi_1''(\overline{x}_1,\overline{z})$, and $g_2(\overline{x}_2) = \varphi_2''(\overline{x}_2,\overline{z})$, we have

$$
\begin{aligned}
& \nu Y.\varphi'(\overline{x}',\overline{z}) \\
\equiv\ & \nu Y.([\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z})] \wedge [\varphi_1''(\overline{x}_1,\overline{z}) \wedge \varphi_2''(\overline{x}_2,\overline{z})]) \\
\equiv\ & \nu Y.[\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z}) \wedge \varphi_1''(\overline{x}_1,\overline{z})] \\
\wedge\ & \nu Y.[\varphi_1'(\overline{x}_1' \oplus Y,\overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y,\overline{z}) \wedge \varphi_2''(\overline{x}_2,\overline{z})],
\end{aligned}
$$

where x_2 does not occur in the first conjunct, and x_1 does not occur in the second.

The other cases are analogous and are left to the reader. □

By repeated application of the lemma above we obtain the following Corollary:

Corollary 4.6. *1. Every $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-formula φ is equivalent to an $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-formula of the form $\varphi_1(\overline{x}_1,\overline{z}) \wedge \varphi_2(\overline{x}_2,\overline{z})$, where $\overline{x}_1$, $\overline{x}_2$ form a partition of $\overline{x}$.*

2. *Every $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-formula ψ is equivalent to an $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-formula of the form $\psi_1(\overline{x}_1,\overline{z}) \vee \psi_2(\overline{x}_2,\overline{z})$, where $\overline{x}_1$, $\overline{x}_2$ form a partition of $\overline{x}$.*

Proposition 4.7. *Every IF$^{*}_{\delta}$ formula, $* \in \{\Diamond, \Box\}$, is equivalent to an IF$^{*}_{\delta}$ formula in normal form.*

Proof. Notice that if a $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-formula φ is non-normal, it must contain a subformula of the form $\nu Y.\varphi'$ which violates Definition 4.4.1, and where φ' is an $(\overline{y} \oplus \overline{X} \oplus Y,\overline{z})$-IF$^{\Box}_{\delta'}$-formula. If in φ', every variable $y \in \overline{y}$ occurs only positively, the trivial substitution given by the identity on $\overline{y}$ in φ' itself would witness the normality. This means that, in the positive generation tree $+\varphi'$, there is at least one leaf $-y$ with $y \in \overline{y}$. I.e., on the branch from each such $-y$ to the root there is an odd number of order-reversing nodes which, as per Definition 4.1, need to be positive occurrence of $\to$ and negative occurrences of $-$, in strict alternation. Thus the first order-reversing node above each such leaf $-y$ is a positive occurrence of $\to$. By the assumption that φ is not in normal form, it follows that at least one of the subformulas rooted at such a node $+\to$ is not a sentence. (Indeed, if all the subformulas $\zeta = \psi^c \to \pi$ rooted at such nodes were sentences, then replacing each of them with a fresh variable $y' \in \overline{y}'$ would give us the required formula φ'' of Definition 4.4.1). Let a *defect* of φ be an occurrence of a $+\to$ node in the scope of a $+\nu Y$ node in $+\varphi$ such that the corresponding subformula $\zeta = \psi^c \to \pi$ is not a sentence and contains negative occurrences of variables in $\overline{y}$. Dually, we define a defect of a $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Diamond$-formula ψ as a positive occurrence of $-$ in the scope of a $+\mu Y$ node in $+\psi$ such that the corresponding subformula $\zeta = \pi - \varphi^c$ is not a sentence and contains negative occurrences of variables in $\overline{y}$.

The proof now proceeds by induction on the set $(\mathsf{defect}(\chi), \chi)$ ordered lexicographically, where χ is an inner formula and $\mathsf{defect}(\chi)$ is the number of defects occurring in χ. The base case is trivial. As for the induction step, we proceed by cases depending on the form of χ. If the main connective of χ is not a fixed point binder, then the claim follows by the induction hypothesis applied to the immediate subformulas. Now suppose φ is a $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$-formula of the form $\nu Y.\varphi'$. Let $\zeta = \psi^c \to \pi$ be a defect of φ. Since, by Definition 4.1, π must be a sentence, all the free variables of ζ occur only in the $\mathrm{IF}_{\delta'\partial}^\Diamond$-formula $\psi^c(\overline{y} \oplus \overline{X} \oplus Y, \overline{z})$. By Corollary 4.6.2, the formula ψ^c is equivalent to one of the form $\psi_1(\overline{y}, \overline{z}) \vee \psi_2(\overline{X} \oplus Y, \overline{z})$, where ψ_1 and ψ_2 are $\mathrm{IF}_{\delta'\partial}^\Diamond$-formulas. Hence, ζ is equivalent to — and hence can be replaced by — $(\psi_1(\overline{y}, \overline{z}) \to \pi) \wedge (\psi_2(\overline{X} \oplus Y) \to \pi)$. Let φ'' be the formula resulting from this replacement in φ. Notice that $(\psi_1(\overline{y}, \overline{z}) \to \pi)$ is an $\mathrm{IF}_{\delta''}^\Box$-sentence, where δ'' is the restriction of δ to $\overline{y}$, and hence, within φ'', no subformula of $(\psi_1(\overline{y}, \overline{z}) \to \pi) \wedge (\psi_2(\overline{X} \oplus Y) \to \pi)$ constitutes a defect. Hence φ'' has at least one defect less that φ, so by the inductive hypothesis φ'', and hence φ, is equivalent to a $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$ formula in normal form. □

Remark 4.8. We observe that an effective procedure for transforming any inner formula into an equivalent one in normal form can be extracted from the proof of Proposition 4.7. In Section 7 we will exploit the fact that such a procedure exists, although we will not describe it in any further detail, limiting ourselves to illustrate it by means of the examples below.

Example 4.9. The formula $\nu X.\neg(x \vee {\sim}X)$ is an $(x, \varnothing)$-$\mathrm{IF}_\delta^\Box$-formula for $\delta = (\partial)$, and it is not in normal form, the subformula $\neg(x \vee {\sim}X)$ being its only defect. The normalization procedure on this subformula amounts to distributing $\neg$ over $\vee$, so as to obtain $\nu X.[\neg x \wedge \neg{\sim}X]$, which is in normal form: indeed, the latter is a substitution instance of $\nu X.[y' \wedge \neg{\sim}X]$ which is a $(y', \varnothing)$-$\mathrm{IF}_{\delta'}^\Box$ with $\delta' = (1)$; moreover, y' has been substituted for the $\mathrm{IF}^\Box$ sentence $\neg x$.

Example 4.10. The formula $\nu X.\Box(X \wedge \neg\mu Y.\Diamond({\sim}X \vee (Y \vee x)))$ is an $(x, \varnothing)$-$\mathrm{IF}_\delta^\Box$ formula for $\delta = (\partial)$, and it is not in normal form, the subformula $\neg\mu Y.\Diamond({\sim}X \vee (Y \vee x))$ being its only defect. The normalization procedure on this subformula involves surfacing the innermost $\vee$ node, by applying associativity of $\vee$ and distributivity of $\Diamond$ over $\vee$, so as to obtain $\neg\mu Y.\Diamond Y \vee (\Diamond{\sim}X \vee \Diamond x)$, to which Lemma 2.3.1 applies with $f(Y) := \Diamond Y$, $g_1(X) := \Diamond{\sim}X$ and $g_2(x) := \Diamond x$, yielding

$$\neg\mu Y.[\Diamond Y \vee (\Diamond{\sim}X \vee \Diamond x)] = \neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge \neg\mu Y.[\Diamond Y \vee \Diamond x].$$

Hence the original formula can be equivalently rewritten as

$$\nu X.\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge \neg\mu Y.[\Diamond Y \vee \Diamond x])),$$

which is in normal form: indeed, it is a substitution instance of the formula $\nu X.\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))$ which is a $(y', \varnothing)$-$\mathrm{IF}^{\Box}_{\delta'}$ with $\delta' = (1)$; moreover, y' has been substituted for the $\mathrm{IF}^{\Box}$-sentence $\neg\mu Y.[\Diamond Y \vee \Diamond x]$.

4.4 Computing the adjoints of normal inner formulas

By Lemma 4.3 we know that the term functions associated with $\mathrm{IF}^{\Box}$- and $\mathrm{IF}^{\Diamond}$-formulas are completely meet- and join-preserving, respectively. In the setting of perfect bi-Heyting algebras this implies that they have left- and right-adjoints, respectively. In the present section we are going to show that these adjoints can be represented, componentwise, as term functions of $\mathrm{IF}^{\blacklozenge}$- and $\mathrm{IF}^{\blacksquare}$-formulas. In fact, in the following lemma we will effectively construct these term functions. To this end, we need to introduce the following notation:

Definition 4.11. For any formula $\varphi(\overline{x}, Y, \overline{z})$ we define $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonleft}(\overline{x}, y, \overline{z})$ for every ordinal κ, as follows: $\varphi^{0\upharpoonright}(\overline{x}, y, \overline{z}) = y = \varphi^{0\downharpoonleft}(\overline{x}, \bot, \overline{z})$. Assuming that $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonleft}(\overline{x}, y, \overline{z})$ have been defined, we let $\varphi^{(\kappa+1)\upharpoonright}(\overline{x}, y, \overline{z}) = \varphi(\overline{x}, \varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z}), \overline{z})$ and $\varphi^{(\kappa+1)\downharpoonleft}(\overline{x}, y, \overline{z}) = \varphi(\overline{x}, \varphi^{\kappa\downharpoonleft}(\overline{x}, y, \overline{z}), \overline{z})$. Assuming that $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonleft}(\overline{x}, y, \overline{z})$ have been defined for every $\kappa < \lambda$, where λ is a limit ordinal, we let $\varphi^{\lambda\upharpoonright}(\overline{x}, y, \overline{z}) = \bigvee_{\kappa<\lambda} \varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\lambda\downharpoonleft}(\overline{x}, y, \overline{z}) = \bigwedge_{\kappa<\lambda} \varphi^{\kappa\downharpoonleft}(\overline{x}, y, \overline{z})$.

Lemma 4.12. *Let $\varphi(\overline{x}, \overline{z})$ and $\psi(\overline{x}, \overline{z})$, respectively, be an $(\overline{x}, \overline{z})$-$IF^{\Box}_{\delta}$ and an $(\overline{x}, \overline{z})$-$IF^{\Diamond}_{\delta}$-formula in normal form, where the arity of $\overline{x}$ is n. Then:*

1. *there exists an n-array $\overline{\psi}(u, \overline{z})$, where u is a fresh variable, given componentwise by $\mathcal{L}^+$-formulas $\psi_i(u, \overline{z})$ for any $1 \leq i \leq n$, such that*
   (a) *for every $1 \leq i \leq n$, the formula $\psi_i(u, \overline{z})$ is an IF_η-formula with η the order type over 1 with $\eta_1 = \delta_i$, and moreover $\psi_i(u, \overline{z})$ is an $IF^{\blacklozenge}_{\eta}$-formula if $\delta_i = 1$ and an $IF^{\blacksquare}_{\eta}$-formula if $\delta_i = \partial$;*
   (b) *in any perfect modal bi-Heyting algebra $\mathbb{C}$ and for all $\overline{a}, b, \overline{c} \in \mathbb{C}$*

$$b \leq \varphi(\overline{a}, \overline{c}) \quad \textit{iff} \quad \overline{\psi}(b, \overline{c}) \leq^{\delta} \overline{a} \quad \textit{iff} \quad \mathop{\&}_{i=1}^{n} \psi_i(b, \overline{c}) \leq^{\delta_i} a_i.$$

2. *there exists an n-array $\overline{\varphi}(u, \overline{z})$, where u is a fresh variable, given componentwise by $\mathcal{L}^+$-formulas $\varphi_i(u, \overline{z})$ fo any $1 \leq i \leq n$, such that*
   (a) *for every $1 \leq i \leq n$, the formula $\varphi_i(u, \overline{z})$ is an IF_η-formula with η the order type over 1 with $\eta_1 = \delta_i$, and moreover $\varphi_i(u, \overline{z})$ is an $IF^{\blacksquare}_{\eta}$-formula if $\delta_i = 1$ and an $IF^{\blacklozenge}_{\eta}$-formula if $\delta_i = \partial$;*
   (b) *in any perfect modal bi-Heyting algebra $\mathbb{C}$ and for all $\overline{a}, b, \overline{c} \in \mathbb{C}$*

$$\psi(\overline{a}, \overline{c}) \leq b \quad \textit{iff} \quad \overline{a} \leq^{\delta} \overline{\varphi}(b, \overline{c}) \quad \textit{iff} \quad \mathop{\&}_{i=1}^{n} a_i \leq^{\delta_i} \varphi_i(b, \overline{c}).$$

Proof. Fix $\overline{a}, b, \overline{c} \in \mathbb{C}$. The proof proceeds by simultaneous induction on φ and ψ. As to the base cases: if φ is $\top$, the the claim holds if we let $\psi_j = \bot$ for every $1 \leq j \leq n$. Dually, if ψ is $\bot$, then the claim holds if we let $\varphi_j = \top$ for every $1 \leq j \leq n$. If φ is x_j for some $1 \leq j \leq n$ such that $\delta_j = 1$, then the claim holds if we let ψ_j be equal to the variable u, and ψ_i be the constant $\bot^{\delta_i}$ for $i \neq j$. Similarly, if

ψ is x_j for some $1 \leq j \leq n$ for which $\delta_j = 1$, then the claim holds if we let φ_j be equal to the variable u, and φ_i be the constant $\top^{\delta_i}$ for $i \neq j$.

If φ is of the form $\varphi^{(1)}(\overline{x}, \overline{z}) \wedge \varphi^{(2)}(\overline{x}, \overline{z})$ we let $\psi_i = \psi_i^{(1)}(u, \overline{z}) \vee^{\delta_i} \psi_i^{(2)}(u, \overline{z})$ for $1 \leq i \leq n$. Indeed, we have

$$
\begin{array}{rll}
b \leq \varphi^{(1)}(\overline{a}, \overline{c}) \wedge \varphi^{(2)}(\overline{a}, \overline{c}) & \text{iff} & b \leq \varphi^{(j)}(\overline{a}, \overline{c}),\ j = 1, 2 \\
& \text{iff} & \psi_i^{(j)}(b, \overline{c}) \leq^{\delta_i} a_i,\ j = 1, 2,\ 1 \leq i \leq n \\
& \text{iff} & \psi_i^{(1)}(b, \overline{c}) \vee^{\delta_i} \psi_i^{(2)}(b, \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n \\
& \text{iff} & \overline{\psi}(b, \overline{c}) \leq^{\delta} \overline{a}.
\end{array}
$$

Moreover, if $\delta_i = 1$, then by the inductive hypothesis $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are $\mathrm{IF}_\eta^{\text{⬙}}$-formulas with $\eta = (1)$, and hence $\psi_i^{(1)}(u, \overline{z}) \vee^{\delta_i} \psi_i^{(2)}(u, \overline{z})$ is $\psi_i^{(1)}(u, \overline{z}) \vee \psi_i^{(2)}(u, \overline{z})$ which is an $\mathrm{IF}_\eta^{\text{⬙}}$-formula. If $\delta_i = \partial$, then by the inductive hypothesis $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are $\mathrm{IF}_\eta^{\text{◩}}$-formulas with $\eta = (\partial)$, and hence $\psi_i^{(1)}(u, \overline{z}) \vee^{\delta_i} \psi_i^{(2)}(u, \overline{z})$ is $\psi_i^{(1)}(u, \overline{z}) \wedge \psi_i^{(2)}(u, \overline{z})$ which is an $\mathrm{IF}_\eta^{\text{◩}}$-formula.

If φ is of the form $\Box\varphi'(\overline{x}, \overline{z})$ we let $\psi_i = \psi_i'(\blacklozenge u/u, \overline{z})$ for $1 \leq i \leq n$. Indeed, we have

$$
\begin{array}{rll}
b \leq \Box\varphi'(\overline{a}, \overline{c}) & \text{iff} & \blacklozenge b \leq \varphi'(\overline{a}, \overline{c}) \\
& \text{iff} & \psi_i'(\blacklozenge b, \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n.
\end{array}
$$

Moreover, if $\delta_i = 1$, then by the inductive hypothesis $\psi_i'(u, \overline{z})$ is an $\mathrm{IF}_\eta^{\text{⬙}}$-formula with $\eta = (1)$, and then, using Definition 4.1, it is not difficult to show that $\psi_i'(\blacklozenge u/u, \overline{z})$ is an $\mathrm{IF}_\eta^{\text{⬙}}$-formula. If $\delta_i = \partial$, then by the inductive hypothesis $\psi_i'(u, \overline{z})$ is an $\mathrm{IF}_\eta^{\text{◩}}$-formula with $\eta = (\partial)$, and then, using Definition 4.1, it is not difficult to show that $\psi_i'(\blacklozenge u/u, \overline{z})$ is an $\mathrm{IF}_\eta^{\text{◩}}$-formula.

Let φ be of the form $\nu Y.\varphi''(\overline{x}', \overline{z})$ with $\overline{x}' = \overline{y} \oplus \overline{X} \oplus Y$, where m and k are the lengths of $\overline{y}$ and $\overline{X}$, respectively. Let $\delta(1)$ and $\delta(2)$ be the restrictions of δ to $\overline{y}$ and $\overline{X}$, respectively. By normality we can assume that $\varphi'' = \varphi'(\overline{\varphi}/\overline{y}', \overline{X}, Y, \overline{z})$, where $\overline{y}'$ is an ℓ-tuple of variables, $\varphi'(\overline{y}' \oplus \overline{X} \oplus Y, \overline{z})$ is an $\mathrm{IF}_{\delta'}^{\Box}$-formula with δ' constantly 1 on $\overline{y}'$ and Y, and restricting to δ on $\overline{X}$, and with $\overline{\varphi} = (\varphi_1, \ldots, \varphi_\ell)$ and φ_j a $(\overline{y}, \overline{z})$-$\mathrm{IF}_\delta^{\Box}$ sentence for every $1 \leq j \leq \ell$. Let $\overline{a}, b, \overline{c} \in \mathbb{C}$ be fixed as above. Let $\overline{a}' = (\varphi_1(\overline{a}, \overline{c}), \ldots, \varphi_\ell(\overline{a}, \overline{c})) \oplus \top^{\delta(2)}$. Then by induction hypothesis on $\varphi'(\overline{y}' \oplus \overline{X} \oplus Y, \overline{z})$, we have formulas $\psi_1'(y, \overline{z}), \ldots, \psi_\ell'(y, \overline{z})$ and $\psi_1''(y, \overline{z}), \ldots, \psi_{k+1}''(y, \overline{z})$ such that

$$
b \leq \varphi'(\overline{a}', \top, \overline{c}) \quad \text{iff} \quad \begin{cases} \psi_j'(b, \overline{c}) \leq \varphi_j(\overline{a}, \overline{c}) & \text{for } 1 \leq j \leq \ell, \quad \text{and} \\ \psi_h''(b, \overline{c}) \leq^{\delta(2)_h} \top^{\delta(2)_h} & \text{for } 1 \leq h \leq k, \quad \text{and} \\ \psi_{k+1}''(b, \overline{c}) \leq \top. & \end{cases}
$$

Moreover, $\psi_{k+1}''(y, \overline{z})$ and $\psi_j'(y, \overline{z})$ for $1 \leq j \leq \ell$ are $\mathrm{IF}_\eta^{\text{⬙}}$-formulas with $\eta = (1)$.

In the following calculation we will abuse notation and write $\varphi^\kappa(\overline{a}', \top, \overline{c})$ for $(\varphi(\overline{a}', w, \overline{c}))^\kappa[\top/w]$, and $\psi'^\kappa(b, \overline{c})$ for $(\psi'(u, \overline{c}))^\kappa[b/u]$.

$$
\begin{array}{lll}
 & b \leq \nu Y.\varphi''(\overline{a}, Y, \overline{c}) & \\
\text{iff} & b \leq \nu Y.\varphi'(\overline{a}', Y, \overline{c}) & \\
\text{iff} & b \leq \bigwedge_{\kappa\geq 0} \varphi'^{\kappa\downharpoonleft}(\overline{a}', \top, \overline{c}) & \text{(Definition 4.11)} \\
\text{iff} & b \leq \varphi'^{\kappa\downharpoonleft}(\overline{a}', \top, \overline{c}) \text{ for all } \kappa \geq 0 & \\
\text{iff} & \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell \text{ and all } \kappa \geq 0 & \text{(Lemma 2.1.3)} \\
\quad\text{and} & \psi'_h(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^{\delta(2)_h} \top^{\delta(2)_h} \text{ for all } 1 \leq h \leq k \text{ and all } \kappa \geq 0 & \\
\quad\text{and} & \psi'_{k+1}(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \top, \text{ for all } \kappa \geq 0 & \\
\text{iff} & \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell \text{ and all } \kappa \geq 0 & \\
\text{iff} & \bigvee_{\kappa\geq 0} \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell & \\
\text{iff} & \psi'_j(b \vee \bigvee_{\kappa\geq 0} \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell & (*) \\
\text{iff} & \psi'_j(b \vee \bigvee_{\kappa\geq 0} \psi''^{\kappa'\upharpoonright}_{k+1}(b, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell & \\
\text{iff} & \psi'_j(b \vee \mu Y.\psi''_{k+1}(b \vee Y, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \text{ for all } 1 \leq j \leq \ell &
\end{array}
$$

To see that the starred equivalence holds, recall that $\psi'_j(u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta}$-formula with $\eta = (1)$, hence by Lemma 4.3 its associated term function is completely join-preserving in $\mathbb{C}$. Applying the induction hypothesis to φ_j, we obtain formulas $\psi_{(j,i)}(u, \overline{z})$, for $1 \leq i \leq n$, such that

$$
\begin{array}{ll}
 & \psi'_j(b \vee \mu Y.\psi''_{k+1}(b \vee Y, \overline{c}), \overline{c}) \leq^1 \varphi_j(\overline{a}, \overline{c}) \\
\text{iff} & \psi_{(j,i)}(\psi'_j(b \vee \mu Y.\psi''_{k+1}(b \vee Y, \overline{c}), \overline{c}), \overline{c}) \leq^{\delta_i} a_i, \text{ for all } 1 \leq i \leq n.
\end{array}
$$

This shows that, for every $1 \leq i \leq n$, we can take $\psi_i(u, \overline{z})$ to be

$$
\bigvee\nolimits^{\delta_i} \left\{ \psi_{(j,i)}(\psi'_j(u \vee \mu Y.\psi''_{k+1}(u \vee Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell \right\}, \tag{4.1}
$$

which proves part (b) of the claim. As to part (a), we begin by recalling that $\psi''_{k+1}(y, \overline{z})$ and $\psi'_j(y, \overline{z})$ for $1 \leq j \leq \ell$ are $\mathrm{IF}^{\Diamond}_{\eta}$-formulas with $\eta = (1)$. Hence $\psi'_j(u \vee \mu Y.\psi''_{k+1}(u \vee Y, \overline{z}), \overline{z})$ is also an $\mathrm{IF}^{\Diamond}_{\eta}$-formula. By the induction hypothesis applied to φ_j, each formula $\psi_{(j,i)}(u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta'}$-formula with $\eta' = (1)$ if $\delta_i = 1$, or an $\mathrm{IF}^{\Box}_{\eta'}$-formula with $\eta' = (\partial)$ if $\delta_i = \partial$. Therefore, reasoning about $\bigvee^{\delta_i}$ in a way analogous to the inductive step for $\wedge$ above, we see that (4.1) is an $\mathrm{IF}^{\Diamond}_{\eta'}$-formula with $\eta' = (1)$ if $\delta_i = 1$, or an $\mathrm{IF}^{\Box}_{\eta'}$-formula with $\eta' = (\partial)$ if $\delta_i = \partial$.

If φ is of the form $\pi(\overline{z}) \to \varphi'(\overline{x}, \overline{z})$, we let $\psi_i = \psi'_i((u \wedge \pi(\overline{z}))/u, \overline{z})$. Indeed, we have

$$
\begin{array}{rll}
b \leq \pi(\overline{c}) \to \varphi'(\overline{a}, \overline{c}) & \text{iff} & b \wedge \pi(\overline{c}) \leq \varphi'(\overline{a}, \overline{c}) \\
 & \text{iff} & \psi'_i(b \wedge \pi(\overline{c}), \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n.
\end{array}
$$

Moreover, if $\delta_i = 1$, then by the inductive hypothesis $\psi'_i(u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta}$-formula with $\eta = (1)$, and then, using Definition 4.1, it is not difficult to show that $\psi'_i((u \wedge \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta}$-formula. If $\delta_i = \partial$, then by the inductive hypothesis $\psi'_i(u, \overline{z})$ is an $\mathrm{IF}^{\Box}_{\eta}$-formula with $\eta = (\partial)$, and then, using Definition 4.1, it is not difficult to show that $\psi'_i((u \wedge \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\Box}_{\eta}$-formula.

If φ is of the form $\pi(\overline{z}) \vee \varphi'(\overline{x}, \overline{z})$, we let $\psi_i = \psi'_i((u - \pi(\overline{z}))/u, \overline{z})$. Indeed, we have

$$
\begin{array}{rll}
b \leq \pi(\overline{c}) \vee \varphi'(\overline{a}, \overline{c}) & \text{iff} & b - \pi(\overline{c}) \leq \varphi'(\overline{a}, \overline{c}) \\
 & \text{iff} & \psi'_i(b - \pi(\overline{c}), \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n.
\end{array}
$$

Moreover, if $\delta_i = 1$, then by the inductive hypothesis $\psi'_i(u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta}$-formula with $\eta = (1)$, and then, using Definition 4.1, it is not difficult to show that $\psi'_i((u - \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\eta}$-formula. If $\delta_i = \partial$, then

by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\text{◩}}_{\eta}$-formula with $\eta = (\partial)$, and then, using Definition 4.1, it is not difficult to show that $\psi'_i((u - \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\text{◩}}_{\eta}$-formula.
If φ is of the form $\psi^c(\overline{x},\overline{z}) \to \pi(\overline{z})$, then by clause (4) of Definition 4.1 $\psi^c(\overline{x},\overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\delta^\partial}$-formula, and hence by the inductive hypothesis there are formulas $\varphi^c_i(u,\overline{z})$, $1 \leq i \leq n$ such that for every $\overline{a}$, $\overline{c}$ and b', we have $\psi^c(\overline{a},\overline{c}) \leq b'$ iff $a_i \leq^{\delta^\partial_i} \varphi^c_i(b',\overline{c})$ for $1 \leq i \leq n$. We let $\psi_i = \varphi^c_i((u \to \pi(\overline{z}))/u, \overline{z})$. Indeed, we have

$$\begin{array}{rll} b \leq \psi^c(\overline{a},\overline{c}) \to \pi(\overline{c}) & \text{iff} & \psi^c(\overline{a},\overline{c}) \leq b \to \pi(\overline{c}) \\ & \text{iff} & a_i \leq^{\delta_i{}^\partial} \varphi^c_i(b \to \pi(\overline{c}), \overline{c}),\ 1 \leq i \leq n \\ & \text{iff} & \varphi^c_i(b \to \pi(\overline{c}), \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n. \end{array}$$

Moreover, if $\delta_i = 1$ (hence $\delta^\partial_i = \partial$), then by the inductive hypothesis applied to $\psi^c(\overline{x},\overline{z})$, which we recall is an $\mathrm{IF}^{\Diamond}_{\delta^\partial}$-formula, $\varphi^c_i(u,\overline{z})$ is an $\mathrm{IF}^{\text{⬘}}_{\eta^\partial}$-formula with $\eta^\partial = (\partial)$, and then, using Definition 4.1, it is not difficult to show that $\varphi^c_i((u \to \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\text{⬘}}_{\eta}$-formula with $\eta = (1)$. If $\delta_i = \partial$ (hence $\delta^\partial_i = 1$), then by the inductive hypothesis $\varphi^c_i(u,\overline{z})$ is an $\mathrm{IF}^{\text{◩}}_{\eta^\partial}$-formula with $\eta^\partial = (1)$, and then, using Definition 4.1, it is not difficult to show that $\psi'_i((u \to \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\text{◩}}_{\eta}$-formula with $\eta = (\partial)$.

Similar proofs can be given in the remaining cases for ψ. □

5 Adjunction rules for normal inner formulas

The following definition is extracted from the proof of Lemma 4.12.

Definition 5.1. For $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n, for each order-type δ over $\overline{x}$, and each $1 \leq i \leq n$, we define maps LA^δ_i and RA^δ_i, sending normal $(\overline{x},\overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$- and $(\overline{x},\overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formulas into $(u,\overline{z})$-$\mathrm{IF}^{\text{⬘}}_{(\delta_i)}$- and $(u,\overline{z})$-$\mathrm{IF}^{\text{◩}}_{(\delta_i)}$-formulas respectively, by the following simultaneous recursion:

$$
\begin{array}{rcl}
\mathsf{LA}_i^\delta(\top) & = & \bot \\
\mathsf{LA}_i^\delta(x_i) & = & u \text{ for } u \in Var - (\overline{x} \cup \overline{z}); \\
\mathsf{LA}_i^\delta(x_j) & = & \bot^{\delta_j} \text{ when } i \neq j; \\
\mathsf{LA}_i^\delta(\Box\varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(\blacklozenge u, \overline{z}); \\
\mathsf{LA}_i^\delta(\varphi_1(\overline{x},\overline{z}) \wedge \varphi_2(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi_1)(u,\overline{z}) \vee^{\delta_i} \mathsf{LA}_i^\delta(\varphi_2)(u,\overline{z}) \ ; \\
\mathsf{LA}_i^\delta(\nu Y.\varphi(\overline{\varphi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})) & = & \bigvee^{\delta_i}\{\mathsf{LA}_i^\delta(\varphi_j)(\mathsf{LA}_j^{\delta'}(\varphi)(u \vee \mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi)(u \vee Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\} \ ; \\
\mathsf{LA}_i^\delta(\pi(\overline{z}) \to \varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(u \wedge \pi(\overline{z}), \overline{z}); \\
\mathsf{LA}_i^\delta(\pi(\overline{z}) \vee \varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(u - \pi(\overline{z}), \overline{z}); \\
\mathsf{LA}_i^\delta(\psi^c(\overline{x},\overline{z}) \to \pi(\overline{z})) & = & \mathsf{RA}_i^{\delta^\partial}(\psi^c)(u \to \pi(\overline{z}), \overline{z});
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{RA}_i^\delta(\bot) & = & \top \\
\mathsf{RA}_i^\delta(x_i) & = & u \text{ for } u \in Var - (\overline{x} \cup \overline{z}); \\
\mathsf{RA}_i^\delta(x_j) & = & \top^{\delta_j} \text{ when } i \neq j; \\
\mathsf{RA}_i^\delta(\Diamond\psi(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\blacksquare u, \overline{z}); \\
\mathsf{RA}_i^\delta(\psi_1(\overline{x},\overline{z}) \vee \psi_2(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi_1)(u,\overline{z}) \wedge^{\delta_i} \mathsf{RA}_i^\delta(\psi_2)(u,\overline{z}); \\
\mathsf{RA}_i^\delta(\mu Y.\psi(\overline{\psi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})) & = & \bigwedge^{\delta_i}\{\mathsf{RA}_i^\delta(\psi_j)(\mathsf{RA}_j^{\delta'}(\psi)(u \wedge \nu Y.\mathsf{RA}_{k+1}^{\delta'}(\psi)(u \wedge Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\} \ ; \\
\mathsf{RA}_i^\delta(\psi(\overline{x},\overline{z}) - \pi(\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\pi(\overline{z}) \vee u, \overline{z}); \\
\mathsf{RA}_i^\delta(\pi(\overline{z}) \wedge \psi(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\pi(\overline{z}) \to u, \overline{z}); \\
\mathsf{RA}_i^\delta(\pi(\overline{z}) - \varphi^c(\overline{x},\overline{z})) & = & \mathsf{LA}_i^{\delta^\partial}(\varphi^c)(\pi(\overline{z}) - u, \overline{z}).
\end{array}
$$

By normality, formulas with νY as main connective are of the form $\nu Y.\varphi(\overline{\varphi}(\overline{y},\overline{z})/\overline{y}', \overline{X}, Y, \overline{z})$ where $\varphi(\overline{y}', \overline{X}, Y, \overline{z})$ is an $\mathrm{IF}_{\delta'}^{\Box}$-formula, the length of $\overline{y}'$ is ℓ, the length of $\overline{y}' \oplus \overline{X} \oplus Y$ is $k+1$, δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\varphi}(\overline{y},\overline{z}) = (\varphi_1(\overline{y},\overline{z}), \ldots, \varphi_\ell(\overline{y},\overline{z}))$ is such that $\varphi_j(\overline{y},\overline{z})$ is a $(\overline{y},\overline{z})$-$\mathrm{IF}_\delta^\Box$-sentence for every $1 \leq j \leq \ell$. Likewise, formulas with μY as main connective are of the form $\mu Y.\psi(\overline{\psi}(\overline{y},\overline{z})/\overline{y}', \overline{X}, Y, \overline{z})$ where $\psi(\overline{y}', \overline{X}, Y, \overline{z})$ is an $\mathrm{IF}_{\delta'}^{\Diamond}$-formula, the length of $\overline{y}'$ is ℓ, the length of $\overline{y}' \oplus \overline{X} \oplus Y$ is $k+1$, δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\psi}(\overline{y},\overline{z}) = (\psi_1(\overline{y},\overline{z}), \ldots, \psi_\ell(\overline{y},\overline{z}))$ is such that ψ_j is a $(\overline{y},\overline{z})$-$\mathrm{IF}_\delta^\Diamond$-sentence for every $1 \leq j \leq \ell$.

The following lemma is just a direct consequence of the definition, but is very useful in simplifying computations.

Lemma 5.2. *1. Let φ be a $(\overline{x},\overline{z})$-$IF_\delta^\Box$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n. If x_i does not occur in φ for some i, then $\mathsf{LA}_i^\delta(\varphi) = \bot^{\delta_i}$.*

2. Let ψ be a $(\overline{x},\overline{z})$-$IF_\delta^\Diamond$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n. If x_i does not occur in ψ for some i, then $\mathsf{RA}_i^\delta(\psi) = \top^{\delta_i}$.

Proof. By simultaneous induction on φ and ψ. The base cases hold by definition. If φ is of the form $\nu Y.\varphi(\overline{\varphi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})$, then by definition

$$\mathsf{LA}_i^\delta(\varphi) = \bigvee^{\delta_i}\left\{\mathsf{LA}_i^\delta(\varphi_j)(\mathsf{LA}_j^{\delta'}(\varphi)(u \vee \mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi)(u \vee Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\right\}.$$

Since x_i does not occur in any formula in $\overline{\varphi}$, by induction hypothesis $\mathsf{LA}_i^\delta(\varphi_j) = \bot^{\delta_i}$ for every $1 \leq j \leq \ell$. Hence $\mathsf{LA}_i^\delta(\varphi) = \bigvee^{\delta_i}\bot^{\delta_i} = \bot^{\delta_i}$. The remaining cases are left to the reader.

□

We are now in a position to give versions of the adjunction rules tailored to normal $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$- and $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-formulas, for which the adjoints are expressible as $\mathcal{L}^+$-term functions:

$$\frac{\eta \leq \varphi(\overline{x},\overline{z})}{\&_{i=1}^{n} \mathsf{LA}_i^{\delta}(\varphi)[\eta/u] \leq^{\delta_i} x_i}\ (\mathrm{IF}_R)$$

where $\varphi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and $\varphi \in (\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$.

$$\frac{\psi(\overline{x},\overline{z}) \leq \eta}{\&_{i=1}^{n} x_i \leq^{\delta_i} \mathsf{RA}_i^{\delta}(\psi)[\eta/u]}\ (\mathrm{IF}_L)$$

where $\psi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and $\psi \in (\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$. The soundness of these rules immediately follows from Lemma 4.12.

The rules above are closed under substitution. In particular, the following reformulations are sound for any propositional variables $\overline{p}$ and any sentences $\overline{\gamma}$:

$$\frac{\eta \leq \varphi(\overline{p}/\overline{x},\overline{\gamma}/\overline{z})}{\&_{i=1}^{n} \mathsf{LA}_i^{\delta}(\varphi)[\eta/u,\overline{\gamma}/\overline{z}] \leq^{\delta_i} p_i}\ (\mathrm{IF}_R^{\sigma})$$

where $\varphi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and φ is a normal $(\overline{x},\overline{z})$-IF$^{\Box}_{\delta}$-sentence, i.e., $\overline{x} = \overline{y}$.

$$\frac{\psi(\overline{p}/\overline{x},\overline{\gamma}/\overline{z}) \leq \eta}{\&_{i=1}^{n} p_i \leq^{\delta_i} \mathsf{RA}_i^{\delta}(\psi)[\eta/u,\overline{\gamma}/\overline{z}]}\ (\mathrm{IF}_L^{\sigma})$$

where $\psi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and ψ is a normal $(\overline{x},\overline{z})$-IF$^{\Diamond}_{\delta}$-sentence, i.e., $\overline{x} = \overline{y}$.

As discussed earlier, the maps LA_i and RA_i explicitly compute the term functions corresponding to the adjoints of normal $(\overline{x},\overline{z})$-inner formulas. By construction, this adjunction is parametric in $\overline{z}$. The next lemma states the syntactic version of order-theoretic facts that hold in such situations generally, and which will be useful in the proof that μ-ALBA is successful on all ϵ-recursive inequalities.

In what follows we will say that is formula φ is δ_i-positive in a variable u if φ is positive in u when $\delta_i = 1$ and negative in u when $\delta_i = \partial$.

Lemma 5.3. *1. Let $\varphi(\overline{x},\overline{z})$ be a normal IF$^{\Box}_{\delta}$-formula in which each $z \in \overline{z}$ occurs at most once. Then, for each $1 \leq i \leq n$, $\mathsf{LA}_i^{\delta}(\varphi)(u,\overline{z})$ is δ_i-positive in u, and for each $z \in \overline{z}$, the polarity of z in $\mathsf{LA}_i^{\delta}(\varphi)(u,\overline{z})$ is the opposite of (respectively, the same as) its polarity in φ if $\delta_i = 1$ (respectively, if $\delta_i = \partial$).*

2. Let $\psi(\overline{x},\overline{z})$ be a normal IF$^{\Diamond}_{\delta}$-formula in which each $z \in \overline{z}$ occurs at most once. Then, for each $1 \leq i \leq n$, $\mathsf{RA}_i^{\delta}(\psi)(u,\overline{z})$ is δ_i-positive in u, and for each $z \in \overline{z}$, the polarity of z in $\mathsf{RA}_i^{\delta}(\psi)(u,\overline{z})$ is the opposite of (respectively, the same as) its polarity in ψ if $\delta_i = 1$ (respectively, if $\delta_i = \partial$).

Proof. By simultaneous induction on φ and ψ. The base cases are trivially true. The cases in which the main connective is $\Box$ or $\wedge$ immediately follow from the induction hypothesis. For φ of the form $\nu Y.\varphi'(\overline{\varphi}(\overline{x},\overline{z}_1)/\overline{y}', Y, \overline{z}_2)$ as in Definition 4.4, we have

$$\begin{array}{rl} & \mathsf{LA}_i^{\delta}(\nu Y.\varphi'(\overline{\varphi}(\overline{x},\overline{z}_1)/\overline{y}',Y,\overline{z}_2)) \\ = & \bigvee^{\delta_i}\left\{\mathsf{LA}_i^{\delta}(\varphi_j)(\mathsf{LA}_j^{\delta'}(\varphi')(u\vee\mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi')(u\vee Y,\overline{z}_2),\overline{z}_2),\overline{z}_1)\mid 1\leq j\leq \ell\right\}, \end{array}$$

with δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\varphi}(\overline{x},\overline{z}_1) = (\varphi_1(\overline{x},\overline{z}_1),\ldots,\varphi_\ell(\overline{x},\overline{z}_1))$ such that φ_j is a $(\overline{x},\overline{z})$-$\mathrm{IF}_\delta^{\Box}$-sentence for every $1 \leq j \leq \ell$. By induction hypothesis, $\mathsf{LA}_i^{\delta}(\varphi_j)(u',\overline{z}_1)$ is δ_i-positive in u', $\mathsf{LA}_{k+1}^{\delta'}(\varphi')(u,\overline{z}_2)$ is δ'_{k+1}-positive (hence positive) in u, and $\mathsf{LA}_j^{\delta'}(\varphi')(u,\overline{z}_2)$ is δ'_j-positive (hence positive) in u for every $1 \leq j \leq \ell$. Hence $\mathsf{LA}_i^{\delta}(\nu Y.\varphi'(\overline{\varphi}(\overline{x},\overline{z}_1)/\overline{y}',Y,\overline{z}_2))$ is δ_i-positive in u. If $z \in \overline{z}_1$, then z occurs in φ_j for some $1 \leq j \leq \ell$, hence the statement follows by application of the induction hypothesis to φ_j. Let $z \in \overline{z}_2$. Since δ' is constantly 1 on $\overline{y}'$ and Y, by induction hypothesis on φ', it follows that z has the opposite polarity in $\mathsf{LA}_j^{\delta'}(\varphi')(u\vee\mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi')(u\vee Y,\overline{z}_2),\overline{z}_2)$ to that which it has in φ'. If $\delta_i = 1$, then $\mathsf{LA}_i^{\delta}(\varphi_j)(u',\overline{z}_1)$ is positive in u', and hence the polarity of z in $\mathsf{LA}_i^{\delta}(\nu Y.\varphi'(\overline{\varphi}(\overline{x},\overline{z}_1)/\overline{y}',Y,\overline{z}_2))$ is the opposite to that it has in φ'. If $\delta_i = \partial$, then $\mathsf{LA}_i^{\delta}(\varphi_j)(u',\overline{z}_1)$ is negative in u', and hence the polarity of z in $\mathsf{LA}_i^{\delta}(\nu Y.\varphi'(\overline{\varphi}(\overline{x},\overline{z}_1)/\overline{y}',Y,\overline{z}_2))$ is the same to that it has in φ'.

For φ of the form $\pi(\overline{z}_1) \to \varphi'(\overline{x},\overline{z}_2)$ we have $\mathsf{LA}_i^{\delta}(\varphi) = \mathsf{LA}_i^{\delta}(\varphi')((u\wedge\pi(\overline{z}_1))/u',\overline{z}_2)$. Then the claims about the polarities of u and $z \in \overline{z}_2$ follows by the inductive hypothesis applied to φ'. If $z \in \overline{z}_1$ then we distinguish two cases: if $\delta_i = 1$ then $\mathsf{LA}_i^{\delta}(\varphi')(u',\overline{z}_2)$ is positive in u' by the induction hypothesis, and since $\pi(\overline{z}_1)$ occurs negatively in φ, the polarity of z in $\mathsf{LA}_i^{\delta}(\varphi')((u\wedge\pi(\overline{z}_1))/u',\overline{z}_2)$ is the opposite of its polarity in φ. If $\delta_i = \partial$ then $\mathsf{LA}_i^{\delta}(\varphi')(u',\overline{z}_2)$ is negative in u' by the induction hypothesis, and since $\pi(\overline{z}_1)$ occurs negatively in φ, the polarity of z in $\mathsf{LA}_i^{\delta}(\varphi')((u\wedge\pi(\overline{z}_1))/u',\overline{z}_2)$ is the same as its polarity in φ.

For φ of the form $\psi^c(\overline{x},\overline{z}_2) \to \pi(\overline{z}_1)$ we have $\mathsf{LA}_i^{\delta}(\varphi) = \mathsf{RA}_i^{\delta^\partial}(\psi^c)((u\to\pi(\overline{z}_1))/u',\overline{z}_2)$. If $\delta_i = 1$, then $\delta_i^\partial = \partial$, and by the inductive hypothesis $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u',\overline{z}_2)$ is negative in u', and hence $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u',\overline{z}_2)$ is positive in u. If $\delta_i = \partial$, then $\delta_i^\partial = 1$, and by the inductive hypothesis $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u',\overline{z}_2)$ is positive in u', and hence $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u',\overline{z}_2)$ is negative in u. If $z \in \overline{z}_2$ and $\delta_i = 1$, then $\delta_i^\partial = \partial$ and hence, by the induction hypothesis, the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u',\overline{z}_2)$ is the same as its polarity in ψ^c, and since ψ^c occurs negatively in $\psi^c(\overline{x},\overline{z}_2) \to \pi(\overline{z}_1)$, we have that the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u',\overline{z}_2)$, and hence in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u',\overline{z}_2)$, is the opposite of its polarity in φ. The case where $z \in \overline{z}_2$ and $\delta_i = \partial$ follows by an order-dual argument. If $z \in \overline{z}_1$ and $\delta_i = 1$ then $\delta_i^\partial = \partial$, and hence by the induction hypothesis, $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u',\overline{z}_2)$ is negative in u'. So because $\pi(\overline{z}_1)$ occurs positively in φ, it occurs negatively in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u',\overline{z}_2)$, and hence the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u',\overline{z}_2)$ is the opposite of its polarity in φ. The case in which $z \in \overline{z}_1$ and $\delta_i = \partial$ follows by an order-dual argument.

The remaining cases are left to the reader. □

6 Examples

In the ensuing examples, for the sake of clarity, we will often write $\mathsf{LA}_{x_i}^{\delta}(\varphi)$ instead of $\mathsf{LA}_i^{\delta}(\varphi)$ where φ is some $(\overline{x},\overline{z})$-$\mathrm{IF}_\delta^{\Box}$-formula. Similarly for $\mathsf{RA}_{x_i}^{\delta}(\psi)$.

Example 6.1. Consider the inequality $\nu X.[\Box(X \wedge \neg\mu Y.[\Diamond(\sim X \vee (Y \vee p))])] \leq \Diamond\Box\neg p$, which is ϵ-recursive for $\epsilon_p = \partial$. Its left-hand side has been discussed in Example 4.10. After first approximation we have:

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.[\Box(X \wedge \neg \mu Y.[\Diamond(\sim X \vee (Y \vee p))])] \ \& \ \Diamond\Box\neg p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.1}$$

No approximation rules are applicable, thus we work toward the application of an appropriate adjunction rule to display the p in the first inequality in the antecedent of the quasi-inequality above. As discussed in Example 4.10, the left-hand side of this inequality is not in normal form, and its normalization was computed there. We thus apply the adjunction rule (IF^{σ}_{R}) to its normalization

$$\varphi = \nu X.[\Box(X \wedge (\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge \neg \mu Y.[\Diamond Y \vee \Diamond p]))].$$

Recall that φ is a substitution instance of the formula $\nu X.\varphi' = \nu X.[\Box(X \wedge (\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))]$, where φ' is a $(y' \oplus X, \varnothing)$-$\mathrm{IF}^{\Box}_{\delta}$-formula with $\delta = (1,1)$. Moreover, y' has been substituted for the $(p, \varnothing)$-$\mathrm{IF}^{\Box}_{\epsilon}$-sentence $\neg\psi = \neg \mu Y.[\Diamond Y \vee \Diamond p]$. Thus,

$$\begin{array}{rcl} \mathsf{LA}^{\epsilon}_{p}(\varphi) & = & \mathsf{LA}^{\epsilon}_{p}(\neg\psi)[(\mathsf{LA}^{\delta}_{y'}(\varphi')[(u \vee \mu X.[\mathsf{LA}^{\delta}_{X}(\varphi')[(u \vee X)/u']])/u'])/u] \\ & = & \mathsf{LA}^{\epsilon}_{p}(\neg\psi)[\mathsf{LA}^{\delta}_{y'}(\varphi')(u \vee \mu X.\mathsf{LA}^{\delta}_{X}(\varphi')(u \vee X))], \end{array}$$

where

$$\mathsf{LA}^{\epsilon}_{p}(\neg\psi)(u) \quad = \quad \mathsf{RA}^{\epsilon^{\partial}}_{p}(\psi)(\neg u/u)$$

and $\psi = \mu Y.[\Diamond Y \vee \Diamond p]$ is of the form $\mu Y.\psi'(p/y', Y, \varnothing)$ such that $\psi'(y', Y, \varnothing) = \Diamond Y \vee \Diamond y'$ is an $\mathrm{IF}^{\Diamond}_{\delta'}$-formula with δ' being the order-type constantly 1 on $y' \oplus Y$. Hence ψ is already in normal form. Thus,

$$\begin{array}{rcl} \mathsf{RA}^{\epsilon^{\partial}}_{p}(\psi)(u) & = & \mathsf{RA}^{\epsilon^{\partial}}_{p}(\mu Y.[\Diamond Y \vee \Diamond p])(u) \\ & = & \mathsf{RA}^{\epsilon^{\partial}}_{p}(p)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\mathsf{RA}^{\delta'}_{Y}(\psi')(u \wedge Y))) \\ & = & \mathsf{RA}^{\epsilon^{\partial}}_{p}(p)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\blacksquare(u \wedge Y))) \\ & = & \mathsf{RA}^{\epsilon^{\partial}}_{p}(p)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y))) \\ & = & \blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)), \end{array}$$

and hence,

$$\mathsf{LA}^{\epsilon}_{p}(\neg\psi)(u) \quad = \quad \mathsf{RA}^{\epsilon^{\partial}}_{p}(\psi)(\neg u/u) \quad = \quad \blacksquare(\neg u \wedge \nu Y.\blacksquare(\neg u \wedge Y)).$$

Next,

$$\begin{array}{rcll} \mathsf{LA}^{\delta}_{y'}(\varphi')(u) & = & \mathsf{LA}^{\delta}_{y'}(\Box(X \wedge (\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y')))(u) & \\ & = & \mathsf{LA}^{\delta}_{y'}(X \wedge (\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u) & \\ & = & (\mathsf{LA}^{\delta}_{y'}(X) \vee \mathsf{LA}^{\delta}_{y'}(\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u) & \\ & = & (\bot \vee (\mathsf{LA}^{\delta}_{y'}(\neg \mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \mathsf{LA}^{\delta}_{y'}(y')))(\blacklozenge u/u) & \text{(lemma 5.2)} \\ & = & (\bot \vee (\bot \vee \mathsf{LA}^{\delta}_{y'}(y')))(\blacklozenge u/u) & \text{(lemma 5.2)} \\ & = & \mathsf{LA}^{\delta}_{y'}(y')(\blacklozenge u/u) & \\ & = & u(\blacklozenge u/u) & \\ & = & \blacklozenge u. & \end{array}$$

$$\begin{array}{rcll}
\mathsf{LA}^{\delta}_{X}(\varphi')(u) & = & \mathsf{LA}^{\delta}_{X}(\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y')))(u) & \\
& = & \mathsf{LA}^{\delta}_{X}(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u) & \\
& = & (\mathsf{LA}^{\delta}_{X}(X) \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u) & \\
& = & (u \vee (\mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \mathsf{LA}^{\delta}_{X}(y')))(\blacklozenge u/u) & \\
& = & (u \vee (\mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \bot))(\blacklozenge u/u) & \text{(lemma 5.2)} \\
& = & (u \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]))(\blacklozenge u/u) & \\
& = & \blacklozenge u \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(\blacklozenge u/u). &
\end{array}$$

$$\mathsf{LA}^{\delta}_{X}(\neg\psi)(u) \quad = \quad \mathsf{RA}^{\delta^{\partial}}_{X}(\psi)(\neg u/u)$$

and $\psi = \mu Y.[\Diamond Y \vee \Diamond{\sim}X]$ is of the form $\mu Y.\psi'({\sim}X/y', Y, \varnothing)$ such that $\psi'(y', Y, \varnothing) = \Diamond Y \vee \Diamond y'$ is an $\mathrm{IF}^{\Diamond}_{\delta'}$-formula with δ' being the order-type constantly 1 on $y' \oplus Y$. Hence ψ is already in normal form. Thus,

$$\begin{array}{rcll}
\mathsf{RA}^{\delta^{\partial}}_{X}(\psi)(u) & = & \mathsf{RA}^{\delta^{\partial}}_{X}(\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(u) & \\
& = & \mathsf{RA}^{\delta^{\partial}}_{X}({\sim}X)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\mathsf{RA}^{\delta'}_{Y}(\psi')(u \wedge Y))) & \\
& = & \mathsf{RA}^{\delta^{\partial}}_{X}({\sim}X)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y))) & \text{(cf. calculation of } \mathsf{RA}^{\epsilon^{\partial}}_{p}(\psi)(u) \text{ above)} \\
& = & ({\sim}u)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y))/u) & (*) \\
& = & {\sim}\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)) &
\end{array}$$

The starred equality above is justified as follows:

$$\begin{array}{rcl}
\mathsf{RA}^{\delta^{\partial}}_{X}({\sim}X)(u) & = & \mathsf{RA}^{\delta^{\partial}}_{X}(\top - X)(u) \\
& = & \mathsf{LA}^{\delta}_{X}(X)(\top - u/u) \\
& = & (u)(\top - u/u) \\
& = & \top - u \\
& = & {\sim}u
\end{array}$$

Thus,

$$\begin{array}{rcl}
\mathsf{LA}^{\delta}_{X}(\varphi')(u) & = & \blacklozenge u \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(\blacklozenge u/u) \\
& = & \blacklozenge u \vee (\mathsf{RA}^{\delta^{\partial}}_{X}(\psi)(\neg u/u))(\blacklozenge u/u) \\
& = & \blacklozenge u \vee (({\sim}\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)))(\neg u/u))(\blacklozenge u/u) \\
& = & \blacklozenge u \vee ({\sim}\blacksquare(\neg u \wedge \nu Y.\blacksquare(\neg u \wedge Y)))(\blacklozenge u/u) \\
& = & \blacklozenge u \vee {\sim}\blacksquare\neg\blacklozenge u \wedge \nu Y.\blacksquare(\neg\blacklozenge u \wedge Y)
\end{array}$$

Finally,

$$\begin{array}{rcl}
\mathsf{LA}^{\epsilon}_{p}(\varphi)(u) & = & \mathsf{LA}^{\epsilon}_{p}(\neg\psi)[\mathsf{LA}^{\delta}_{y'}(\varphi')(u \vee \mu X.\mathsf{LA}^{\delta}_{X}(\varphi')(u \vee X))] \\
& = & \mathsf{LA}^{\epsilon}_{p}(\neg\psi)[\blacklozenge(u \vee \mu X.(\blacklozenge(u \vee X) \vee {\sim}\blacksquare\neg\blacklozenge(u \vee X) \wedge \nu Y.\blacksquare(\neg\blacklozenge(u \vee X) \wedge Y)))] \\
& = & \blacksquare(\neg w \wedge \nu Y.\blacksquare(\neg w \wedge Y))(\blacklozenge(u \vee \mu X.(\blacklozenge(u \vee X) \vee {\sim}\blacksquare\neg\blacklozenge(u \vee X) \wedge \nu Y.\blacksquare(\neg\blacklozenge(u \vee X) \wedge Y)))/w).
\end{array}$$

Thus, applying (IF^{σ}_{R}) to the normalized inequality transforms (6.1) into

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathsf{LA}^{\epsilon}_{p}(\varphi)(\mathbf{i}/u) \leq p \ \& \ \Diamond\Box\neg p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape, since $\mathsf{LA}^{\epsilon}_{p}(\varphi)(\mathbf{i}/u)$ is p-free. Now applying (RA) yields the quasi-inequality

$$\forall \mathbf{i}\forall \mathbf{m}[\Diamond\Box\neg \mathsf{LA}^{\epsilon}_{p}(\varphi)(\mathbf{i}/u) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

from which all propositional variables have been eliminated, and which can be further rewritten as

$$\forall \mathbf{i}[\mathbf{i} \leq \Diamond\Box\neg \mathsf{LA}^{\epsilon}_{p}(\varphi)(\mathbf{i}/u)].$$

Example 6.2. Consider the inequality

$$\Diamond\mu X.[(p \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge \mu Z.{\sim}(\Box p \wedge \neg Z))) \rightarrow \Diamond\Box\Box p]] \leq \Diamond\Box p.$$

which, as discussed in Example 3.5, is ϵ-recursive with $\epsilon_p = 1$. After first approximation we have:

$$\forall p\forall \mathbf{i}\forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mu X.[(p\vee X)\vee{\sim}\nu Y.[\Diamond(X\vee{\sim}((Y\wedge p)\wedge\mu Z.{\sim}(\Box p\wedge\neg Z))) \rightarrow \Diamond\Box\Box p]]\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

Applying ($\Diamond$ Appr) to surface the inner skeleton of the first inequality in the antecedent of the quasi-inequality above yields:

$$\forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mathbf{j}\ \&\ \mathbf{j} \leq \psi\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{6.2}$$

with $\psi = \mu X.[(p \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge \mu Z.{\sim}(\Box p \wedge \neg Z))) \rightarrow \Diamond\Box\Box p]]$. Now notice that $\psi = \psi'(\varphi_1/x_1, \varphi_2/x_2, \gamma/z)$, where

$$\begin{array}{rcl}\psi'(x_1, x_2, z) & = & \mu X.[(x_1 \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge x_2)) \rightarrow z]],\\ \varphi_1 & = & p,\\ \varphi_2 & = & \mu Z.{\sim}(\Box p \wedge \neg Z),\\ \gamma & = & \Diamond\Box\Box p.\end{array}$$

Moreover, ψ' is an (x_1, x_2, z)-$\mathrm{IF}^{\Diamond}_{(1,\partial)}$ formula. Hence, by Lemma 4.3, its associated term function is completely join-preserving as a map $\mathbb{A}\times\mathbb{A}^{\partial} \rightarrow \mathbb{A}$, for any perfect modal bi-Heyting algebra $\mathbb{A}$; that is, the inequality $\mathbf{j} \leq \psi$ satisfies the appropriate order-theoretic conditions for the application of the rule (μ-A). Hence, after this application, the inequality $\mathbf{j} \leq \psi$ is equivalently replaced with the following disjunction:

$$\exists \mathbf{j}'[\mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \gamma/z)\ \&\ \mathbf{j}' \leq \varphi_1] \parr \exists \mathbf{n}[\mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \gamma/z)\ \&\ \varphi_2 \leq \mathbf{n}]$$

At this point we transform the quasi-inequality obtained from (6.2) by performing the replacement above, into the conjunction of two quasi-inequalities, by distributing &s over $\parr$ in the antecedent so as to make $\parr$ the main connective of the antecedent, and then distributing $\Rightarrow$ over $\parr$. This gives us:

$$\forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{j}'\forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mathbf{j}\ \&\ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \gamma/z)\ \&\ \mathbf{j}' \leq \varphi_1\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{6.3}$$

and

$$\forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i} \leq \Diamond\mathbf{j}\ \&\ \mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \gamma/z)\ \&\ \varphi_2 \leq \mathbf{n}\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.4}$$

Recalling that $\varphi_1 = p$ and $\gamma = \Diamond\Box\Box p$, the quasi-inequality (6.3) is

$$\forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{j}'\forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mathbf{j}\ \&\ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \Diamond\Box\Box p/z)\ \&\ \mathbf{j}' \leq p\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape (cf. page 7), hence we can apply (RA) to it and obtain

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{j}' \forall \mathbf{m} [(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \Diamond\Box\Box\mathbf{j}'/z) \ \& \ \Diamond\Box\mathbf{j}' \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

where all propositional variables have been eliminated, and hence can be translated into FO+LFP as discussed in Section 1.2. Turning our attention to (6.4), we note that only occurrence of p for which we want to solve is in the inequality $\varphi_2 \leq \mathbf{n}$. Recalling that $\varphi_2 = \mu Z.{\sim}(\Box p \wedge \neg Z)$ we work towards the application of an appropriate adjunction rule. As it stands, φ_2 is not a substitution instance of a normal $\mathrm{IF}^{\Diamond}_{\delta'}$-formula, thus we need to normalize it. Indeed, the normalization consists in distributing $\sim$ over $\wedge$, transforming φ_2 into $\mu Z.[{\sim}\Box p \vee {\sim}\neg Z] = \mu Z.\psi''({\sim}\Box p/y')$ with $\psi''(y', Z) = y' \vee {\sim}\neg Z$ which is a normal $\mathrm{IF}^{\Diamond}_{(1)}$-formula. Hence we may apply $(\mathrm{IF}^{\sigma}_{L})$ which yields

$$p \leq \mathsf{RA}^{(1)}_p(\varphi_2)[\mathbf{n}/u],$$

where

$$\begin{aligned}
\mathsf{RA}^{(1)}_p(\varphi_2)(u) &= \mathsf{RA}^{(1)}_p(\mu Z.\psi''({\sim}\Box p/y', Z)) \\
&= \mathsf{RA}^{(1)}_p({\sim}\Box p)(\mathsf{RA}^{(1)}_{y'}(\psi'')(u \wedge \nu Z.[\mathsf{RA}^{(1)}_Z(\psi'')(u \wedge Z)])),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{RA}^{(1)}_Z(\psi'')(u) &= \mathsf{RA}^{(1)}_Z(y' \vee {\sim}\neg Z)(u) \\
&= (\mathsf{RA}^{(1)}_Z(y') \wedge \mathsf{RA}^{(1)}_Z({\sim}\neg Z))(u) \\
&= (\top \wedge \mathsf{RA}^{(1)}_Z({\sim}\neg Z))(u) && \text{(Lemma 5.2)} \\
&= \mathsf{RA}^{(1)}_Z(\top - \neg Z)(u) \\
&= \mathsf{LA}^{(\partial)}_Z(\neg Z)((\top - u)/u) \\
&= (\mathsf{RA}^{(1)}_Z(Z)(\neg u/u))((\top - u)/u) \\
&= (u(\neg u/u))((\top - u)/u) \\
&= \neg{\sim}u.
\end{aligned}$$

$$\begin{aligned}
\mathsf{RA}^{(1)}_{y'}(\psi'')(u) &= \mathsf{RA}^{(1)}_{y'}(y' \vee {\sim}\neg Z)(u) \\
&= (\mathsf{RA}^{(1)}_{y'}(y') \wedge \mathsf{RA}^{(1)}_{y'}({\sim}\neg Z))(u) \\
&= (u \wedge \top)(u) && \text{(Lemma 5.2)} \\
&= u.
\end{aligned}$$

$$\begin{aligned}
\mathsf{RA}^{(1)}_p({\sim}\Box p)(u) &= \mathsf{RA}^{(1)}_p(\top - \Box p)(u) \\
&= \mathsf{LA}^{(\partial)}_p(\Box p)((\top - u)/u) \\
&= (\mathsf{LA}^{(\partial)}_p(p)(\blacklozenge u/u))((\top - u)/u) \\
&= (u(\blacklozenge u/u))((\top - u)/u) \\
&= \blacklozenge u((\top - u)/u) \\
&= \blacklozenge{\sim}u.
\end{aligned}$$

Therefore,

$$\begin{array}{rcl}\mathsf{RA}^{(1)}_p(\varphi_2)(u) & = & \mathsf{RA}^{(1)}_p(\sim\Box p)(\mathsf{RA}^{(1)}_{y'}(\psi'')(u\wedge \nu Z.[\mathsf{RA}^{(1)}_Z(\psi'')(u\wedge Z)])),\\ & = & \blacklozenge\sim(u\wedge \nu Z.\neg\sim(u\wedge Z)).\end{array}$$

Thus (6.4) becomes

$$\forall p\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i}\leq \Diamond\mathbf{j}\ \&\ \mathbf{j}\leq \psi'(\bot/x_1,\mathbf{n}/x_2,\gamma/z)\ \&\ p\leq \blacklozenge\sim(\mathbf{n}\wedge\nu Z.\neg\sim(\mathbf{n}\wedge Z))\ \&\ \Diamond\Box p\leq \mathbf{m})\Rightarrow \mathbf{i}\leq \mathbf{m}],$$

which is in Ackermann shape. Applying the Ackermann rule (LA) and recalling that $\gamma = \Diamond\Box\Box p$, we obtain

$$\forall \mathbf{i}\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{i}\leq \Diamond\mathbf{j}\ \&\ \mathbf{j}\leq \psi'(\bot/x_1,\mathbf{n}/x_2,\Diamond\Box\Box\blacklozenge\sim(\mathbf{n}\wedge\nu Z.\neg\sim(\mathbf{n}\wedge Z))/z)\ \&\ \Diamond\Box\blacklozenge\sim(\mathbf{n}\wedge\nu Z.\neg\sim(\mathbf{n}\wedge Z))\leq \mathbf{m})\Rightarrow \mathbf{i}\leq \mathbf{m}],$$

where all occurring propositional variables have been eliminated.

7 Success on recursive μ-inequalities

The aim of the present section is to show that the enhanced version of ALBA is successful on ϵ-recursive inequalities.

7.1 Preprocess, first approximation and approximation

Indeed, let $\eta\leq\beta$ be an ϵ-recursive inequality. We proceed as in ALBA and preprocess this inequality by applying splitting and ($\top$) and ($\bot$) exhaustively. This might produce multiple inequalities, on each of which we proceed separately. On each such inequality, denoted again $\eta\leq\beta$, we proceed to first approximation, which yields the following quasi-inequality:

$$\forall\overline{p}\forall\mathbf{i}\forall\mathbf{m}[(\mathbf{i}\leq\eta\ \&\ \beta\leq\mathbf{m})\Rightarrow\mathbf{i}\leq\mathbf{m}]. \tag{7.1}$$

Because its consequent is always pure, we only concentrate on its antecedent. Since the outer skeleton of β and η is built exactly as the outer part of an inductive modal formula, the ordinary approximation rules can be applied so as to surface the inner skeleton. So we can equivalently rewrite $\mathbf{i}\leq\eta\ \&\ \beta\leq\mathbf{m}$ as the conjunction of a set of inequalities which, whenever they contain critical variables in the scope of fixed points occurring as skeleton nodes, are of the form

$$\mathbf{i}\leq\mu X.\psi'(\overline{p})\qquad\text{and}\qquad \nu X.\varphi'(\overline{p})\leq\mathbf{m}, \tag{7.2}$$

where $\mu X.\psi'(\overline{p})$ and $\nu X.\varphi'(\overline{p})$ are sentences. (For the critical branches which do not contain such fixed points, we further proceed by exhaustively applying the approximation rules as in ALBA).

Proposition 7.1. *1. The inequality $\mathbf{i}\leq\mu X.\psi'$ in (7.2) is of the form $\mathbf{i}\leq\mu X.\psi(\overline{\varphi}/\overline{y},\overline{\gamma}/\overline{z})$, where $\mu X.\psi(\overline{y},X,\overline{z})$ is an $(\overline{y},\overline{z})$-$IF^{\Diamond}_{\delta}$-formula for some order-type δ over $\overline{y}$;*

2. the inequality $\nu X.\varphi'\leq\mathbf{m}$ in (7.2) is of the form $\nu X.\varphi(\overline{\psi}/\overline{y},X,\overline{\gamma}/\overline{z})\leq\mathbf{m}$, where $\nu X.\varphi(\overline{y},X,\overline{z})$ is an $(\overline{y},\overline{z})$-$IF^{\Box}_{\delta}$-formula for some order-type δ over $\overline{y}$.

Proof. Notice that preprocessing, first approximation and ordinary approximation rules do not involve fixed points. Hence a proof very similar to that given for [9, Lemmas 10.4 and 10.6] proves that $+\mu X.\psi'$ and $-\nu X.\varphi'$ are non trivially ϵ-recursive. Hence the statement immediately follows from Lemma 7.2 below. □

Lemma 7.2. *1. If $+\psi'$ is non-trivially ϵ-recursive, and the P_3-paths of all critical branches are of length 0, then ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$IF^{\Diamond}_{\delta}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$, some order-type δ over $\overline{x}$, and $\overline{\varphi}$ and $\overline{\gamma}$ $\mathcal{L}$-sentences.*

2. If $-\varphi'$ is non-trivially ϵ-recursive, and the P_3-paths of all critical branches are of length 0, then φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$IF^{\Box}_{\delta}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$, some order-type δ over $\overline{x}$, and $\overline{\psi}$ and $\overline{\gamma}$ $\mathcal{L}$-sentences.

Proof. Let us define the *skeleton depth* of an ϵ-recursive generation tree $*\xi$, with $* \in \{+, -\}$, to be the maximum length of the P_2 paths in $*\xi$ leading to critical variables. The proof proceeds by simultaneous induction on the skeleton depths of $+\psi'$ and $-\varphi'$.

If the depth of $+\psi'$ is 0, then the critical branches will consist only of PIA nodes, i.e., $+\psi'$ is non-trivially ϵ-PIA, and by Definition 3.1.1 ψ' is a sentence; hence we let $\psi = x_1$ which is $\mathrm{IF}^{\Diamond}_{\delta}$ with $\delta = (1)$, and $\varphi_1 = \psi'$. Analogously for the base case of $-\varphi'$.

As for the induction step, let us suppose that the depth of $+\psi'$ is $k + 1$, and that the statement is true for generation trees satisfying the assumptions and of depth not greater than k. We proceed by cases, depending on the form of $+\psi'$.

If $+\psi'$ is of the form $+\mu X.\psi'_1$, then by the induction hypothesis applied to $+\psi'_1$, we have that ψ'_1 is of the form $\psi_1(\overline{\varphi}/\overline{y}, \overline{X}'\overline{\gamma}/\overline{z})$ where $\overline{X}' = \overline{X} \oplus X$ and $\psi_1(\overline{x}', \overline{z})$ is an $(\overline{x}', \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula for $\overline{x}' = \overline{y} \oplus \overline{X}' = \overline{x} \oplus X$ and some order-type $\delta' = \delta \oplus 1$ over $\overline{x} \oplus X$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Hence we let $\psi = \mu X.\psi_1(\overline{x}, X, \overline{z})$ which is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula.

If $+\psi'$ is of the form $+(\psi'_1 \vee \psi'_2)$, then, by the induction hypothesis applied to $+\psi'_1$ and $+\psi'_2$, we have that ψ'_1 and ψ'_2 are of the form $\psi_1(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ and $\psi_2(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, respectively, satisfying the statement for some order-types δ over $\overline{x} = \overline{y} \oplus \overline{X}$. We let $\psi = \psi_1(\overline{x}, \overline{z}) \vee \psi_2(\overline{x}, \overline{z})$, which is $\mathrm{IF}^{\Diamond}_{\delta}$. Hence ψ' is of the form $\psi = \psi_1(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}) \vee \psi_2(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, as required.

If $+\psi'$ is of the form $+(\chi - \varphi')$ with χ a sentence and $\epsilon^{\partial}(+\chi)$, then $-\varphi'$ is ϵ-recursive, and hence, by the induction hypothesis, φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta^{\partial}}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Then we let $\psi = z - \varphi_1(\overline{x}, \overline{z})$, which is $(\overline{x}, \overline{z} \oplus z)$-$\mathrm{IF}^{\Diamond}_{\delta}$, where z is a fresh variable. Hence ψ' is of the form $(z - \varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}_1))[\chi/z]$ as required.

If $+\psi'$ is of the form $+(\psi' - \chi)$ with χ a sentence and $\epsilon(+\chi)$, then $+\psi'$ is ϵ-recursive, and hence by the induction hypothesis ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Then we let $\psi = \psi(\overline{x}, \overline{z}) - z$ which is $(\overline{x}, \overline{z} \oplus z)$-$\mathrm{IF}^{\Diamond}_{\delta}$, where z is a fresh variable. Hence ψ' is of the form $(\psi(\overline{\psi}/\overline{y}, \overline{\gamma}/\overline{z}) - z)[\chi/z]$ as required.

The remaining cases are left to the reader. □

Remark 7.3. Actually, Lemma 7.2 can be strengthened to the following:

1. Let ψ' be such that $+\psi'$ is non-trivially ϵ-recursive, and the P_3-paths of all critical branches are of length 0. Then ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula,

for $\overline{x} = \overline{y} \oplus \overline{X}$ and some order-type δ over $\overline{x}$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Moreover, if $\overline{y} = (y_1, \ldots, y_n)$ then, for each $1 \leq i \leq n$, $+\varphi_i$ is ϵ-PIA if $\delta_i = 1$ and $-\varphi_i$ is ϵ-PIA if $\delta_i = \partial$. Finally $\epsilon^\partial(+\psi(\overline{x}, \overline{\gamma}/\overline{z}))$.

2. Let φ' be such that $-\varphi'$ is non-trivially ϵ-recursive, and the P_3-paths of all critical branches are of length 0. Then φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$-formula, for $\overline{x} = \overline{y} \oplus \overline{X}$ and some order-type δ over $\overline{x}$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Moreover, if $\overline{y} = (y_1, \ldots, y_n)$ then, for each $1 \leq i \leq n$, $-\psi_i$ is ϵ-PIA if $\delta_i = 1$ and $+\psi_i$ is ϵ-PIA if $\delta_i = \partial$. Finally $\epsilon^\partial(-\varphi(\overline{x}, \overline{\gamma}/\overline{z}))$.

Hence, Proposition 7.1 can be strengthened in an analogous way.

The proof of the enhanced Lemma 7.2 is essentially a refined version of the induction in the original proof. The base case as it stands already verifies this strengthening. In particular, for $+\psi'$ we have $\psi(\overline{x}, \overline{\gamma}/\overline{z}) = x_1$ and $\epsilon^\partial(x_1)$.

We illustrate the rest of the induction by considering the case when $+\psi'$ is of the form $+(\chi - \varphi')$ with χ a sentence and $\epsilon^\partial(+\chi)$. Then $-\varphi'$ is non-trivially ϵ-recursive, and hence, by the strengthened induction hypothesis, φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta^\partial}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Moreover, $\overline{y} = (y_1, \ldots, y_n)$ and for every $1 \leq i \leq n$, the generation tree $-\psi_i$ is non-trivially ϵ-PIA if $(\delta^\partial)_i = 1$ (i.e., $\delta_i = \partial$) and $+\psi_i$ is non-trivially ϵ-PIA if $\delta^\partial_i = \partial$ (i.e., $\delta_i = 1$). Then we let $\psi = z - \varphi(\overline{x}, \overline{z})$, which is $(\overline{x}, \overline{z} \oplus z)$-$\mathrm{IF}^{\Diamond}_{\delta}$, where z is a fresh variable. Moreover, for $1 \leq i \leq n$ we let $\varphi_i = \psi_i$. Hence ψ' is of the form $(z - \varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}))[\chi/z]$, with $\psi_1 \ldots \psi_n$ playing the role of $\varphi_1 \ldots \varphi_n$. Finally, $\epsilon^\partial(\chi - \varphi(\overline{x}, \overline{\gamma}/\overline{z}))$, since $\epsilon^\partial(+\chi)$, and the induction hypothesis implies that $\epsilon(+\varphi(\overline{x}, \overline{\gamma}/\overline{z}))$.

Proposition 7.1 and Lemma 4.3 together say that the approximation rules (μ^δ-A) and (ν^δ-A) can be applied to the inequalities (7.2), respectively.[9] In addition to this, by the enhancement of Proposition 7.1 discussed in remark 7.3, we can assume w.l.o.g. that any inequality sitting in the antecedents of the quasi-inequalities produced by these rule applications and containing a critical branch is of the form

$$\mathbf{j} \leq \varphi \quad \text{or} \quad \psi \leq \mathbf{n}, \tag{7.3}$$

where φ and ψ are sentences (see Definition 3.1.1), and moreover $+\varphi$ and $-\psi$ are non-trivially ϵ-PIA. Lemma 7.4 in the next subsection, together with the fact that φ and ψ are sentences, ensures that the appropriate adjunction rules (IF^σ_R) and (IF^σ_L) are respectively applicable to these inequalities.

7.2 Application of adjunction rules

If ϵ and δ are order-types over $\overline{p}$ and $\overline{x}$ respectively, and $\overline{p}$ is not longer than $\overline{x}$ and has length n, then we abuse terminology and say that δ restricts to ϵ if $\epsilon_i = \delta_i$ for each $1 \leq i \leq n$.

Lemma 7.4. *Let ϵ be an order-type over $\overline{p}$.*

[9] Note that applying one of these approximation rules within the antecedent of a quasi-inequality may split that quasi-inequality into the conjunction of several quasi-inequalities, on each of which we proceed separately. See e.g. Example 2.5.

1. *Let $\varphi'(\overline{p}, \overline{X})$ be such that $+\varphi'$ is non-trivially ϵ-PIA. Then $\varphi'(\overline{p}, \overline{X})$ is of the form $\varphi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\varphi(\overline{x}, \overline{z})$ is a normal $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula with $\overline{x} = \overline{y} \oplus \overline{X}$ and δ is an order-type over $\overline{x}$ which restricts to ϵ over $\overline{y}$. Moreover, $\epsilon^{\partial}(\gamma) \prec +\varphi'$ for each $\gamma \in \overline{\gamma}$. Finally each $z \in \overline{z}$ occurs at most once in $\varphi(\overline{x}, \overline{z})$.*

2. *Let $\psi'(\overline{p}, \overline{X})$ be such that $-\psi'$ is non-trivially ϵ-PIA. Then $\psi'(\overline{p}, \overline{X})$ is of the form $\psi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is a normal $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula with $\overline{x} = \overline{y} \oplus \overline{X}$ and δ is an order-type over $\overline{x}$ which restricts to ϵ over $\overline{y}$. Moreover, $\epsilon^{\partial}(\gamma) \prec -\psi'$ for each $\gamma \in \overline{\gamma}$. Finally each $z \in \overline{z}$ occurs at most once in $\psi(\overline{x}, \overline{z})$.*

Proof. It is sufficient to show that the formulas $\varphi(\overline{x}, \overline{z})$ and $\psi(\overline{x}, \overline{z})$ in the statement of the lemma are $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$- and $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formulas, respectively, since normality will then follow from Proposition 4.7.

Let us define the *PIA depth* of a non-trivial ϵ-PIA generation tree $*\xi$, with $* \in \{+, -\}$, to be the maximum length of its critical branches. The proof proceeds by simultaneous induction on the PIA depths of $+\varphi'$ and $-\psi'$.

If the depth of $+\varphi'$ is 0, then $\varphi' = p_1$ such that $\epsilon_1 = 1$, so we let $\varphi = y_1$ which is IF$^{\Box}_{\delta}$ with $\delta = (1)$.

Analogously for the base case of $-\psi'$.

As for the induction step, let us suppose that the depth of $+\varphi'$ is $k+1$ and that the statement is true for generation trees satisfying the assumptions and of depth not greater than k. We proceed by cases depending on the form of $+\varphi'$.

If $+\varphi'$ is of the form $+\nu X.\varphi'_1(\overline{p}, \overline{X}')$ for $\overline{X}' = \overline{X} \oplus X$, then by the induction hypothesis applied to $+\varphi'_1$, we have that $\varphi'_1(\overline{p}, \overline{X}')$ is of the form $\varphi_1(\overline{p}/\overline{y}, \overline{X}', \overline{\gamma}/\overline{z})$, where $\varphi_1(\overline{x}', \overline{z})$ is an $(\overline{x}', \overline{z})$-IF$^{\Box}_{\delta'}$ for $\overline{x}' = \overline{y} \oplus \overline{X}' = \overline{x} \oplus X$ with $\delta' = \delta \oplus 1$ where δ is an order-type over $\overline{x}$ which restricts to ϵ over $\overline{y}$, and each $z \in \overline{z}$ occurs at most once. Moreover, $\epsilon^{\partial}(\gamma) \prec +\varphi'_1$ for each $\gamma \in \overline{\gamma}$. Hence we let $\varphi = \nu X.\varphi_1(\overline{x} \oplus X, \overline{z})$, which is an $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula, in which each $z \in \overline{z}$ occurs at most once.

If $+\varphi'$ is of the form $+(\varphi'_1(\overline{p}, \overline{X}) \wedge \varphi'_2(\overline{p}, \overline{X}))$, then by the induction hypothesis applied to $+\varphi'_1$ and $+\varphi'_2$ we have that φ'_j is of the form $\varphi_j(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ which satisfies the statement for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$ which restricts to ϵ over $\overline{y}$. We let $\varphi = \varphi_1(\overline{x}, \overline{z}) \wedge \varphi_2(\overline{xz})$, which is IF$^{\Box}_{\delta}$. Hence φ' is of the form $\varphi_1(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}) \wedge \varphi_2(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, as required.

If $+\varphi'$ is of the form $+(\psi'(\overline{p}, \overline{X}) \to \chi)$ with $\epsilon^{\partial}(+\chi)$, then $-\psi'$ is non-trivially ϵ-PIA, and hence by the induction hypothesis ψ' is of the form $\psi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta^{\partial}}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ and δ an order-type over $\overline{x}$ which restricts to ϵ over $\overline{y}$, in which each $z \in \overline{z}$ occurs at most once. Moreover, $\epsilon^{\partial}(\gamma) \prec -\psi'$ for each $\gamma \in \overline{\gamma}$. Then we let $\varphi = \psi(\overline{x}, \overline{z}) \to z$, where z is a fresh variable, which is $(\overline{x}, \overline{z} \oplus z)$-IF$^{\Box}_{\delta}$. Hence φ' is of the form $(\psi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}) \to z)[\chi/z]$ as required. Finally, $\epsilon^{\partial}(\gamma) \prec -\psi' \prec +\varphi'$ for each $\gamma \in \overline{\gamma}$, and also $\epsilon^{\partial}(+\chi)$ implies that $\epsilon^{\partial}(\chi) \prec +\varphi'$.

The remaining cases are left to the reader. □

Analyzing lemma 7.4 above we note that it guarantees us the following:

1. To every inequality $\mathbf{j} \leq \varphi$ with $+\varphi$ non-trivially ϵ-PIA, of a suitable instance of the (IF$^{\sigma}_{R}$) rule can be applied.

2. By a 'suitable' instance we mean one given in terms of a normal $(\overline{y}, \overline{z})$-IF$^{\Box}_{\delta}$-formula with $\delta = \epsilon$ and in which all and only the ϵ-critical variable occurrences in $+\varphi$ are substituted for y-positions.

3. Consequently, all non-critical variable occurrences are in the $\overline{\gamma}$.

An analogous order-dual list of facts holds for inequalities $\psi \leq \mathbf{m}$ with $-\psi$ non-trivially ϵ-PIA.

7.3 The ϵ-Ackermann shape

Applying suitable instances of the rules (IF^{σ}_{R}) and (IF^{σ}_{L}) to the inequalities $\mathbf{j} \leq \varphi$ and $\psi \leq \mathbf{m}$ in (7.3), respectively, yields, for each $1 \leq i \leq n$,

$$\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}] \leq^{\epsilon_i} p_i \quad \text{and} \quad p_i \leq^{\epsilon_i} \mathsf{RA}^{\delta}_{i}(\psi)[\mathbf{m}/u, \overline{\gamma}/\overline{z}] \tag{7.4}$$

respectively. Lemma 5.3 implies that the polarity of the $\overline{\gamma}$ remains invariant under such applications. Indeed, we know that $\epsilon^{\partial}(\gamma) \prec +\varphi$ for each $\gamma \in \overline{\gamma}$. For each $1 \leq i \leq n$, if $\delta_i = 1$, then Lemma 5.3 implies that γ occurs in $\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$ with the opposite polarity to what it had in φ. Moreover, the rule yields $\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}] \leq^{1} p_i$, hence $\epsilon^{\partial}(\gamma) \prec -\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$. If $\delta_i = \partial$, then Lemma 5.3 implies that γ occurs in $\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$ with the same polarity to what it had in φ. Moreover, the rule yields $\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}] \leq^{\partial} p_i$, i.e, $p_i \leq \mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$, hence $\epsilon^{\partial}(\gamma) \prec +\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$. The application of ($\mathrm{IF}^{\sigma}_{L}$) to $\psi \leq \mathbf{m}$ is analyzed order-dually.

Since the only occurrences of $\overline{p}$ in $\mathsf{LA}^{\delta}_{i}(\varphi)[\mathbf{i}/u, \overline{\gamma}/\overline{z}]$ and in $\mathsf{RA}^{\delta}_{i}(\psi)[\mathbf{m}/u, \overline{\gamma}/\overline{z}]$ are of course the ones sitting in the $\overline{\gamma}$, the polarity invariance discussed above implies that the displayed inequalities in (7.4) are of the form

$$\alpha(\overline{p}) \leq p_i \quad \text{or} \quad p_j \leq \alpha'(\overline{p}), \tag{7.5}$$

with $\epsilon_i = 1$, $\epsilon_j = \partial$, $\epsilon^{\partial}(-\alpha(\overline{p}))$, and $\epsilon^{\partial}(+\alpha'(\overline{p}))$.

Definition 7.5. Given an order-type ϵ over $\overline{p}$, a set of $\mathcal{L}^{+}$-inequalities is in *ϵ-Ackermann shape* if each inequality in the set is of one of the following forms:

1. $\alpha(\overline{p}) \leq p_i$ with $\epsilon_i = 1$ and $\epsilon^{\partial}(-\alpha(\overline{p}))$;
2. $p_j \leq \alpha'(\overline{p})$ if $\epsilon_j = \partial$ and $\epsilon^{\partial}(+\alpha'(\overline{p}))$;
3. $\gamma \leq \gamma'$ with $\epsilon^{\partial}(-\gamma)$ and $\epsilon^{\partial}(+\gamma')$.

Taking stock of the reduction process up to this point, we see that the obtained system is now in ϵ-Ackermann shape. Indeed, through the application of approximation rules, the ϵ-critical occurrences in the antecedent of (7.1) have been ripped out, were made to sit in inequalities of the form (7.3), and then displayed in inequalities as in (7.5), which are clearly in one of the required shape of Definition 7.5.1 or 7.5.2. Moreover, the approximation rules produced, besides the inequalities of the form (7.3), also inequalities of the form $\mathbf{i} \leq \psi(\overline{x}, \gamma/\overline{z})$ and $\varphi(\overline{x}, \gamma/\overline{z}) \leq \mathbf{m}$, which, by remark 7.3, are of the form prescribed in Definition 7.5.3.

Exactly in the same way as was shown in [9], applying the Ackermann rule to a set of inequalities in ϵ-Ackermann shape produces a set of inequalities which is again in ϵ-Ackermann shape. Hence all occurrences of propositional variables may be eliminated by repeated applications of the Ackermann rule. Thus we shown that this procedure applied to an ϵ-recursive inequality in input produces a set of pure quasi-inequalities in $\mathcal{L}^{+}$, each of which can be equivalently translated into FO+LFP.

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