A new proof of a Paley—Wiener type theorem for the Jacobi transform

Tom Koornwinder

1. Introduction

Jacobi functions $\varphi_\lambda(t)$ of order (α, β) are the eigenfunctions of the differential operator $(\Delta(t))^{-1}(d/dt)(\Delta(t)\, d/dt)$, $\Delta(t)=(e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}$, such that $\varphi_\lambda(0)=1$, $\varphi_\lambda'(0)=0$. The Jacobi transform

$$f^\wedge(\lambda) = (2^{1/2}/\Gamma(\alpha+1))\int_0^\infty f(t)\varphi_\lambda(t)\Delta(t)\,dt, \tag{1.1}$$

which generalizes the Mehler-Fok transform, was studied by Titchmarsh [23, §4. 17], Olevskiĭ [21], Braaksma and Meulenbeld [2], Flensted—Jensen [9], [11, §2 and §12] and Flensted—Jensen and Koornwinder [12]. Some papers by Chébli [3], [4], [5] deal with a larger class of integral transforms which includes the Jacobi transform. An even more general class was considered by Braaksma and De Snoo [24].

In the present paper short proofs will be given of a Paley—Wiener type theorem and the inversion formula for the Jacobi transform. The L^2-theory, i.e. the Plancherel theorem, is then an easy consequence. These results were earlier obtained by Flensted—Jensen [9], [11, §2] and by Chébli [5]. However, to prove the Paley—Wiener theorem these two authors needed the L^2-theory, which can be obtained as a corollary of the Weyl—Stone—Titchmarsh—Kodaira theorem about the spectral decomposition of a singular Sturm—Liouville operator (cf. for instance Dunford and Schwartz [6, Chap. 13, §5]). The proofs presented here exploit the properties of Jacobi functions as hypergeometric functions and no general theorem needs to be invoked. Furthermore, it turns out that the Paley—Wiener theorem, which was proved by Flensted—Jensen [11, §2] for real α, β, $\alpha>-1$, holds for all complex values of α and β.

The key formula in this paper is a generalized Mehler formula

$$(\Gamma(\alpha+1))^{-1}\Delta(t)\varphi_\lambda(t) = \pi^{-1/2}\int_0^t \cos\lambda s\, A(s,t)\,ds, \tag{1.2}$$

where for $\mathrm{Re}\,\alpha > \mathrm{Re}\,\beta > -\frac{1}{2}$, $A(s,t)$ is given as an integral of elementary functions. Substituting (1.2) in (1.1) we can write the Jacobi transform $f^{\wedge}$ as the Fourier-cosine transform of $F(f)$, where the mapping F consists of two successive Weyl type fractional integral transforms. Thus the Jacobi transform is factorized as the product of three integral transforms with elementary kernels and the Paley—Wiener theorem follows from the mapping properties of these elementary transforms.

For certain discrete values of α and β the mapping F has a geometric and group-theoretic interpretation as a Radon transform on rank one symmetric spaces (cf. Helgason [16, Chap, 1,2]). For integer of half integer values of α and β such that $\alpha \geqq \beta \geqq -\frac{1}{2}$ a similar interpretation was given by Flensted—Jensen [10] on certain pseudo-Riemannian symmetric spaces. A large class of integral transforms for which the corresponding mapping F is positive was examined by Chébli [5]. Finally, Flensted—Jensen and Ragozin [13] wrote a note on the analogue of (1.2) for spherical functions on non-compact symmetric spaces of arbitrary rank.

In section 2 of this paper some properties and formulas for Jacobi functions are given. Section 3 contains the proof of the Paley—Wiener theorem for all complex α and β. Formula (1.2) is the only result on Jacobi functions which is needed there. In section 4 the inversion formula is derived by using the Paley—Wiener theorem, some estimates for Jacobi functions and a formula for Jacobi functions of the second kind which is dual to (1.2). The paper concludes with some remarks, in particular about the Plancherel theorem and about Paley—Wiener type theorems for the Hankel transform and for Jacobi series.

Notation. This is mainly similar to the notation used in [12]. For reasons of elegance and in order to avoid singularities if $\alpha = -1, -2, \ldots$, some constant factors have been changed. If no confusion is possible the indices α, β denoting the order may be deleted.

2. Jacobi functions

Consider for $\alpha, \beta, \lambda \in \mathbf{C}$ (the set of all complex numbers) and $0 < t < \infty$ the differential equation

$$(2.1) \qquad (\Delta_{\alpha,\beta}(t))^{-1} \frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right) = -(\lambda^2+\varrho^2)u(t),$$

where $\varrho = \alpha+\beta+1$ and

$$(2.2) \qquad \Delta_{\alpha,\beta}(t) = (e^t - e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}.$$

By substituting $z=-(\sinh t)^2$ in (2.1) a hypergeometric differential equation is obtained (cf. [7, 2.1 (1)]) with parameters $\frac{1}{2}(\varrho+i\lambda), \frac{1}{2}(\varrho-i\lambda), \alpha+1$. Hence, if $\alpha\neq-1, -2, -3, \ldots$ then the function

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F(\tfrac{1}{2}(\varrho+i\lambda), \tfrac{1}{2}(\varrho-i\lambda); \alpha+1; -(\sinh t)^2) \tag{2.3}$$

is the solution of (2.1) which satisfies $\varphi_\lambda(0)=1$, $\varphi_\lambda'(0)=0$. Here the hypergeometric function $F(a, b; c; z)$ denotes the unique analytic continuation for $z\notin[1, \infty)$ of the power series

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} z^n, \quad |z|<1.$$

Note that $(\Gamma(\alpha+1))^{-1}\varphi_\lambda^{(\alpha,\beta)}(t)$ is an entire function of α, β and λ (also for $\alpha=-1, -2, \ldots$).

For $\lambda\neq-i, -2i, -3i, \ldots$ another solution of (2.1) (cf. [7, 2.9 (9)]) is given by the function

$$\Phi_\lambda^{(\alpha,\beta)}(t) = (e^t-e^{-t})^{i\lambda-\varrho}\cdot \tag{2.4}$$
$$\cdot F(\tfrac{1}{2}(-\alpha+\beta+1-i\lambda), \tfrac{1}{2}(\alpha+\beta+1-i\lambda); 1-i\lambda; -(\sinh t)^{-2}).$$

This solution is characterized by the property that $\Phi_\lambda(t)=e^{(i\lambda-\varrho)t}(1+o(1))$ for $t\to\infty$. The functions $\varphi_\lambda(t)$ and $\Phi_\lambda(t)$ are called Jacobi functions of the first and second kind, respectively.

Using [7, 2.10 (2) and 2.10 (5)] we obtain for non-integer λ the identity

$$\pi^{1/2}(\Gamma(\alpha+1))^{-1}\varphi_\lambda(t) = \tfrac{1}{2}c(\lambda)\Phi_\lambda(t)+\tfrac{1}{2}c(-\lambda)\Phi_{-\lambda}(t), \tag{2.5}$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1}\Gamma(\frac{1}{2}i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}. \tag{2.6}$$

Note that for real λ, α, β, $\overline{c(\lambda)}=c(-\lambda)$.

It follows easily from (2.1) and the definitions of $\varphi_\lambda(t)$, $\Phi_\lambda(t)$, $\Delta(t)$ and $c(\lambda)$ that

$$\begin{cases} \varphi_{2\lambda}^{(-1/2,-1/2)}(t) = \cos\lambda t, & \Phi_\lambda^{(-1/2,-1/2)}(t) = e^{i\lambda t}, \\ \Delta_{-1/2,-1/2}(t) = 1, & c_{-1/2,-1/2}(\lambda) = 1, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \varphi_{2\lambda}^{(\alpha,\alpha)}(t) = \varphi_\lambda^{(\alpha,-1/2)}(2t), & \Phi_{2\lambda}^{(\alpha,\alpha)}(t) = \Phi_\lambda^{(\alpha,-1/2)}(2t), \\ \Delta_{\alpha,\alpha}(t) = \Delta_{\alpha,-1/2}(2t), & c_{\alpha,\alpha}(2\lambda) = c_{\alpha,-1/2}(\lambda). \end{cases} \tag{2.8}$$

The first two formulas of (2.8) can also be interpreted as quadratic transformations for hypergeometric functions, cf. [7, 2.11 (2) and 2.11 (26)].

Application of [7, 2.8 (20) and 2.8 (27)] gives the differentiation formulas

$$(2.9)\qquad (\Gamma(\alpha+1))^{-1}\frac{d\varphi_\lambda^{(\alpha,\beta)}(t)}{dt}=$$
$$=-\tfrac{1}{4}\left((\alpha+\beta+1)^2+\lambda^2\right)(\Gamma(\alpha+2))^{-1}\sinh 2t\;\varphi_\lambda^{(\alpha+1,\beta+1)}(t)$$

and

$$(2.10)\qquad (\Gamma(\alpha+2))^{-1}\frac{d}{dt}\left[(\sinh 2t)^{-1}\Delta_{\alpha+1,\beta+1}(t)\,\varphi_\lambda^{(\alpha+1,\beta+1)}(t)\right]=$$
$$=16(\Gamma(\alpha+1))^{-1}\Delta_{\alpha,\beta}(t)\,\varphi_\lambda^{(\alpha,\beta)}(t).$$

Next we derive some useful integration formulas for Jacobi functions. It follows from Bateman's integral [7, 2.4 (2)] and the identity

$$(2.11)\qquad F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$$

(cf. [7, 2.1 (23)]) that for $y>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}c>0$

$$(2.12)\qquad (\Gamma(c+\mu))^{-1}y^{c+\mu-1}(1+y)^{a+b-c+\mu}F(a+\mu,b+\mu;c+\mu;-y)=$$
$$=\frac{1}{\Gamma(c)\Gamma(\mu)}\int_0^y x^{c-1}(1+x)^{a+b-c}F(a,b;c;-x)\,(y-x)^{\mu-1}\,dx.$$

It follows from Askey and Fitch [1, (2.10)] that for $x>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}b>0$

$$(2.13)\qquad \Gamma(b)x^{-b}F(a,b;c;-x^{-1})=$$
$$=\frac{\Gamma(b+\mu)}{\Gamma(\mu)}\int_x^\infty y^{-b-\mu}F(a,b+\mu;c;-y^{-1})\,(y-x)^{\mu-1}\,dy.$$

Translating (2.12) and (2.13) in terms of Jacobi functions we obtain

$$(2.14)\qquad (\Gamma(\alpha+\mu+1))^{-1}\Delta_{\alpha+\mu,\beta+\mu}(t)\,\varphi_\lambda^{(\alpha+\mu,\beta+\mu)}(t)=$$
$$=\frac{2^{3\mu+1}\sinh 2t}{\Gamma(\alpha+1)\Gamma(\mu)}\int_0^t \Delta_{\alpha,\beta}(s)\,\varphi_\lambda^{(\alpha,\beta)}(s)\,(\cosh 2t-\cosh 2s)^{\mu-1}\,ds,$$

where $t>0$, $\operatorname{Re}\mu>0$, $\operatorname{Re}\alpha>-1$, and

$$(2.15)\qquad (c_{\alpha,\beta}(-\lambda))^{-1}\Phi_\lambda^{(\alpha,\beta)}(s)=$$
$$=\frac{2^{3\mu+1}}{c_{\alpha+\mu,\beta+\mu}(-\lambda)\Gamma(\mu)}\int_s^\infty \Phi_\lambda^{(\alpha+\mu,\beta+\mu)}(t)\,(\cosh 2t-\cosh 2s)^{\mu-1}\sinh 2t\,dt,$$

where $s>0$, $\operatorname{Re}\mu>0$, $\operatorname{Im}\lambda>-\operatorname{Re}(\alpha+\beta+1)$.

The inetgrals (2.14) and (2.15) connect Jacobi functions of order (α,β) with functions of order $(\alpha-\beta-\frac{1}{2},-\frac{1}{2})$ and Jacobi functions of order $(\alpha-\beta-\frac{1}{2},\alpha-\beta-\frac{1}{2})$ with functions of order $(-\frac{1}{2},-\frac{1}{2})$. Hence, by (2.7), (2.8), (2.14) and (2.15) we conclude that for $\operatorname{Re}\alpha>\operatorname{Re}\beta>-\frac{1}{2}$

$$(2.16)\qquad (\Gamma(\alpha+1))^{-1}\Delta(t)\,\varphi_\lambda(t)=\pi^{-1/2}\int_0^t\cos\lambda s\,A(s,t)\,ds$$

and

$$e^{i\lambda s} = \left(c(-\lambda)\right)^{-1}\int_s^\infty \Phi_\lambda(t)A(s,t)\,dt, \quad \operatorname{Im}\lambda > 0, \tag{2.17}$$

where the kernel is given by

$$A_{\alpha,\beta}(s,t) = \tag{2.18}$$

$$= \frac{2^{3\alpha+5/2}\sinh 2t}{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\int_s^t (\cosh 2t-\cosh 2w)^{\beta-1/2}(\cosh w-\cosh s)^{\alpha-\beta-1}\sinh w\,dw.$$

By substituting $\tau=(\cosh t-\cosh w)/(\cosh t-\cosh s)$ in (2.18) and using Euler's integral [7, 2.1 (10)] we obtain

$$A_{\alpha,\beta}(s,t) = 2^{3\alpha+2\beta+3/2}\left(\Gamma(\alpha+\tfrac{1}{2})\right)^{-1}(\sinh 2t)(\cosh t)^{\beta-1/2}\cdot \tag{2.19}$$

$$\cdot(\cosh t-\cosh s)^{\alpha-1/2}F\left(\tfrac{1}{2}+\beta,\ \tfrac{1}{2}-\beta;\ \alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}\right).$$

Combination of (2.19) and (2.11) gives

$$A_{\alpha,\beta}(s,t) = 2^{\alpha+2\beta+5/2}\left(\Gamma(\alpha+\tfrac{1}{2})\right)^{-1}(\sinh 2t)(\cosh t)^{\beta-\alpha}\cdot \tag{2.20}$$

$$\cdot(\cosh 2t-\cosh 2s)^{\alpha-1/2}F\left(\alpha+\beta,\ \alpha-\beta;\ \alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}\right).$$

Note that for $0\leqq s<t$ the argument of the hypergeometric functions in (2.19) and (2.20) has its value between 0 and $\frac{1}{2}$. Hence these hypergeometric functions are bounded functions in s and t. By analytic continuation with respect to α and β and by using the expressions (2.19) or (2.20) for the kernel it follows that formula (2.16) is valid for $\operatorname{Re}\alpha>-\frac{1}{2}$ and formula (2.17) holds if $\operatorname{Re}\alpha>-\frac{1}{2}$, $\operatorname{Im}\lambda>0$. It is clear from (2.19) and (2.20) that $A_{\alpha,\beta}(s,t)>0$ if $0\leqq s<t$, $\alpha>-\frac{1}{2}$ and $|\beta|\leqq \max(\frac{1}{2},\alpha)$.

From (2.16) and (2.20) we have the integral representation

$$\varphi_\lambda^{(\alpha,\beta)}(t) = 2^{-\alpha+3/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{1}{(\sinh t)^{2\alpha}(\cosh t)^{\alpha+\beta}}\cdot \tag{2.21}$$

$$\cdot\int_0^t \cos\lambda s(\cosh 2t-\cosh 2s)^{\alpha-1/2}F\left(\alpha+\beta,\ \alpha-\beta;\ \alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}\right)ds,$$

valid for $\operatorname{Re}\alpha>-\frac{1}{2}$. In view of (2.9), formula (2.21) in the case of order $(\alpha+1,\beta+1)$ gives an integral representation for $d\varphi_\lambda^{(\alpha,\beta)}(t)/dt$. This last integral can be rewritten by using integration by parts and by [7, 2.8 (27)]. Thus we obtain the integral representation

$$\frac{d\varphi_\lambda^{(\alpha,\beta)}(t)}{dt} = -2^{-\alpha+3/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{(\alpha+\beta+1)^2+\lambda^2}{\lambda}\cdot \tag{2.22}$$

$$\cdot\frac{1}{(\sinh t)^{2\alpha+1}(\cosh t)^{\alpha+\beta}}\int_0^t \sin\lambda s \sinh s(\cosh 2t-\cosh 2s)^{\alpha-1/2}\cdot$$

$$\cdot F\left(\alpha+\beta+1,\ \alpha-\beta-1;\ \alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}\right)ds,$$

which is also valid for $\mathrm{Re}\,\alpha>-\frac{1}{2}$.

We shall need some estimates which are essentially due to Flensted—Jensen [9, Theorem 2], [11, §2], but which will be stated here for all $\alpha,\beta\in\mathbf{C}$. The proof of Lemma 2.3 below is different from the proof given in [9].

Lemma 2.1. *For each $\alpha,\beta\in\mathbf{C}$ and $\delta>0$ there exists a positive constant K such that for all $t\geqq\delta$ and all $\lambda\in\mathbf{C}$ with $\mathrm{Im}\,\lambda\geqq 0$*

$$|\Phi_\lambda^{(\alpha,\beta)}(t)|\leqq Ke^{-(\mathrm{Im}\,\lambda+\mathrm{Re}\,\varrho)t}.$$

Lemma 2.2. *For each $\alpha,\beta\in\mathbf{C}$ and $r>0$ there exists a positive constant K such that if $\lambda\in\mathbf{C}$, $\mathrm{Im}\,\lambda\geqq 0$ and λ is at distance larger than r from the poles of $(c_{\alpha,\beta}(-\lambda))^{-1}$ then*

$$|c_{\alpha,\beta}(-\lambda)|^{-1}\leqq K(1+|\lambda|)^{\alpha+1/2}.$$

Lemma 2.1 follows by extending the proof of [9, Lemma 7] to the case of complex α and β. Lemma 2.2 follows from (2.6) and Stirling's formula.

Lemma 2.3. *For each $\alpha,\beta\in\mathbf{C}$ and for each non-negative integer n there exists a positive constant K such that for all $t\geqq 0$ and all $\lambda\in\mathbf{C}$*

$$\left|(\Gamma(\alpha+1))^{-1}\frac{d^n}{dt^n}\varphi_\lambda^{(\alpha,\beta)}(t)\right|\leqq K(1+|\lambda|)^{n+k}(1+t)e^{(|\mathrm{Im}\,\lambda|-\mathrm{Re}\,\varrho)t},$$

where $k=0$ if $\mathrm{Re}\,\alpha>-\frac{1}{2}$ and $k=[\frac{1}{2}-\mathrm{Re}\,\alpha]$ if $\mathrm{Re}\,\alpha\leqq-\frac{1}{2}$.

Proof. Consider first the case that $n=0$ and $\mathrm{Re}\,\alpha>-\frac{1}{2}$. It follows from (2.21) that

$$|\varphi_\lambda^{(\alpha,\beta)}(t)|\leqq \text{const.}\, e^{(|\mathrm{Im}\,\lambda|+\mathrm{Re}(\alpha-\beta))t}\cdot$$

$$\cdot(\sinh t\cosh t)^{-2\mathrm{Re}\,\alpha}\int_0^t(\cosh 2t-\cosh 2s)^{\mathrm{Re}\,\alpha-1/2}ds=$$

$$=\text{const.}\, e^{(|\mathrm{Im}\,\lambda|+\mathrm{Re}(\alpha-\beta))t}\varphi_0^{(\mathrm{Re}\,\alpha,\mathrm{Re}\,\alpha)}(t).$$

Applying [7, 2.10(7)] we have the estimate

$$\varphi_0^{(\mathrm{Re}\,\alpha,\mathrm{Re}\,\alpha)}(t)\leqq\text{const.}\,(1+t)e^{-(2\mathrm{Re}\,\alpha+1)t}.$$

By combining the last two equalities the lemma is proved for $n=0$. The estimate in the case that $n=1$, $\operatorname{Re}\alpha>-\frac{1}{2}$ and $|\lambda|<1$ follows from (2.9) and the estimate for $\varphi_\lambda^{(\alpha+1,\beta+1)}(t)$. If $n=1$, $\operatorname{Re}\alpha>-\frac{1}{2}$ and $|\lambda|\geqq 1$ then we conclude from (2.22) that

$$\left|\frac{d}{dt}\varphi_\lambda^{(\alpha,\beta)}(t)\right| \leqq \text{const.}\,(1+|\lambda|)e^{(|\operatorname{Im}\lambda|+\operatorname{Re}(\alpha-\beta))t}\varphi_0^{(\operatorname{Re}\alpha,\operatorname{Re}\alpha)}(t) \leqq$$

$$\leqq \text{const.}\,(1+|\lambda|)(1+t)e^{(|\operatorname{Im}\lambda|-\operatorname{Re}\varrho)t}.$$

The case that $n=0$, 1 and Re $\alpha\leqq-\frac{1}{2}$ follows by complete induction with respect to $k=[\frac{1}{2}-\operatorname{Re}\alpha]$ where formulas (2.9) and (2.10) are used. Finally we prove the case that $n=2, 3, \ldots$ by a complete induction with respect to n using the formula

$$\big(\Gamma(\alpha+1)\big)^{-1}\frac{d^n}{dt^n}\varphi_\lambda^{(\alpha,\beta)}(t) = -(\varrho^2+\lambda^2)\big(\Gamma(\alpha+1)\big)^{-1}\frac{d^{n-2}}{dt^{n-2}}\varphi_\lambda^{(\alpha,\beta)}(t)+$$

$$+\tfrac{1}{2}(\varrho^2+\lambda^2)\big(\Gamma(\alpha+2)\big)^{-1}\frac{d^{n-2}}{dt^{n-2}}[(\varrho\cosh 2t+\alpha-\beta)\varphi_\lambda^{(\alpha+1,\beta+1)}(t)].$$

This formula follows by differentiating the formula

$$\frac{d^2}{dt^2}\varphi_\lambda^{(\alpha,\beta)}(t) = (\varrho^2+\lambda^2)\left[\frac{\varrho\cosh 2t+\alpha-\beta}{2(\alpha+1)}\varphi_\lambda^{(\alpha+1,\beta+1)}(t)-\varphi_\lambda^{(\alpha,\beta)}(t)\right],$$

which is a consequence of (2.1) and (2.9). □

3. A Paley—Wiener type theorem

Let C_0^∞ be the class of all even infinitely differentiable functions on $\mathbf{R}$ (the set of all real numbers) with compact support. Let $\mathscr{H}$ be the class of even, entire, rapidly decreasing functions of exponential type, i.e., $g\in\mathscr{H}$ if and only if g is an even and entire analytic function on $\mathbf{C}$ and there exist positive constants A and K_n ($n=0, 1, 2, \ldots$) such that for all $\lambda\in\mathbf{C}$ and for all $n=0, 1, 2, \ldots$

$$|g(\lambda)| \leqq K_n(1+|\lambda|)^{-n}e^{A|\operatorname{Im}\lambda|}. \tag{3.1}$$

Let for $f\in C_0^\infty$ and $\operatorname{Re}\alpha>-1$ the Fourier—Jacobi transform $f\to f^\wedge_{\alpha,\beta}$ be defined by

$$f^\wedge_{\alpha,\beta}(\lambda) = \big(2^{1/2}/\Gamma(\alpha+1)\big)\int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt. \tag{3.2}$$

Clearly $f^{\wedge}_{\alpha,\beta}(\lambda)$ is analytic in $\alpha, \beta, \lambda \in \mathbf{C}$ with $\operatorname{Re}\alpha > -1$. Substitution of (2.10) in (3.2) and repeated integration by parts gives

$$(3.3)\qquad f^{\wedge}_{\alpha,\beta}(\lambda) = \frac{(-1)^n}{2^{4n}\Gamma(\alpha+n+1)} \int_0^\infty \left\{\left(\frac{1}{\sinh 2t}\frac{d}{dt}\right)^n f(t)\right\} \cdot$$
$$\cdot \varphi_\lambda^{(\alpha+n,\beta+n)}(t) \Delta_{\alpha+n,\beta+n}(t)\, dt, \quad n = 0, 1, 2, \ldots.$$

This formula defines the analytic continuation of $f^{\wedge}_{\alpha,\beta}(\lambda)$ for $\operatorname{Re}\alpha > -n-1$. Hence $f^{\wedge}_{\alpha,\beta}(\lambda)$ is an entire function of α, β, λ.

If $\alpha = \beta = -\frac{1}{2}$ then (3.2) reduces to the Fourier-cosine transform

$$(3.4)\qquad f^{\wedge}_{-1/2,-1/2}(\lambda) = (2/\pi)^{1/2} \int_0^\infty f(t) \cos \lambda t\, dt.$$

Theorem 3.1. (Paley and Wiener). *The Fourier-cosine transform is a bijection from C_0^∞ onto $\mathscr{H}$.*

For a proof see for instance Hörmander [17, Theorem 1.7.7]. In this section we shall generalize theorem 3.1 to general complex values of α and β.

Let for $f \in C_0^\infty$ and $\operatorname{Re}\alpha > -\frac{1}{2}$ the mapping $f \to F_{\alpha,\beta}(f)$ be defined by

$$(3.5)\qquad \big(F_{\alpha,\beta}(f)\big)(s) = \int_s^\infty f(t) A_{\alpha,\beta}(s,t)\, dt, \quad s > 0.$$

Note that $\big(F_{\alpha,\beta}(f)\big)(s)$ is analytic in α and β. In particular, if $\operatorname{Re}\alpha > \operatorname{Re}\beta > -\frac{1}{2}$ then by (2.18) we have

(3.6)
$$\big(F_{\alpha,\beta}(f)\big)(s) = \frac{2^{3\alpha+3/2}}{\Gamma(\alpha-\beta)} \int_s^\infty \left[\frac{1}{\Gamma(\beta+\frac{1}{2})} \int_w^\infty f(t)(\cosh 2t - \cosh 2w)^{\beta-1/2} d(\cosh 2t)\right] \cdot$$
$$\cdot (\cosh w - \cosh s)^{\alpha-\beta-1} d(\cosh w).$$

Combining (2.16), (3.2) and (3.5) we obtain that for $f \in C_0^\infty$ and $\operatorname{Re}\alpha > -\frac{1}{2}$

$$(3.7)\qquad f^{\wedge}_{\alpha,\beta}(\lambda) = (2/\pi)^{1/2} \int_0^\infty \big(F_{\alpha,\beta}(f)\big)(s) \cos \lambda s\, ds.$$

This means that the Jacobi transform of order (α, β) of f is the cosine transform of $F_{\alpha,\beta}(f)$.

To analyze the transform $F_{\alpha,\beta}$ consider the Weyl fractional integral transform $\mathscr{W}_\mu$ which is for $a \in \mathbf{R}$, $g \in C_0^\infty([a, \infty))$ and $\operatorname{Re}\mu > 0$ defined by

$$(3.8)\qquad \big(\mathscr{W}_\mu(g)\big)(y) = \big(\Gamma(\mu)\big)^{-1} \int_y^\infty g(x)(x-y)^{\mu-1}\, dx$$

(cf. [8, Chap. 13]). Here $C_0^\infty([a, \infty))$ denotes the class of infinitely differentiable functions on the interval $[a, \infty)$ (right differentiable in a) with compact support.

Repeated integration by parts in (3.8) gives

$$(3.9)\quad (\mathcal{W}_\mu(g))(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \left(\frac{d^n}{dx^n} g(x)\right)(x-y)^{\mu+n-1}\,dx, \quad n = 0, 1, 2, \ldots.$$

By (3.8) and (3.9) $(\mathcal{W}_\mu(g))(y)$ is defined as an entire function in μ, continuous in $(\mu, y)\in \mathbf{C}\times[a, \infty)$. Clearly, the function $\mathcal{W}_\mu(g)$ has also compact support and, since $(\mathcal{W}_\mu(g))'=\mathcal{W}_\mu(g')$ we conclude that $\mathcal{W}_\mu(g)\in C_0^\infty([a, \infty))$. It is an easy exercise to prove that $\mathcal{W}_0=\mathrm{id}$, $\mathcal{W}_{-1}(g)=-g'$, $\mathcal{W}_\mu\circ\mathcal{W}_\nu=\mathcal{W}_{\mu+\nu}$. In particular, $\mathcal{W}_\mu\circ\mathcal{W}_{-\mu}=\mathrm{id}=\mathcal{W}_{-\mu}\circ\mathcal{W}_\mu$. This proves the following theorem.

Theorem 3.2. *For all $a\in\mathbf{R}$ and $\mu\in\mathbf{C}$ the mapping $\mathcal{W}_\mu$, defined by* (3.9), *is bijective from $C_0^\infty([a, \infty))$ onto itself.*

Let us next define for $f\in C_0^\infty$, $\operatorname{Re}\mu>0$, $\sigma>0$, $s\geqq 0$

$$(3.10)\quad (\mathcal{W}_\mu^\sigma(f))(s) = (\Gamma(\mu))^{-1}\int_s^\infty f(t)\,(\cosh\sigma t-\cosh\sigma s)^{\mu-1}d(\cosh\sigma t).$$

Again we can extend $(\mathcal{W}_\mu^\sigma(f))(s)$ so an entire function of μ by

(3.11)
$$(\mathcal{W}_\mu^\sigma(f))(s) = \frac{(-1)^n}{\Gamma(\mu+n)}\int_s^\infty \left(\frac{d^n}{d(\cosh\sigma t)^n} f(t)\right)(\cosh\sigma t-\cosh\sigma s)^{\mu+n-1}d(\cosh\sigma t),$$

$$n = 0, 1, 2, \ldots, \quad \operatorname{Re}\mu>-n.$$

Let $f(t)=g(\cosh\sigma t)$. Then $f\in C_0^\infty$ if and only if $g\in C_0^\infty([1, \infty))$. Hence it follows from theorem 3.2 that for each $\mu\in\mathbf{C}$ the mapping $\mathcal{W}_\mu^\sigma$ is bijective from C_0^∞ onto itself. The inverse mapping of $\mathcal{W}_\mu^\sigma$ is $\mathcal{W}_{-\mu}^\sigma$. Applying this result to (3.6) we obtain

Corollary 3.3. *If $f\in C_0^\infty$ then $(F_{\alpha,\beta}(f))(s)$ has an analytic continuation to an entire function in α and β which is given by*

$$(3.12)\quad F_{\alpha,\beta}(f) = 2^{3\alpha+3/2}\mathcal{W}_{\alpha-\beta}^1\circ\mathcal{W}_{\beta+1/2}^2(f).$$

For all $\alpha, \beta\in\mathbf{C}$ the mapping $F_{\alpha,\beta}$ is bijective from C_0^∞ onto itself. The inverse mapping is given by

$$(3.13)\quad f = 2^{-3\alpha-3/2}\mathcal{W}_{-\beta-1/2}^2\circ\mathcal{W}_{\beta-\alpha}^1\circ F_{\alpha,\beta}(f).$$

Combination of Theorem 3.1, corollary 3.3 and formula (3.7) gives the Paley—Wiener type theorem for the Jacobi-transform.

Theorem 3.4. *For all $\alpha, \beta\in\mathbf{C}$ the mapping $f\to f_{\alpha,\beta}^\wedge$ is bijective from C_0^∞ onto $\mathcal{H}$.*

4. The inversion formula

It is well-known that the inversion formula for the cosine transform is given by

$$f(t) = (2/\pi)^{1/2} \int_0^\infty f^{\wedge}_{-1/2,-1/2}(\lambda) \cos \lambda t \, d\lambda, \tag{4.1}$$

where $f \in C_0^\infty$ and $f^{\wedge}_{-1/2,-1/2}(\lambda)$ is defined by (3.4). Substituting $\cos \lambda t = \frac{1}{2} e^{i\lambda t} + \frac{1}{2} e^{-i\lambda t}$ and changing the path of integration in (4.1) we also have

$$f(t) = (2\pi)^{-1/2} \int_{i\eta-\infty}^{i\eta+\infty} f^{\wedge}_{-1/2,-1/2}(\lambda) e^{i\lambda t} \, d\lambda, \tag{4.2}$$

where η is an arbitrary real number. In this section we shall generalize (4.1) and (4.2) to inversion formulas for the Jacobi transform.

Let for $g \in \mathscr{H}$, $t > 0$ and $\alpha, \beta \in \mathbf{C}$

$$g^{\vee}_{\alpha,\beta}(t) = (2\pi)^{-1/2} \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda) \Phi_\lambda^{(\alpha,\beta)}(t) \left(c_{\alpha,\beta}(-\lambda)\right)^{-1} d\lambda, \tag{4.3}$$

where $\eta \geqq 0$, $\eta > -\mathrm{Re}(\alpha+\beta+1)$, $\eta > -\mathrm{Re}(\alpha-\beta+1)$, i.e., $\left(c_{\alpha,\beta}(-\lambda)\right)^{-1}$ is a regular function of λ for $\mathrm{Im}\,\lambda \geqq \eta$. Let for $g \in \mathscr{H}$, A be a positive constant such that the estimates (3.1) hold and choose $\delta > 0$. Then by lemmas 2.1 and 2.2 there is a positive constant K such that for all $t \geqq \delta$ and all $\lambda \in \mathbf{C}$ with $\mathrm{Im}\,\lambda \geqq 0$ and λ outside arbitrary small neighborhoods of the poles of $\left(c_{\alpha,\beta}(-\lambda)\right)^{-1}$ we have

$$\left| g(\lambda) \Phi_\lambda^{(\alpha,\beta)}(t) \left(c_{\alpha,\beta}(-\lambda)\right)^{-1} \right| \leqq K e^{-\varrho t} (1+|\lambda|)^{-2} e^{(A-t)\,\mathrm{Im}\,\lambda}. \tag{4.4}$$

It follows that the integral in (4.3) absolutely converges and that its value does not depend on the choice of η. In particular, if $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$ then we can put $\eta = 0$ in (4.3) and by (2.5) we obtain

$$g^{\vee}_{\alpha,\beta}(t) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty \frac{g(\lambda) \varphi_\lambda^{(\alpha,\beta)}(t)}{c_{\alpha,\beta}(\lambda) c_{\alpha,\beta}(-\lambda)} \, d\lambda. \tag{4.5}$$

Lemma 4.1. *Let* $\mathrm{Re}\,\alpha > -\frac{1}{2}$ *and* $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$. *If* $g \in \mathscr{H}$ then $g^{\vee}_{\alpha,\beta} \in C_0^\infty$ *and* $(g^{\vee}_{\alpha,\beta})^{\wedge}_{\alpha,\beta} = g$.

Proof. It follows from (4.3) and (4.4) by letting $\eta \to \infty$ that $g^{\vee}_{\alpha,\beta}(t) = 0$ if $t > A$. It is clear from (4.5) that $g^{\vee}_{\alpha,\beta}$ is even. The estimates from lemmas 2.2 and 2.3 and formula (3.1) show that

$$\left| g(\lambda) \left(\frac{d^n}{dt^n} \varphi_\lambda^{(\alpha,\beta)}(t) \right) \left(c_{\alpha,\beta}(\lambda) c_{\alpha,\beta}(-\lambda)\right)^{-1} \right| \leqq$$

$$\leqq \mathrm{const.}\, (1+t) e^{-(\mathrm{Re}\,\varrho) t} (1+\lambda)^{-2},$$

uniformly if $\lambda, t \geqq 0$. Hence, by (4.5), $g^{\vee}_{\alpha,\beta} \in C_0^\infty$. To prove the second part of the theorem observe that for $\eta > 0$ and $s > 0$

$$\big(F_{\alpha,\beta}(g^{\vee}_{\alpha,\beta})\big)(s) =$$
$$= (2\pi)^{-1/2} \int_s^\infty A_{\alpha,\beta}(s,t)\,dt \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda)\,\Phi_\lambda^{(\alpha,\beta)}(t)\,\big(c_{\alpha,\beta}(-\lambda)\big)^{-1}\,d\lambda =$$
$$= (2\pi)^{-1/2} \int_{i\eta-\infty}^{i\eta+\infty} \Big[\int_s^\infty \Phi_\lambda^{(\alpha,\beta)}(t)\,\big(c_{\alpha,\beta}(-\lambda)\big)^{-1} A_{\alpha,\beta}(s,t)\,dt\Big]\, g(\lambda)\,d\lambda,$$

where the interchanging of integrals is allowed by Fubini's theorem, in view of (4.4) and the estimate

$$|A_{\alpha,\beta}(s,t)| \leqq \text{const.}\, e^{\varrho t}(t-s)^{\alpha-1/2}, \quad t > s > 0,$$

which is evident from (2.20). Inserting (2.17) we find that

$$\big(F_{\alpha,\beta}(g^{\vee}_{\alpha,\beta})\big)(s) = (2\pi)^{-1/2} \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda)\,e^{i\lambda s}\,d\lambda.$$

By inverting this formula it follows that

$$(g^{\vee}_{\alpha,\beta})^{\wedge}_{\alpha,\beta}(\lambda) = (2/\pi)^{1/2} \int_0^\infty \big(F_{\alpha,\beta}(g^{\vee}_{\alpha,\beta})\big)(s)\cos\lambda s\,ds = g(\lambda). \quad \square$$

Theorem 4.2. *Let $\alpha, \beta \in \mathbf{C}$. Then $f \in C_0^\infty$ and $g = f^{\wedge}_{\alpha,\beta}$ if and only if $g \in \mathscr{H}$ and $f = g^{\vee}_{\alpha,\beta}$.*

Proof. In view of theorem 3.4 it is sufficeint to prove that $(f^{\wedge}_{\alpha,\beta})^{\vee}_{\alpha,\beta}(t) = f(t)$ if $f \in C_0^\infty$, $t > 0$ and $\alpha, \beta \in \mathbf{C}$. By theorem 4.1 this is true for $\operatorname{Re}\alpha > -\frac{1}{2}$, $|\operatorname{Re}\beta| < \operatorname{Re}(\alpha+1)$. By (3.3) and (4.3) $(f^{\wedge}_{\alpha,\beta})^{\vee}_{\alpha,\beta}(t)$ is an entire function of α and β. Hence the theorem follows by analytic continuation. $\square$

5. Some remarks

Remark 1. Suppose that $\big(c_{\alpha,\beta}(-\lambda)\big)^{-1}$ has N poles $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that $\operatorname{Im}\lambda_n > 0$. Then a formula similar to (4.5) can be derived with additional terms of the type $c_n^{(\alpha,\beta)} g(\lambda_n)\,\varphi_{\lambda_n}^{(\alpha,\beta)}(t)$, $n = 1, 2, \ldots, N$ (cf. Flensted—Jensen [11, §2]). Complications arise if some pole of $\big(c_{\alpha,\beta}(-\lambda)\big)^{-1}$ is not simple or lies on the real axis or coincides with a pole of $\big(c_{\alpha,\beta}(\lambda)\big)^{-1}$.

Remark 2. Let $f \in C_0^\infty$ and $g \in \mathscr{H}$. Suppose for convience that $\big(c_{\alpha,\beta}(-\lambda)\big)^{-1}$ has no poles for $\operatorname{Im}\lambda \geqq 0$, i.e., $|\operatorname{Re}\beta| < \operatorname{Re}(\alpha+1)$. Then it is clear from (3.2) and (4.5) that

$$\int_0^\infty f(t)\,g^{\vee}(t)\,\Delta(t)\,dt = \int_0^\infty f^{\wedge}(\lambda)\,g(\lambda)\,\big(c(\lambda)c(-\lambda)\big)^{-1}\,d\lambda.$$

Here Fubini's theorem is used together with the estimates of lemmas 2.2 and 2.3 and formula (3.1). It follows by theorem 4.2 that for $f_1, f_2 \in C_0^\infty$

$$\int_0^\infty f_1(t) f_2(t) \Delta(t)\, dt = \int_0^\infty f_1^\wedge(\lambda) f_2^\wedge(\lambda) \left(c(\lambda)c(-\lambda)\right)^{-1} d\lambda. \tag{5.1}$$

Remark 3. For real α and β, $|\beta| < \alpha+1$, formula (5.1) implies Parseval's formula

$$\int_0^\infty f_1(t) \overline{f_2(t)} \Delta(t)\, dt = \int_0^\infty f_1^\wedge(\lambda) \overline{f_2^\wedge(\lambda)}\, |c(\lambda)|^{-2} d\lambda \tag{5.2}$$

where $f_1, f_2 \in C_0^\infty$. Hence, since C_0^∞ is dense in $L^2(\Delta)$ and $\mathscr{H}$ is dense in $L^2(|c(\lambda)|^{-2})$, the Jacobi transform can be extended to an isometric mapping from $L^2(\Delta)$ onto $L^2(|c(\lambda)|^{-2})$. This gives an alternative proof for the Plancherel theorem obtained by Flensted—Jensen [9, Prop. 3]. If $(c(-\lambda))^{-1}$ has poles for $\operatorname{Im}\lambda > 0$ then a discrete spectrum must be added (cf. [11, §2]).

Remark 4. A Paley—Wiener type theorem for the Hankel transform can be proved by similar methods as in section 3. Let $\mathscr{J}_\alpha(t)$ be a solution of the differential equation $u''(t) + (2\alpha+1)t^{-1}u'(t) + u(t) = 0$, $\alpha \neq -1, -2, \ldots$, such that $\mathscr{J}_\alpha(0) = 1$, $\mathscr{J}_\alpha'(0) = 0$. Then $\mathscr{J}_\alpha(t) = 2^\alpha \Gamma(\alpha+1) t^{-\alpha} J_\alpha(t)$, where $J_\alpha(t)$ is a Bessel function. If Re $\alpha > -\frac{1}{2}$ then it follows from the Poisson integral representation

$$\mathscr{J}_\alpha(t) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi e^{it\cos\varphi} (\sin\varphi)^{2\alpha} d\varphi$$

that

$$t^{2\alpha} \mathscr{J}_\alpha(\lambda t) = \frac{2\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})} \int_0^t \cos\lambda s\, (t^2 - s^2)^{\alpha-1/2} ds. \tag{5.3}$$

Define for $f \in C_0^\infty$ and Re $\alpha > -1$ the Hankel transform by

$$f^\wedge(\lambda) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty f(t) \mathscr{J}_\alpha(\lambda t) t^{2\alpha+1} dt. \tag{5.4}$$

Then

$$f^\wedge(\lambda) = (2/\pi)^{1/2} \int_0^\infty \cos\lambda s\, ds \frac{1}{2^{\alpha+1/2}\Gamma(\alpha+\frac{1}{2})} \cdot \tag{5.5}$$

$$\cdot \int_s^\infty f(t)\,(t^2 - s^2)^{\alpha-1/2} d(t^2), \quad \operatorname{Re}\alpha > -\tfrac{1}{2}.$$

Formula (5.5) is analogous to (3.6) and (3.7) and it can be used in a similar way.

Remark 5. For certain discrete values of α and β Jacobi functions are the spherical functions on non-compact symmetric spaces of rank one. In this context many formulas and results of [9] and the present paper were earlier obtained. Formula (3.7) corresponds to Helgason [15, (9)]. The function $e^{-\varrho s}(F_{\alpha,\beta}(f))(s)$ has a geometric interpretation as a Radon transform, where f is a radial function on the symmetric space (cf. Helgason [16, Chap. 1, 2]). The Paley—Wiener theorem for the spherical Fourier transform on non-compact symmetric spaces of rank one was first proved by Helgason [15].

Remark 6. Formulas (2.16) and (2.18) generalize the classical Mehler—Dirichlet formula (cf. Mehler [20])

$$P_\nu(\cos\theta) = \frac{2^{1/2}}{\pi}\int_0^\theta \frac{\cos(\nu+\frac{1}{2})\varphi}{(\cos\varphi-\cos\theta)^{1/2}}\,d\varphi,$$

where $P_\nu(x)$ is a Legendre function. These formulas can also be obtained from the Laplace type integral representation

$$(5.6)\qquad \varphi_\lambda^{(\alpha,\beta)}(t) = \frac{2\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\int_0^1\int_0^\pi |\cosh t+\sinh t\; re^{i\psi}|^{i\lambda-\varrho}\cdot$$

$$\cdot(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\psi)^{2\beta}\,dr\,d\psi,\quad t>0,\quad \operatorname{Re}\alpha>\operatorname{Re}\beta>-\tfrac{1}{2}$$

(cf. [18, (4)], [9, (3.5)]) by substituting first $\cosh t+\sinh t\; re^{i\psi}=e^s e^{i\chi}$ and next $\cosh w=\cos\chi\cosh t$. A general method of transforming integrals of type (5.6) into integrals of type (2.16) is discussed in [19, §5].

Remark 7. Let $R_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial. Then

$$R_n^{(\alpha,\beta)}(\cos\theta) = F(-n,n+\alpha+\beta+1;\alpha+1;\sin^2\tfrac{1}{2}\theta) = \varphi^{(\alpha,\beta)}_{(n+\alpha+\beta+1)i}(\tfrac{1}{2}i\theta).$$

Analogous to (2.16), (2.18) and (2.19) we obtain

$$(5.7)\qquad R_n^{(\alpha,\beta)}(\cos\theta) = \frac{2^{\alpha-2\beta-1/2}\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(\sin\tfrac{1}{2}\theta)^{-2\alpha}(\cos\tfrac{1}{2}\theta)^{-2\beta}\cdot$$

$$\cdot\int_0^\theta(\cos\psi-\cos\theta)^{\beta-1/2}\sin\tfrac{1}{2}\psi\,d\psi\int_0^\varphi\cos\big(n+\tfrac{1}{2}(\alpha+\beta+1)\big)\varphi\cdot$$

$$\cdot(\cos\tfrac{1}{2}\varphi-\cos\tfrac{1}{2}\psi)^{\alpha-\beta-1}\,d\varphi,\quad \operatorname{Re}\alpha>\operatorname{Re}\beta>-\tfrac{1}{2},\quad 0<\theta<\pi,$$

$$(5.8)\qquad R_n^{(\alpha,\beta)}(\cos\theta) = \frac{2^{\alpha-1/2}\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+\frac{1}{2})}(\sin\tfrac{1}{2}\theta)^{-2\alpha}(\cos\tfrac{1}{2}\theta)^{-\beta-1/2}\cdot$$

$$\cdot\int_0^\theta\cos\big(n+\tfrac{1}{2}(\alpha+\beta+1)\big)\varphi\;F\left(\tfrac{1}{2}+\beta,\,\tfrac{1}{2}-\beta;\,\alpha+\tfrac{1}{2};\,\frac{\cos\frac{1}{2}\theta-\cos\frac{1}{2}\varphi}{2\cos\frac{1}{2}\theta}\right)d\varphi,$$

$$\operatorname{Re}\alpha>-\tfrac{1}{2},\quad 0<\theta<\pi.$$

Quadratic transformation of the hypergeometric function in (5.8) by means of [7, 2.11 (22)]gives another integral representation for $R_n^{(\alpha,\beta)}(\cos\theta)$, which was independently obtained by Gasper [14] in a quite different way.

Remark 8. Suppose that f is an even C^∞-function on $(-\pi, \pi)$ with compact support. If f is expanded in a Fourier—Jacobi series with respect to $R_n^{(\alpha,\beta)}(\cos\theta)$ $(\alpha>\beta>-\frac{1}{2})$ then the Fourier coefficients are given by

(5.9)
$$f^\wedge(n) = \big(\Gamma(\alpha+1)\big)^{-1}\int_0^\pi f(\theta)\,R_n^{(\alpha,\beta)}(\cos\theta)\,(\sin\tfrac{1}{2}\theta)^{2\alpha+1}(\cos\tfrac{1}{2}\theta)^{2\beta+1}\,d\theta,$$
$$n = 0, 1, 2, \ldots.$$

Substitution of (5.7) in (5.9) gives

$$(5.10)\qquad f^\wedge(n) = \frac{2^{\alpha-2\beta-3/2}}{\pi^{1/2}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\int_0^\pi \cos\big(n+\tfrac{1}{2}(\alpha+\beta+1)\big)\varphi\,d\varphi\,\cdot$$
$$\cdot\int_{\psi=\varphi}^\pi (\cos\tfrac{1}{2}\varphi-\cos\tfrac{1}{2}\psi)^{\alpha-\beta-1}\,d(\cos\tfrac{1}{2}\psi)\int_{\theta=\psi}^\pi f(\theta)\,(\cos\psi-\cos\theta)^{\beta-1/2}\,d(\cos\theta).$$

In the same way as in section 3 we can write

$$(5.11)\qquad f^\wedge(n) = \int_0^\pi \cos\big(n+\tfrac{1}{2}(\alpha+\beta+1)\big)\varphi\;\big(F(f)\big)(\varphi)\,d\varphi,$$

where the mapping F is a bijection from the class of even C^∞-functions on $(-\pi, \pi)$ with compact support onto itself. Then the function $f^\wedge$ is well-defined and analytic for all complex values of its argument. Now the classical Paley—Wiener theorem implies a Paley—Wiener type theorem for Jacobi series.

Theorem 5.1. *Let $\alpha>\beta>-\frac{1}{2}$. The function $f^\wedge$ is the Fourier—Jacobi transform of an even C^∞-function on $(-\pi, \pi)$ with compact support if and only if there is a function $g\in\mathscr{H}$ such that $A<\pi$ in* (3.1) *and* $f^\wedge(n)=g\big(n+\frac{1}{2}(\alpha+\beta+1)\big)$, $n=0, 1, 2, \ldots$.

Since g is of exponential type less than π an application of Carlson's theorem (cf. Titchmarsh [23, §5. 81]) shows that g is uniquely determined by $f^\wedge_{\alpha,\beta}(n)$, $n=$ $=0, 1, 2, \ldots$. Just as in section 3 theorem 5.1 remains valid for all $\alpha, \beta\in\mathbf{C}$. R. Askey informed me that in the case $\alpha=\beta=0$ this theorem is due to Beurling (unpublished).

References

1. Askey, R. and Fitch, J., Integral representations for Jacobi polynomials and some applications. *J. Math. Anal. Appl.*, **26** (1969), 411—437.
2. Braaksma, B. L. J. and Meulenbeld, B., Integral transforms with generalized Legendre functions as kernels. *Compositio Math.*, **18** (1967), 235—287.
3. Chébli, H., Sur la positivité des opérateurs de "translation généralisée" associés à un opérateur de Sturm—Liouville sur [0, ∞[. *C. R. Acad. Sci. Paris*, A **275** (1972), 601—604.
4. Chébli, H., Positivité des opérateurs de translation généralisée associés à un opérateur de Sturm-Liouville sur]0, ∞[. *Séminaire de Théorie Spectrale, année 1972—73, Institut de Recherche Mathématique Avancée, Strasbourg.*

5. CHÉBLI, H., Sur un théorème de Paley—Wiener associé à la decomposition spectrale d'un opérateur de Sturm—Liouville sur]0, ∞[. *J. Functional Anal.*, **17** (1974), 447—461.
6. DUNFORD, N. and SCHWARTZ, J. T., *Linear Operators, Vol. II.* Interscience Publishers, New York, 1963.
7. ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F. G., *Higher Transcendental Functions, Vol. I.* McGraw-Hill, New York, 1953.
8. ERDÉLYI, A., MAGNUS, W., OBERHETTIGER, F. and TRICOMI, F. G., *Tables of integral transforms, Vol. II.* McGraw-Hill, New York, 1954.
9. FLENSTED-JENSEN, M. Paley—Wiener type theorems for a differential operator connected with symmetric spaces. *Ark. Mat.*, **10** (1972), 143—162.
10. FLENSTED-JENSEN, M., Spherical functions on rank one symmetric spaces and generalizations. *Proceedings Symposia Pure Mathematics, Vol. 26, American Mathematical Society, Providence (R. I.)*, 1973.
11. FLENSTED-JENSEN, M., The spehrical functions on the universal covering of $SU(n-1,1)/SU(n-1)$. *Københavns Universitet Mat. Institut Preprint Series,* (1973), No. 1.
12. FLENSTED-JENSEN, M. and KOORNWINDER, T. H., The convolution structure for Jacobi function expansions. *Ark. Mat.*, **11** (1973), 245—262.
13. FLENSTED-JENSEN, M. and RAGOZIN, D. L., Spherical functions are Fourier transforms of L^1-functions. *Ann. Sci. Ecole Norm. Sup. (4)*, **6** (1973), 457—458.
14. GASPER, G., Formulas of the Dirichlet-Mehler type. *Prepublication* (1973).
15. HELGASON, S., An analogue of the Paley—Wiener theorem for the Fourier transform on certain symmetric spaces. *Math. Ann.*, **165** (1966), 297—308.
16. HELGASON, S., A duality for symmetric spaces with applications to group representations. *Advances in Math.*, **5** (1970), 1—154.
17. HÖRMANDER, L., *Linear partial differential operators.* Springer-Verlag, Berlin, 1963.
18. KOORNWINDER, T. H., The addition formula for Jacobi polynomials, I. Summary of results. *Nederl. Akad. Wetensch. Proc. A*, **75** = *Indag. Math.*, **34** (1972), 188—191.
19. KOORNWINDER, T. H., Jacobi polynomials, II. An analytic proof of the product formula. *SIAM J. Math. Anal.*, **5** (1974), 125—137.
20. MEHLER, F. G., Ueber eine mit den Kugel und Cylinderfunctionen verwnadte Function und ihre Anwendung in der Theorie der Elektricitätsvertheilung. *Math. Ann.*, **18** (1881), 161—194.
21. OLEVSKIĬ, M. N. On the representation of an arbitrary function in the form of an integral with a kernel containing a hypergeometric function. *Dokl. Akad. Nauk. SSSR.*, **69** (1949), 11—14 (Russian).
22. TITCHMARSH, E. C. *The theory of functions.* Oxford University Press, second edition, 1939.
23. TITCHMARSH, E. C. *Eigenfunction expansions associated with second-order differential equations, Vol. I.* Oxford University Press, 1946.
24. BRAAKSMA, B. L. J. and DE SNOO, H. S. V., Generalized translation operators associated with a singular differential operator. *"Proceedings of a Conference on the Theory of Ordinary and Partial Differential Equations held in Dundee, Scotland, March 1974"*, Lecture Notes in Math., Springer-Verlag. (To appear.)

Received September 17, 1974

T. Koornwinder
Mathematisch Centrum
2:e Boerhaavestraat 49
AMSTERDAM (Oost), Holland