# On Zeilberger's algorithm and its $q$-analogue: a rigorous description

Tom H. Koornwinder

*Universiteit van Amsterdam*
*Korteweg-de Vries Institute for Mathematics*
*Plantage Muidergracht 24*
*1018 TV Amsterdam, The Netherlands*
*email:* `thk@science.uva.nl`

ABSTRACT. Gosper's and Zeilberger's algorithms for summation of terminating hypergeometric series as well as the $q$-versions of these algorithms are described in a very rigorous way. The paper is a companion to Maple V procedures implementing these algorithms. It concludes with the help information for these procedures.

KEYWORDS. Gosper's algorithm, Zeilberger's algorithm, explicit summation formulas, terminating hypergeometric series, terminating $q$-hypergeometric series, computer algebra, Maple.

NOTE. This research was mostly done at CWI, Amsterdam.

## 1. Introduction

In 1978 Gosper [5] published an algorithm for indefinite summation of terminating hypergeometric series. Procedures incorporating this algorithm were for instance included long ago in the standard library of Maple V . Around 1990 Zeilberger [17], [18] showed that definite summation of terminating hypergeometric series can often be reduced in an algorithmic way to Gosper's indefinite summation. He wrote a long Maple procedure implementing his algorithm and he kindly made available his code to all interested people. Zeilberger's algorithm turned out to have a $q$-version, for which Zeilberger wrote a less widely circulated Maple procedure.

The author [8] wrote a critical survey paper about Maple's potential to handle hypergeometric series. There he also briefly described Gosper's and Zeilberger's algorithms. It is the purpose of the present paper to describe these two algorithms as well as their $q$-versions in a very rigorous way.

A companion to this paper are two Maple V procedures, called `zeilb` and `qzeilb`, implementing the Zeilberger and $q$-Zeilberger algorithm, respectively. These procedures are highly rewritten versions of the original procedures written by Zeilberger (see [19] for the most reecent versions of Zeilberger's procedures). It is the intention that this Maple code matches the rigor of the present paper. Thus the present paper together with the source code should convince the reader that the output produced by the procedures can be trusted. Furthermore, input and output are arranged in such a way that evaluation formulas of terminating hypergeometric or $q$-hypergeometric series as given in Bailey [1] respectively Gasper & Rahman [4] (in particular Appendix II) can be compared very easily with the results appearing on the computer screen. The source codes are available on the web page `http://www/science.uva.nl/~thk/art/zeilbalgo/`. Since the first version of this paper appeared, much further work has been done and more powerful implementations of the $(q)$-Zeilberger algorithm have been written:

- Petkovšek, Wilf & Zeilberger wrote a book [14] about Zeilberger's and related algorithms. Maple implementations [19] are available (procedures `EKHAD` and `qEKHAD`).

Version of 2 January 2001

- Paule & Schorn [12] implemented Zeilberger's algorithm for Mathematica, while Paule & Riese [11] implemented its $q$-analogue for Mathematica. See their web page `http://www.risc.uni-linz.ac.at/research/combinat/risc/software/` for downloads.
- W. Koepf wrote a book [6] about Zeilberger's and related algorithms. Maple implementations [7] are available. (packages `hsum` and `qsum`). The `qsum` algorithm is documented in a paper by Böing & Koepf [2].

The contents of this paper are as follows. In Section 2 the idea of Zeilberger's algorithm is explained by a simple example. Section 3 describes Gosper's algorithm, Section 4 Zeilberger's algorithm and Section 5 the $q$-versions. Finally Section 6 provides the help information for the functions `zeilb` and `qzeilb`.

Definitely not included in this paper is the theoretical background concerning holonomic systems (cf. Zeilberger [16] and Cartier [3]) and further generalizations of the method of Zeilberger's algorithm (cf. for instance Petkovšek [13] and Wilf & Zeilberger [15]).

A slightly shortened version of this paper appeared in [9]. The present version of the paper will be regularly updated together with the Maple procedures `zeilb` and `qzeilb`. The present version of these Maple procedures can be used under Maple 6 and under Maple V, Release 4 or 5. Some minor bugs have been corrected.

*Acknowledgements.* I thank Doron Zeilberger for sending me his preprints and the source codes of his Maple procedures. George Gasper provoked me to use Minton's summation formula as a testing case. André Heck was always very helpful in answering questions about Maple. Peter Paule, Axel Schorn, Wolfram Koepf and René Swarttouw gave useful comments concerning my Maple implementation.

## 2. A simple example

Consider the Chu-Vandermonde summation formula

$$ {}_2F_1(-n,b;c;1) := \sum_{k=0}^{n} \frac{(-n)_k\,(b)_k}{(c)_k\,k!} = \frac{(c-b)_n}{(c)_n}\,, \quad n=0,1,2,\ldots, \tag{2.1} $$

and its special case for $c := -n$:

$$ \sum_{k=0}^{n} \frac{(b)_k}{k!} = \frac{(b+1)_n}{n!}\,, \quad n=0,1,2,\ldots\;. \tag{2.2} $$

Here the *shifted factorial* is defined by

$$ (a)_k := \frac{\Gamma(a+k)}{\Gamma(k)}, \quad k \in \mathbb{Z}. \tag{2.3} $$

In identity (2.2) there is an arbitrary upper boundary $n$ for the summation, while the summand is independent of $n$. We call it *indefinite summation*. Verification of (2.2) is straightforward by checking that

$$ \frac{(b+1)_n}{n!} - \frac{(b+1)_{n-1}}{(n-1)!} = \frac{(b)_n}{n!}\,, \quad n=1,2,\ldots\,; \quad \frac{(b+1)_0}{0!} = \frac{(b)_0}{0!}\,. \tag{2.4} $$

However, in identitity (2.1), the summand depends on the upper boundary $n$ of summation. There would be no explicit evaluation for an arbitrary upper boundary $m$. It works just for upper boundary $n$ or for any upper boundary $m = n, n+1, \ldots$ or $\infty$ (since the terms in (2.1) with $k > n$ vanish). So $n$ is a natural upper boundary for the summation. We call this *definite summation*. Observe also that verification of (2.1) is not as straightforward as was possible for (2.2) by means of (2.4).

In fact, we can find an indefinite summation formula which implies (2.1). Put

$$\Sigma(n) := \sum_{k=0}^{n} A(n,k), \tag{2.5}$$

where

$$A(n,k) := \frac{(-n)_k\,(b)_k}{(c)_k\,k!}. \tag{2.6}$$

We want to prove that

$$\Sigma(n) = (c-b)_n/(c)_n\,, \tag{2.7}$$

or equivalently, that $\Sigma(0) = 1$ and

$$\Sigma(n) + \sigma(n)\,\Sigma(n-1) = 0, \quad n = 1, 2, \ldots, \tag{2.8}$$

where

$$\sigma(n) := -\frac{c-b+n-1}{c+n-1}\,. \tag{2.9}$$

Now the indefinite summation formula

$$\sum_{k=0}^{m} \big(A(n,k) + \sigma(n)\,A(n-1,k)\big) = \frac{(-n+1)_m\,(b)_{m+1}}{(c+n-1)\,(c)_m\,m!} \tag{2.10}$$

can immediately be proved by checking that

$$\frac{(-n+1)_m\,(b)_{m+1}}{(c+n-1)\,(c)_m\,m!} - \frac{(-n+1)_{m-1}\,(b)_m}{(c+n-1)\,(c)_{m-1}\,(m-1)!} = A(n,m) + \sigma(n)\,A(n-1,m), \quad m = 1, 2, \ldots, \tag{2.11}$$

and

$$\left.\frac{(-n+1)_m\,(b)_{m+1}}{(c+n-1)\,(c)_m\,m!}\right|_{m=0} = A(n,0) + \sigma(n)\,A(n-1,0). \tag{2.12}$$

(Note that, in (2.6), (2.9), (2.10), (2.11) and (2.12), $n$ can be arbitrarily complex. It can be considered as a parameter, just as $b$ and $c$.) Now (2.10) for $m = n$ yields

$$\Sigma(n) + \sigma(n)\,\Sigma(n-1) = \sum_{k=0}^{n} A(n,k) + \sigma(n) \sum_{k=0}^{n-1} A(n-1,k) = \sum_{k=0}^{n} \big(A(n,k) + \sigma(n)\,A(n-1,k)\big) = 0, \tag{2.13}$$

since the right-hand side of (2.10) vanishes for $m = n$. Hence we obtain (2.8).

Note that (2.10) can be rewritten as an indefinite summation for a certain hypergeometric series which is truncated arbitrarily:

$$\sum_{k=0}^{m} \frac{(-n)_k\,(b)_k\,(bn(n+c-b-1)^{-1}+1)_k}{(c)_k\,(bn(n+c-b-1)^{-1})_k\,k!} = \frac{(-n+1)_m\,(b+1)_m}{(c)_m\,m!}\,.$$

**3. Gosper's algorithm**

Let $\mathbb{F}$ be the field of rational functions in some fixed number of indeterminates (not including $k$) over $\mathbb{Q}$. Let $\mathbb{F}(k)$ denote the field of rational functions in $k$ over $\mathbb{F}$ and let $\mathbb{F}[k]$ be the ring of polynomials in $k$ over $\mathbb{F}$. Let $a(k)$ $(k = 0, 1, \ldots)$ be a sequence of nonzero elements of $\mathbb{F}$ such that $a(k)/a(k-1)$ is in $\mathbb{F}(k)$. Call a sequence $s(k)$ $(k = -1, 0, 1, \ldots)$ in $\mathbb{F}$ an *indefinite sum* for the $a(k)$ if

$$s(n) - s(m) = \sum_{k=m+1}^{n} a(k), \quad n, m = -1, 0, 1, \ldots, \quad m < n, \tag{3.1}$$

or, equivalently,

$$s(k) - s(k-1) = a(k), \quad k = 0, 1, \ldots \,. \tag{3.2}$$

Then the $s(k)$ are unique up to a constant term. Gosper's algorithm will do the following:

1) It determines whether there is a solution $s(k)$ to (3.2), nonzero for $k = 0, 1, 2, \ldots$, such that $s(k-1)/s(k)$ is rational in $k$ over $\mathbb{F}$.
2) If the answer to 1) is positive then it will produce this solution explicitly.

In order to justify the algorithm we need a few lemmas.

**Lemma 3.1.** Let $b(k)$ be a nonzero element of $\mathbb{F}(k)$. Then there are elements $p(k)$, $r_1(k)$ and $r_2(k)$ of $\mathbb{F}[k]$, unique up to a constant factor, such that

$$b(k) = \frac{p(k)}{p(k-1)} \, \frac{r_1(k)}{r_2(k)}\,, \tag{3.3}$$

$$\gcd(r_1(k), r_2(k+j)) = 1 \tag{3.4}$$

for all integers $j \ge 0$, and

$$\gcd(r_1(k), p(k-1)) = 1 = \gcd(r_2(k), p(k)). \tag{3.5}$$

**Proof** We first prove the existence statement. Suppose that, for some $i = 1, 2, \ldots$, identity (3.3) holds together with (3.4) for $j = 0, 1, \ldots, i-1$ and with (3.5). This is certainly possible with $i := 1$ and $p(k) := 1$. We now describe a successive rewriting of $p(k)$, $r_1(k)$, $r_2(k)$ such that this process comes to an end and the end result has the desired properties. If $r_1(k)$ has a prime factor $\gamma(k)$ such that $\gamma(k-i)$ is a factor of $r_2(k)$ then put $\widetilde{r_1}(k) := r_1(k)/\gamma(k)$, $\widetilde{r_2}(k) := r_2(k)/\gamma(k-i)$, $\widetilde{p}(k) := p(k)\,\gamma(k)\,\gamma(k-1)\ldots\gamma(k-i+1)$. Then (3.3), (3.4) for $j = 0, 1, \ldots, i-1$, and (3.5) still hold when $p(k)$, $r_1(k)$, $r_2(k)$ are replaced by $\widetilde{p}_k$, $\widetilde{r_1}(k)$, $\widetilde{r_2}(k)$, respectively. In order to see this for (3.5) observe that any common factor of $\widetilde{p}(k)$ and $\widetilde{r_2}(k)$ must be $\gamma(k-j)$ for some $j = 0, 1, \ldots, i-1$. But this cannot be a factor of $r_2(k)$ while $\gamma(k)$ is a factor of $r_1(k)$, Similarly, any common factor of $\widetilde{p}(k-1)$ and $\widetilde{r_1}(k)$ must be $\gamma(k-j)$ for some $j = 1, 2, \ldots, i$. But this cannot be a factor of $r_1(k)$ while $\gamma(k-i)$ is a factor of $r_2(k)$.

Next we prove the unicity statement. Suppose $p(k), r_1(k), r_2(k)$ and $\widetilde{p}(k), \widetilde{r_1}(k), \widetilde{r_2}(k)$ are two triples satisfying (3.3), (3.4) for all integers $j \ge 0$, and (3.5). Then

$$p(k)\,r_1(k)\,\widetilde{p}(k-1)\,\widetilde{r_2}(k) = \widetilde{p}(k)\,\widetilde{r_1}(k)\,p(k-1)\,r_2(k).$$

Suppose $\gamma(k)$ is a prime factor occurring in $p(k)$ with higher multiplicity than in $\widetilde{p}(k)$. We may assume that for all positive integers $i$ the prime factor $\gamma(k+i)$ does not occur with higher multiplicity in $p(k)$ than in $\widetilde{p}(k)$. Then $\gamma(k)$ must be a factor of $\widetilde{r_1}(k)$. Let $j \ge 0$ be the maximal integer such that $\gamma(k-j)$ occurs in $p(k)$ with higher multiplicity than in $\widetilde{p}(k)$. Then $\gamma(k-j-1)$ must be a factor of $\widetilde{r_2}(k)$. Thus $\gcd(r_1(k), r_2(k+j+1)) \ne 1$, which is a contradiction. Similarly we show that no prime factor occurs in $\widetilde{p}(k)$ with higher multiplicity than in $p(k)$. Thus $p(k)$ equals $\widetilde{p}(k)$ up to a constant factor. Hence $r_1(k)/r_2(k)$ and $\widetilde{r_1}(k)/\widetilde{r_2}(k)$ are equal up to a constant factor, which implies the unicity statement. □

**Lemma 3.2.** Let $b(k)$ be in $\mathbb{F}(k)$ such that $b(k)$ is a nonzero element of $\mathbb{F}$ for each $k = 1, 2, \dots$ . Let $b(k)$ be written as (3.3), where $p(k), r_1(k), r_2(k)$ are in $\mathbb{F}[k]$, satisfy (3.4) for all integers $j \geq 0$, and also satisfy

$$\gcd(r_1(k), p(k-1)) \neq 0 \neq \gcd(r_2(k), p(k)) \quad \text{for } k = 1, 2, \dots . \tag{3.6}$$

Then $p(k) \neq 0$ in $\mathbb{F}$ for $k = 0, 1, 2, \dots$ and $r_1(k), r_2(k) \neq 0$ in $\mathbb{F}$ for $k = 1, 2, \dots$ .

**Proof** Clearly, because of (3.3), (3.4) and (3.6), $r_1(k)$ and $r_2(k)$ must be nonzero for $k = 1, 2, \dots$ . If $p(k) = 0$ for some $k = 0, 1, 2, \dots$ then there will be a highest nonnegative integer $j$ for which $p(j) = 0$. Then $b(k)$ will have a pole at $k = j + 1$, which is contrary to the assumption. □

We now assume that, for each integer $k \geq 0$, $a(k)$ is a nonzero element of $\mathbb{F}$ and that

$$\frac{a(k)}{a(k-1)} = \frac{p(k)}{p(k-1)} \frac{r_1(k)}{r_2(k)}, \quad k = 1, 2, \dots, \tag{3.7}$$

such that (3.4) holds for all integers $j \geq 0$. Assume also that $p(k) \neq 0$ in $\mathbb{F}$ for $k = 0, 1, 2, \dots$ and $r_1(k), r_2(k) \neq 0$ in $\mathbb{F}$ for $k = 1, 2, \dots$ . Because of Lemma 3.2 these inequalities will be certainly satisfied if (3.5) or the weaker (3.6) are valid. It follows from (3.7) that

$$\frac{a(k+1)\, r_2(k+1)}{p(k+1)} = \frac{a(k)\, r_1(k+1)}{p(k)}, \quad k = 0, 1, 2, \dots . \tag{3.8}$$

Let $s(k)$ and $f(k)$ be elements of $\mathbb{F}$ defined for $k = -1, 0, 1, \dots$ such that

$$s(k) = \frac{r_2(k+1)\, a(k+1)}{p(k+1)}\, f(k), \quad k = -1, 0, 1, \dots . \tag{3.9}$$

In view of (3.8) we also have

$$s(k) = \frac{r_1(k+1)\, a(k)}{p(k)}\, f(k), \quad k = 0, 1, 2, \dots . \tag{3.10}$$

Note that, for $k = 0, 1, 2, \dots$, we have $s(k) \neq 0$ iff $f(k) \neq 0$ (because the other factors in (3.9) are nonzero). We will always assume that $s(k)$ and $f(k)$ are nonzero elements of $\mathbb{F}$ for $k = 0, 1, 2, \dots$ .

**Lemma 3.3.** Under the above assumptions the identities

$$s(k) - s(k-1) = a(k) \tag{3.11}$$

and

$$r_1(k+1)\, f(k) - r_2(k)\, f(k-1) = p(k) \tag{3.12}$$

are equivalent for each $k = 0, 1, 2, \dots$ .

**Proof** Identity (3.11) can be equivalently written as

$$\frac{r_2(k+1)\, a(k+1)}{p(k+1)}\, f(k) - \frac{r_2(k)\, a(k)}{p(k)}\, f(k-1) = a(k)$$

and identity (3.12) can be equivalently written as

$$\frac{r_1(k+1)\, a(k)}{p(k)}\, f(k) - \frac{r_2(k)\, a(k)}{p(k)}\, f(k-1) = a(k)$$

(use that $a(k)$ and $p(k)$ are nonzero for $k = 0, 1, 2, \dots$). Now apply (3.8). □

**Lemma 3.4.** Under the earlier assumptions, let (3.11) and (3.12) be valid for $k = 0, 1, 2, \ldots$ . Then $s(k-1)/s(k)$ is in $\mathbb{F}(k)$ iff $f(k)$ is in $\mathbb{F}(k)$.

**Proof** By (3.11) and (3.10) we have

$$1 - \frac{s(k-1)}{s(k)} = \frac{a(k)}{s(k)} = \frac{p(k)}{r_1(k+1)} \frac{1}{f(k)} \,. \qquad \square$$

**Lemma 3.5.** Under the earlier assumptions, let $f(k)$ be in $\mathbb{F}(k)$ such that (3.12) holds. Then $f(k)$ is in $\mathbb{F}[k]$.

**Proof** Assume that $f(k)$ is not a polynomial. Then

$$f(k) = \frac{c(k)}{d(k)} \,,$$

where $c(k)$ and $d(k)$ are polynomials in $k$ over $\mathbb{F}$ without common factors and $d(k)$ has positive degree and

$$d(k)\, d(k-1)\, p(k) = r_1(k+1)\, c(k)\, d(k-1) - r_2(k)\, c(k-1)\, d(k). \tag{3.13}$$

Let $j$ be the largest integer such that

$$g(k) := \gcd(d(k), d(k+j)) \neq 1.$$

This $j$ exists and $j \geq 0$. Then

$$\gcd(d(k-1), d(k+j)) = 1 = \gcd(d(k-1), g(k))$$

and

$$\gcd(d(k-j-1), d(k)) = 1 = \gcd(g(k-j-1), d(k)).$$

Now $g(k)$ divides $d(k)$ and is relatively prime to $d(k-1)$ and $c(k)$. Hence, by (3.13), $g(k)$ divides $r_1(k+1)$, so $g(k-1)$ divides $r_1(k)$. Also, $g(k-j-1)$ divides $d(k-1)$ and is relatively prime to $d(k)$ and $c(k-1)$. Hence, by (3.13), $g(k-j-1)$ divides $r_2(k)$, so $g(k-1)$ divides $r_2(k+j)$. Thus $r_1(k)$ and $r_2(k+j)$ have a common factor of positive degree, contradicting (3.4). $\square$

In the following we will mean by $\deg(g(k))$ the degree of a polynomial $g(k)$ and we will put this equal to $-1$ if $g(k) = 0$.

**Lemma 3.6.** Under the earlier assumptions, let $f(k)$ be a nonzero element of $\mathbb{F}[k]$ and a solution of (3.12). Then:

(a) If $\deg\big(r_1(k+1) + r_2(k)\big) \leq \deg\big(r_1(k+1) - r_2(k)\big)$ then

$$\deg(f(k)) = \deg(p(k)) - \deg\big(r_1(k+1) - r_2(k)\big).$$

(b) If $l := \deg\big(r_1(k+1) + r_2(k)\big) > \deg\big(r_1(k+1) - r_2(k)\big)$ then let $e_l$ be the coefficient of $k^l$ in $r_1(k+1) + r_2(k)$ and $d_{l-1}$ be the coefficient of $k^{l-1}$ in $r_1(k+1) - r_2(k)$.

(b1) If $-2d_{l-1}/e_l$ is not a nonnegative integer then

$$\deg(f(k)) = \deg(p(k)) - l + 1.$$

(b2) If $-2d_{l-1}/e_l$ is a nonnegative integer then

$$\deg(f(k)) \leq \max\{-2d_{l-1}/e_l \,,\ \deg(p(k)) - l + 1\}.$$

**Proof** Rewrite (3.12) as

$$p(k) = \big(r_1(k+1) - r_2(k)\big)\,\frac{f(k)+f(k-1)}{2} + \big(r_1(k+1) + r_2(k)\big)\,\frac{f(k)-f(k-1)}{2}\,.$$

By our assumptions, $p(k)$ is a nonzero polynomial and $r_1(k+1) - r_2(k)$ and $r_1(k+1) + r_2(k)$ will not be both equal to zero. Case (a) is now evident. In case (b) let $f(k)$ have degree $m$ with coefficient $c_m$ of $k^m$. Then

$$p(k) = (d_{l-1} + \tfrac{1}{2}\,m\,e_l)\,c_m\,k^{l+m-1} + \mathcal{O}(k^{l+m-2}).$$

Cases (b1) and (b2) are now evident. $\square$

**Lemma 3.7.** Under the earlier assumptions, if (3.12) has solutions $f(k)$ belonging to $\mathbb{F}[k]$ then these form a zero or one dimensional set. In case of dimension one, the solution space has the form $f_0(k) + c\,f_1(k)$, where $f_0(k)$ is some special polynomial solution of (3.12), $f_1(k)$ ia a nonzero polynomial solution of

$$r_1(k+1)\,f_1(k) - r_2(k)\,f_1(k-1) = 0, \tag{3.14}$$

and $c$ is an arbitrary element of $\mathbb{F}$. If such a solution $f_1(k)$ of (3.14) exists then $f_1(k) \neq 0$ for $k = -1, 0, 1\ldots$ and $r_2(0) \neq 0$.

**Proof** Clearly, if (3.12) has two distinct polynomial solutions then their difference $f_1(k)$ is a nonzero polynomial solution of (3.14), unique up to a constant factor. If $f_1(k) = 0$ for some $k = -1, 0, 1, \ldots$ or if $r_2(0) = 0$ then $f_1(k) = 0$ for infinitely many values of $k$, which would contradict that $f_1(k)$ is a nonzero polynomial. $\square$

We can now describe the successive steps of Gosper's algorithm. Let $a(k)$ be given.

*Step 1.* Check that $a(k)$, for $k = 0, 1, 2, \ldots$, is a nonzero element of $\mathbb{F}$. Also check that $a(k)/a(k-1)$ is in $\mathbb{F}(k)$.

*Step 2.* Determine $p(k), r_1(k), r_2(k)$ in (3.7) by the algorithm given in the proof of Lemma 3.1 (existence statement).

*Step 3.* Find, by Lemma 3.6, an upper bound $d$ for the degree of a nonzero polynomial $f(k)$ satisfying (3.12). If $d$ is negative then there will be no solution $s(k)$ of (3.11) with the desired properties.

*Step 4.* Put

$$f(k) := \sum_{i=0}^{d} f_i\,k^i, \tag{3.15}$$

where the $f_i$ are yet unknown elements of $\mathbb{F}$. Find the most general solution of the system of linear equations in the $f_i$ obtained by putting the coefficients of the various powers of $k$ in

$$r_1(k+1)\,f(k) - r_2(k)\,f(k-1) - p(k) \tag{3.16}$$

equal to 0. If no solution is found then there will be no solution $s(k)$ of (3.11) with the desired properties. Otherwise, the solution space may have dimension 0 or 1.

*Step 5.* In case the solution space has dimension 0, check if $f(k) \neq 0$ for $k = 0, 1, 2, \ldots$. When this is not the case, there will be no solution of (3.11) with the desired properties.

*Step 6.* Obtain the desired solution(s) $s(k)$ of (3.11) from (3.9). Then

$$s(n) - s(-1) = \sum_{k=0}^{n} a(k). \tag{3.17}$$

Our Maple program implements Gosper's algorithm for

$$a(k) := \frac{(\alpha_1)_k \dots (\alpha_r)_k}{(\beta_1)_k \dots (\beta_s)_k} \frac{z^k}{k!}, \tag{3.18}$$

being the coefficients of a truncated hypergeometric series. Here $\alpha_1, \dots, \alpha_r$, $\beta_1, \dots, \beta_s$ and $z$ are elements of $\mathbb{F}$. In order that $a(k)$ is in $\mathbb{F}$ for each $k = 0, 1, 2, \dots$, we require that

$$\beta_1, \dots, \beta_s \neq 0, -1, -2, \dots . \tag{3.19}$$

Also, in order that $a(k) \neq 0$ for $k = 0, 1, 2, \dots$, we require that

$$\alpha_1, \dots, \alpha_r \neq 0, -1, -2, \dots \quad \text{and} \quad z \neq 0. \tag{3.20}$$

Now $a(k)/a(k-1)$ is certainly in $\mathbb{F}(k)$:

$$\frac{a(k)}{a(k-1)} = \frac{(\alpha_1 + k - 1) \dots (\alpha_r + k - 1)\, z}{(\beta_1 + k - 1) \dots (\beta_s + k - 1)\, k}.$$

**Remark 3.8.** If, in (3.18),

$$\alpha_1, \dots, \alpha_r \neq 1, 2, 3, \dots \tag{3.21}$$

then $k$ will be a factor of $r_2(k)$, so $r_2(0) = 0$ and $s(-1) = 0$ by (3.9). Hence, for such $\alpha_i$'s, (3.11) has at most one solution $s(k)$ and such a solution will satisfy $s(-1) = 0$. We will always make this assumption (3.21).

**Example 3.9.** Consider (2.2), so

$$a(k) := \frac{(b)_k}{k!}, \tag{3.22}$$

$\mathbb{F} := \mathbb{Q}(b)$ and conditions (3.19), (3.20) and (3.21) are satisfied. From

$$\frac{a(k)}{a(k-1)} = \frac{b + k - 1}{k} \tag{3.23}$$

we get

$$p(k) = 1, \quad r_1(k) = b + k - 1, \quad r_2(k) = k.$$

Hence

$$r_1(k+1) + r_2(k) = 2k + b, \quad r_1(k+1) - r_2(k) = b,$$

so we are in the case (b1) of Lemma 3.6 and every nonzero solution $f(k)$ of (3.12) will have degree 0. Equation (3.12) becomes

$$(b + k)\, f(k) - k\, f(k-1) = 1,$$

so $f(k) = b^{-1}$, which is nonzero for $k = 0, 1, 2, \dots$. Now equation (3.9) yields

$$s(k) = \frac{k+1}{b} \frac{(b)_{k+1}}{(k+1)!} \quad k = -1, 0, 1, \dots .$$

Thus, indeed, $s(-1)=0$ and

$$s(k)=\frac{(b+1)_k}{k!}\,,\quad k=0,1,2,\ldots\ .$$

**Example 3.10.** We continue the previous example, but we now assume that $b$ in (3.22) is a fixed positive integer. Then $\mathbb{F}=\mathbb{Q}$ and condition (3.21) is no longer satisfied. We can rewrite (3.22) as

$$a(k)=\frac{(k+1)_{b-1}}{(b-1)!},$$

which is a polynomial of degree $b-1$ in $k$. (In the previous example, $a(k)$ was certainly not polynomial in $k$.) From (3.23) we now get

$$p(k)=\frac{(k+1)_{b-1}}{(b-1)!}=a(k),\quad r_1(k)=1,\quad r_2(k)=1.$$

We are now in case (b2) of Lemma 3.6 and find that $f(x)$ must have degree $\le b$. Equation (3.12) becomes

$$f(k)-f(k-1)=\frac{(k+1)_{b-1}}{(b-1)!},$$

Its general solution is

$$f(k)=\frac{(k+1)_b}{b!}+\text{const.}\ .$$

Then (3.9) becomes

$$s(k)=f(k).$$

We obtain

$$s(n)-s(-1)=\frac{(b+1)_n}{n!}\,.$$

It is curious to observe that the specialization of (3.22) to some special positive integer value of $b$ causes Gosper's algorithm to solve a system of $b+1$ instead of 1 linear equations. For large $b$ this will consume much more computing time.

## 4. Zeilberger's algorithm

Let $\mathbb{F}$ be the field of rational functions in some fixed number of indeterminates (not including $k$ and $n$) over $\mathbb{Q}$. Let $A(n,k)$ be such that

(i) $A(n,k)\in\mathbb{F}$ for $n,k=0,1,2,\ldots$ ;
(ii) $A(n,k)$ is a nonzero element of $\mathbb{F}(n)$ for $k=0,1,2,\ldots$ ;
(iii) $A(n,k)=0$ for integer $n,k$ with $0\le n<k$;
(iv) $A(n,k)/A(n,k-1)$ is in $\mathbb{F}(n,k)$;
(v) $A(n,k)/A(n-1,k)$ is in $\mathbb{F}(n,k)$.

Put

$$\Sigma(n):=\sum_{k=0}^{n}A(n,k),\quad n=0,1,2,\ldots\ . \tag{4.1}$$

Fix $l = 1, 2, \ldots$ . Zeilberger's algorithm will search for $\sigma_j(n)$ $(j = 1, 2, \ldots, l)$ in $\mathbb{F}(n)$ such that $\Sigma(n)$ satisfies the $l$th order recurrence

$$\Sigma(n) + \sum_{j=1}^{l} \sigma_j(n)\, \Sigma(n-j) = 0, \quad n = l, l+1, l+2, \ldots \,. \tag{4.2}$$

In particular, if such a recurrence can be found for $l = 1$ then $\Sigma(n)$ can be obtained by iteration of (4.2) from the starting value $\Sigma(0) = A(0, 0)$.

Zeilberger's algorithm reduces the problem to Gosper's algorithm as follows. Let the $\sigma_j(n)$ be yet undetermined elements of $\mathbb{F}(n)$. Put

$$a(k) := A(n, k) + \sum_{j=1}^{l} \sigma_j(n)\, A(n-j, k). \tag{4.3}$$

Then $a(k)$ $(k = 0, 1, \ldots)$ is a sequence of elements of $\mathbb{F}(n)$. Assume that the $\sigma_j(n)$ are such that the $a(k)$ are nonzero elements of $\mathbb{F}(n)$. From (4.3) we obtain

$$\frac{a(k)}{a(k-1)} = \frac{1 + \sum_{j=1}^{l} \sigma_j(n) A(n-j, k)/A(n, k)}{1 + \sum_{j=1}^{l} \sigma_j(n) A(n-j, k-1)/A(n, k-1)}\, \frac{A(n, k)}{A(n, k-1)}\,, \tag{4.4}$$

so $a(k)/a(k-1)$ is in $\mathbb{F}(n)(k)$.

Now suppose Gosper's algorithm has supplied explicit $\sigma_j(n)$ and an explicit solution $s(n) = S(n, k)$ of (3.2), where $s(k)$ is, for each $k = -1, 0, 1, \ldots$, an element of $\mathbb{F}(n)$, nonzero if $k = 0, 1, 2, \ldots$ . (In a moment we will discuss the details of this application of Zeilberger's algorithm.) Suppose that $s(-1) = 0$. Then, by (3.17),

$$S(n, m) = s(m) = \sum_{k=0}^{m} a(k) = \sum_{k=0}^{m} A(n, k) + \sum_{j=1}^{l} \sigma_j(n) \sum_{k=0}^{m} A(n-j, k), \quad m = 0, 1, 2, \ldots \,. \tag{4.5}$$

Suppose that also $S(n, n) = 0$. Then, by (4.1) and Assumption (iii) on the $A(n, k)$, the case $m = n$ of (4.5) yields (4.2).

We now discuss the details of the application of Gosper's algorithm. Write

$$\frac{A(n, k)}{A(n-1, k)} = \frac{B(n, k)}{C(n, k)}\,, \tag{4.6}$$

where $B(n, k)$ and $C(n, k)$ are coprime elements of $\mathbb{F}[n, k]$, and

$$\frac{A(n, k)}{A(n, k-1)} = \frac{D(n, k)}{E(n, k)}\,, \tag{4.7}$$

where $D(n, k)$ and $E(n, k)$ are coprime elements of $\mathbb{F}[n, k]$. Then $B(n, k)$, $C(n, k)$, $D(n, k)$ and $E(n, k)$ are nonzero elements of $\mathbb{F}(n)$ for $k = 0, 1, 2, \ldots$ . We obtain from (4.4) that

$$\frac{a(k)}{a(k-1)} = \frac{p_0(k)}{p_0(k-1)}\, \frac{r_{10}(k)}{r_{20}(k)}\,, \tag{4.8}$$

where

$$p_0(k) := \prod_{i=0}^{l-1} B(n-i,k) + \sum_{j=1}^{l} \sigma_j(n) \prod_{i=0}^{j-1} C(n-i,k) \prod_{i=j}^{l-1} B(n-i,k), \tag{4.9}$$

$$r_{10}(k) := D(n,k) \prod_{i=0}^{l-1} B(n-i,k-1), \tag{4.10}$$

$$r_{20}(k) := E(n,k) \prod_{i=0}^{l-1} B(n-i,k). \tag{4.11}$$

Then $r_{10}(k)$ and $r_{20}(k)$ are elements of $\mathbb{F}(n)[k]$ and nonzero elements of $\mathbb{F}(n)$ for each $k = 1, 2, \ldots$. Now use the algorithm of Lemma 3.1 in order to write

$$\frac{r_{10}(k)}{r_{20}(k)} = \frac{p_1(k)}{p_1(k-1)} \, \frac{r_1(k)}{r_2(k)}\,, \tag{4.12}$$

where $p_1(k), r_1(k), r_2(k)$ are elements of $\mathbb{F}(n)[k]$, such that

$$\gcd(r_1(k), r_2(k+j)) = 1$$

for all integers $j \geq 0$ and

$$\gcd(r_1(k), p_1(k-1)) = 1 = \gcd(r_2(k), p_1(k)).$$

Put

$$p(k) := p_0(k)\, p_1(k). \tag{4.13}$$

Then (4.8), (4.12) and (4.13) yield (3.7).

Note that, by (4.9), (4.3) and (4.6)

$$p_0(k) = \frac{a(k)}{A(n,k)} \prod_{i=0}^{l-1} B(n-i,k).$$

Thus, by (3.9) and (4.13):

$$S(n,k) = s(k) = \frac{r_2(k+1)\, f(k)\, A(n,k+1)}{p_1(k+1)\, \prod_{i=0}^{l-1} B(n-i,k+1)}\,, \quad k = -1, 0, 1, \ldots\,. \tag{4.14}$$

We can now describe the successive steps of Zeilberger's algorithm. Let $A(n,k)$ be given.

*Step 1.* Check conditions (i)–(v) of the beginning of this section. Write $A(n,k)/A(n,k-1)$ and $A(n,k)/A(n-1,k)$ as in (4.6) and (4.7).

*Step 2.* Determine $p_1(k), r_1(k), r_2(k)$ in (4.12) by the algorithm of Lemma 3.1. Check if $r_2(0) = 0$, otherwise the algorithm fails. Determine $p(k)$ (with yet undetermined $\sigma_j$) by (4.13) and (4.9).

*Step 3.* Find by Lemma 3.6 an upper bound $d$ for the degree over $\mathbb{F}(n)$ of a solution $f(k)$ of (3.12) which lies in $\mathbb{F}(n)[k]$. (Here we take for $\deg(p(k))$ the degree of $p(k)$ with yet undetermined $\sigma_j(n)$, so, with a priori knowledge of the $\sigma_j(n)$, $d$ might have been lower.) If $d$ is negative then the algorithm fails.

*Step 4.* Now substitute (3.15) in (3.16) and obtain a system of linear equations over $\mathbb{F}(n)$ in the $f_i$ $(i = 0, \ldots, d)$ and $\sigma_i$ $(i = 1, \ldots, l)$ by putting the coefficients of the various powers of $k$ in (3.16) to 0. Solve this system of equations. If no solution is found then the algorithm fails. Otherwise, the solution space may have dimension 0 or higher. In case of higher dimension, we will have some free parameters with which we extend the field $\mathbb{F}$.

*Step 5.* With the solutions from Step 4 substituted, we have to reevaluate some expressions in order to be sure that the conditions are still valid under which Gosper's algorithm works. Check if $p(k)$ is a non-zero element of $\mathbb{F}(n)$ for $k = 0, 1, 2, \ldots$ and if $a(0)$ is a nonzero element of $\mathbb{F}(n)$. Then, by (3.7), $a(k)$ is a nonzero element of $\mathbb{F}(n)$ for $k = 0, 1, 2, \ldots$. Check if $f(k) \neq 0$ for $k = 0, 1, 2, \ldots$. If one of the checks gives negative answer then the algorithm fails.

*Step 6.* Obtain the solution $s(k) = S(n, k)$ of (3.11) from (3.9). Then $s(-1) = 0$ by (3.9) since $r_2(0) = 0$. Now we have to check if posssibly for certain $n = l, l+1, l+2, \ldots$ the $f_i(n)$ and $\sigma_i(n)$ have poles. Suppose there are no poles for integer $n > n_0 \geq l - 1$. Consider (4.5) only for such $n$. Check if possibly $S(n, n) \neq 0$ for some integer $n > n_0$. We can do this by inspection of (3.9). Because of (4.3) and Assumption (iii) on $A(n, k)$ we have $a(k+1)|_{k=n} = 0$ for integer $n > n_0$. Thus we have to check if possibly $r_2(k+1)|_{k=n}$ has a pole or $p(k+1)|_{k=n}$ has a zero for integer $n > n_0$. If this is the case then we can get integer $n_1 \geq n_0$ such that $S(n, n) = 0$ for $n > n_1$. Define $\Sigma(n)$ by (4.1). Then the recurrence (4.2) is valid for $n > n_1$.

Our Maple program implements Zeilberger's algorithm for sums (4.1) with

$$A(n,k) := \frac{(-n)_k}{k!} \frac{(\alpha_2 + i_2 n)_k \ldots (\alpha_r + i_r n)_k}{(\beta_1 + j_1 n)_k \ldots (\beta_s + j_s n)_k} z^k,$$

being the coefficients of a hypergeometric series

$${}_rF_s \left[ \begin{matrix} -n, \alpha_2 + i_2 n, \ldots, \alpha_r + i_r n \\ \beta_1 + j_1 n, \ldots \beta_s + j_s n \end{matrix} ; z \right].$$

We assume that no upper indices coincide with lower indices of the hypergeometric function, that $\alpha_2, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ and $z$ are elements of $\mathbb{F}$ and that $i_2, \ldots, i_r, j_1, \ldots, j_s \in \mathbb{Z}$. In order that $A(n, k)$ is in $\mathbb{F}$ for $n, k = 0, 1, 2, \ldots$ we require that

$$\beta_t \notin \mathbb{Z} \quad \text{if } j_t = -1, -2, \ldots \tag{4.15}$$

and

$$\beta_t \neq 0, -1, -2, \ldots \quad \text{if } j_t = 0. \tag{4.16}$$

In order that $A(n, k) \neq 0$ as element of $\mathbb{F}(n)$ for $k = 0, 1, 2, \ldots$ we require that $z \neq 0$ and

$$\alpha_t \neq 0, -1, -2, \ldots \quad \text{if } i_t = 0. \tag{4.17}$$

Now $A(n, k) = 0$ for integer $n, k$ with $0 \leq n < k$. We get

$$\frac{A(n,k)}{A(n,k-1)} = \frac{-n+k-1}{k} \frac{(\alpha_2 + i_2 n + k - 1) \ldots (\alpha_r + i_r n + k - 1)}{(\beta_1 + j_1 n + k - 1) \ldots (\beta_s + j_s n + k - 1)} z \tag{4.18}$$

and

$$\frac{A(n,k)}{A(n-1,k)} = \frac{n}{n-k} \prod_{t=2}^{r} \frac{(\alpha_t + i_t(n-1) + k)_{i_t}}{(\alpha_t + i_t(n-1))_{i_t}} \prod_{t=1}^{s} \frac{(\beta_t + j_t(n-1))_{j_t}}{(\beta_t + j_t(n-1) + k)_{j_t}}, \tag{4.19}$$

where the shifted factorial $(a)_k$ is defined by (2.3), also for negative integer $k$. Clearly, the right hand sides of (4.18) and (4.19) are elements of $\mathbb{F}(n)(k)$.

We can now perform Step 1 and Step 2. In order to obtain $r_2(0) = 0$ in Step 2 we require that

$$\alpha_t \neq 1, 2, \ldots \quad \text{if } i_t = 0. \tag{4.20}$$

Next we can perform Steps 3, 4, 5 and 6.

**Example 4.1.** Consider (2.1), so $A(n,k)$ is given by (2.6), $\mathbb{F} := \mathbb{Q}(b,c)$ and conditions (4.15), (4.16), (4.17) and (4.20) are satisfied. Let the desired order of recurrence $l$ be equal to 1. Then

$$a(k) := A(n,k) + \sigma_1(n)\, A(n-1,k),$$

$$\frac{A(n,k)}{A(n-1,k)} = \frac{n}{n-k},$$

$$\frac{A(n,k)}{A(n,k-1)} = \frac{(-n+k-1)\,(b+k-1)}{(c+k-1)\,k}.$$

Then we get (3.7) with

$$p(k) = -n + \sigma_1(n)\,(k-n), \quad r_1(k) = (k-n-1)\,(k+b-1), \quad r_2(k) = k\,(k+c-1).$$

We are in the case (b1) of Lemma 3.6 and find that $\deg(f(k)) \le 0$. So $f(k) = f_0(n)$. We have to solve

$$(r_1(k+1) - r_2(k))\, f_0(n) - p(k) = 0,$$

i.e. the system

$$\begin{cases} (b-c-n+1)\, f_0(n) - \sigma_1(n) = 0, \\ \qquad -nbf_0(n) + n + n\sigma_1(n) = 0. \end{cases}$$

As unique solution we find

$$f_0(n) = \frac{1}{c+n-1}, \quad \sigma_1(n) = -\,\frac{n+c-b-1}{n+c-1}.$$

Now all checks of Step 5 have positive answer and in Step 6 we find that $n_0$ and $n_1$ are equal to 0. Thus we obtain (2.8) with $\sigma(n)$ given by (2.9) and hence we obtain (2.7). We also obtain (2.10) from (4.14).

**Example 4.2.** Let $m_1, \ldots, m_p$ be nonnegative integers and let $n$ be integer such that $n \ge m_1 + \cdots + m_p$. Minton [10] showed that

$${}_{p+2}F_{p+1}\left[\begin{matrix} -n, b, c_1+m_1, \ldots, c_p+m_p \\ b+1, c_1, \ldots, c_p \end{matrix}; 1\right] = \frac{n!}{(b+1)_n}\, \frac{(c_1-b)_{m_1} \ldots (c_p-b)_{m_p}}{(b_1)_{m_1} \ldots (b_p)_{m_p}}.$$

Put the left-hand side equal to $\Sigma(n)$. Then

$$\frac{\Sigma(n)}{\Sigma(n-1)} = \frac{n}{b+n}, \quad n > m_1 + \cdots + m_p.$$

So we have the complication here that the evaluation of the ratio $\Sigma(n)/\Sigma(n-1)$ is not valid for the lowest values of $n$, up to $m_1 + \cdots + m_p$. Let us analyse this with Zeilberger's algorithm in the simple special case

$$\Sigma(n) := {}_3F_2\left[\begin{matrix} -n, b, c+1 \\ b+1, c \end{matrix}; 1\right] = \sum_{k=0}^{n} A(n,k),$$

with

$$A(n,k) := \frac{(-n)_k}{k!}\, \frac{b(c+k)}{(b+k)c}.$$

Then $\mathbb{F} := \mathbb{Q}(b, c)$, and conditions (4.15), (4.16), (4.17) and (4.20) are satisfied. Let the desired order of recurrence $l$ be equal to 1. Then

$$a(k) := A(n, k) + \sigma_1(n)\, A(n-1, k),$$

$$\frac{A(n,k)}{A(n-1,k)} = \frac{n}{n-k}\,, \qquad \frac{A(n,k)}{A(n,k-1)} = \frac{-n+k-1}{k}\,\frac{(c+k)\,(b+k-1)}{(b+k)\,(c+k-1)}\,.$$

Then we get (4.8) with

$$p_0(k) = -n + \sigma_1(n)\,(k-n), \quad r_{10}(k) = n\,(-n+k-1)\,(c+k)\,(b+k-1),$$
$$r_{20}(k) = n\,k\,(c+k-1)\,(b+k).$$

Hence we get (3.7) with

$$p(k) = (c+k)\,(-n + \sigma_1(n)\,(k-n)), \quad r_1(k) = (-n+k-1)\,(b+k-1), \quad r_2(k) = k\,(b+k).$$

We are in the case (b1) of Lemma 3.6 and find that $\deg(f(k)) \le 1$. Solution of the resulting equation (3.12) yields

$$\sigma_1(n) = -\frac{n}{b+n}\,, \quad f(k) = \frac{c}{n+b} + \frac{nk}{(n+b)\,(n-1)}\,.$$

Now all checks of Step 5 have positive answer. In Step 6 we find that $f_1(n)$ has a pole at $n = 1$. We obtain $n_0 = 1 = n_1$. Thus

$$\frac{\Sigma(n)}{\Sigma(n-1)} = -\sigma_1(n) = \frac{n}{b+n} \quad \text{for } n = 2, 3, \ldots\,.$$

## 5. The $q$-case

Consider the $q$-Chu-Vandermonde summation formula

$${}_2\phi_1(q^{-n}, b; c; q, q) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k\,(b;q)_k}{(c;q)_k\,(q;q)_k}\, q^k = \frac{(c/b;q)_n}{(c;q)_n}\, b^n, \quad n = 0, 1, 2, \ldots\,, \tag{5.1}$$

and its special case for $c := q^{-n}$:

$$\sum_{k=0}^{n} \frac{(b;q)_k}{(q;q)_k}\, q^k = \frac{(bq;q)_n}{(q;q)_n}\,, \quad n = 0, 1, 2, \ldots\,. \tag{5.2}$$

Here the *$q$-shifted factorial* is defined by

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty}\,, \quad k \in \mathbb{Z}; \qquad (a;q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j). \tag{5.3}$$

As with (2.2), formula (5.2) is an indefinite summation which can be verified immediately. Formula (5.1) is a definite summation which can be treated along similar lines as (2.1). Write the sum in (5.1) as (2.5) with

$$A(n,k) := \frac{(q^{-n};q)_k\,(b;q)_k}{(c;q)_k\,(q;q)_k}\, q^k, \quad k = 0, 1, 2, \ldots\,.$$

Then $A(n,k)=0$ for integer $n,k$ such that $k>n$. Since (5.1) evidently holds for $n=0$, the general case of (5.1) would follow from (2.8) with

$$\sigma(n):=\frac{-b+q^{n-1}c}{1-q^{n-1}c}\,.$$

Now the indefinite summation formula

$$\sum_{k=0}^{m}\bigl(A(n,k)+\sigma(n)\,A(n-1,k)\bigr)=\frac{(q^{-n+1};q)_m\,(b;q)_{m+1}}{(1-cq^{n-1})\,(c;q)_m\,(q;q)_m} \tag{5.4}$$

can be immediately be proved by checking that

$$\frac{(q^{-n+1};q)_m\,(b;q)_{m+1}}{(1-cq^{n-1})\,(c;q)_m\,(q;q)_m}-\frac{(q^{-n+1};q)_{m-1}\,(b;q)_m}{(1-cq^{n-1})\,(c;q)_{m-1}\,(q;q)_{m-1}}=A(n,m)+\sigma(n)\,A(n-1,m)$$

for $m=1,2,\ldots$, and that

$$\left.\frac{(q^{-n+1};q)_m\,(b;q)_{m+1}}{(1-cq^{n+1})\,(c;q)_m\,(q;q)_m}\right|_{m=0}=A(n,0)+\sigma(n)\,A(n-1,0).$$

Note that, in the above formulas, $q^n$ can be treated as a complex parameter. Since the right-hand side of (5.4) vanishes for $m=n$, (2.8) will follow by (2.13).

Surprisingly, Gosper's and Zeilberger's algorithms can be carried over to the $q$-case almost unchanged. Let us briefly indicate which adaptations have to be made in our descriptions of these algorithms.

In §3, $\mathbb{F}$ will now be the field of rational functions in some fixed number of indeterminates including $q$ (but not including $k$) over $\mathbb{Q}$. The $q$-Gosper algorithm will look for solutions $s(k)$ to (3.2), nonzero for $k=0,1,2,\ldots$ such that $s(k-1)/s(k)$ is rational in $q^k$ over $\mathbb{F}$. Throughout in §3 replace $\mathbb{F}(k)$ by $\mathbb{F}(q^k)$ and $\mathbb{F}[k]$ by $\mathbb{F}[q^k]$.

In Lemma 3.1, $p(k)$ will be only unique up to a factor which is a constant times some power of $q^k$.

In Lemma 3.5 the conclusion will be that $f(k)$ is in $\mathbb{F}[q^k,q^{-k}]$, i.e. a Laurent polynomial in $q^k$ over $\mathbb{F}$.

In the reformulation of Lemma 3.6 let $\deg(g(k))$ and $\mathrm{ldeg}(g(k))$ mean the highest occurring degree $m_2$ and the lowest occurring degree $m_1$ in a nonzero Laurent polynomial

$$g(k):=\sum_{j=m_1}^{m_2}c_j\,q^{jk},\quad m_1,m_2\in\mathbb{Z},\quad m_1\le m_2,\quad c_{m_1}\ne 0\ne c_{m_2}.$$

We now have:

**Lemma 5.1.** Under the suitably reformulated assumptions of §3, let $f(k)$ be a nonzero element of $\mathbb{F}[q^k,q^{-k}]$ and a solution of (3.12). Then:
(a) If $\mathrm{ldeg}(r_1(k))\ne\mathrm{ldeg}(r_2(k))$ then

$$\mathrm{ldeg}(f(k))=\mathrm{ldeg}(p(k))-\min\{\mathrm{ldeg}(r_1(k)),\mathrm{ldeg}(r_2(k))\}.$$

(b) If $l := \mathrm{ldeg}(r_1(k)) = \mathrm{ldeg}(r_2(k))$ then let $d_l$ and $e_l$ be the coefficients of $q^{kl}$ in $r_1(k)$ respectively $r_2(k)$.

(b1) If ${}^q\log(e_l/d_l) \notin \mathbb{Z}$ then

$$\mathrm{ldeg}(f(k)) = \mathrm{ldeg}(p(k)) - l.$$

(b2) If ${}^q\log(e_l/d_l) \in \mathbb{Z}$ then

$$\mathrm{ldeg}(f(k)) \geq \min\{{}^q\log(e_l/d_l), \mathrm{ldeg}(p(k))\} - l.$$

Also:

(a)$'$ If $\deg(r_1(k)) \neq \deg(r_2(k))$ then

$$\deg(f(k)) = \deg(p(k)) - \max\{\deg(r_1(k)), \deg(r_2(k))\}.$$

(b)$'$ If $l := \deg(r_1(k)) = \deg(r_2(k))$ then let $d_l$ and $e_l$ be the coefficients of $q^{kl}$ in $r_1(k)$ respectively $r_2(k)$.

(b1)$'$ If ${}^q\log(e_l/d_l) \notin \mathbb{Z}$ then

$$\deg(f(k)) = \deg(p(k)) - l.$$

(b2)$'$ If ${}^q\log(e_l/d_l) \in \mathbb{Z}$ then

$$\deg(f(k)) \leq \max\{{}^q\log(e_l/d_l), \deg(p(k))\} - l.$$

**Remark 5.2.** We may relax the assumptions on $p(k)$ by allowing that $p(k)$ is a Laurent polynomial instead of an ordinary polynomial in $q^k$. Suppose $m$ is the lower bound found for $\mathrm{ldeg}(f(k))$ in Lemma 5.1. Now put

$$\widetilde{f}(k) := q^{-mk} f(k), \quad \widetilde{p}(k) := q^{-mk} p(k), \quad \widetilde{r_1}(k) := r_1(k), \quad \widetilde{r_2}(k) := q^{-m} r_2(k).$$

Then (3.7) and (3.12) are still satisfied with $\widetilde{f}(k)$, $\widetilde{p}(k)$, $\widetilde{r_1}(k)$, $\widetilde{r_2}(k)$ instead of $f(k)$, $p(k)$, $r_1(k)$, $r_2(k)$, and $\widetilde{f}(k)$ is in $\mathbb{F}[q^k]$ but $\widetilde{p}(k)$ is possibly in $\mathbb{F}[q^k, q^{-k}]$. If $\widetilde{p}(k)$ has terms with negative powers of $q^k$ them (3.12) will have no solution.

Steps 1–6 in §3 can now be performed in the $q$-case with the obvious minor adaptations. In particular, in Step 3 we determine a lower bound for $\mathrm{ldeg}(f(k))$, then rewrite $f(k)$, $p(k)$, $r_1(k)$, $r_2(k)$ as in Remark 5.2, and finally find an upper bound $d$ for $\deg(f(k))$. If $\mathrm{ldeg}(\widetilde{p}(k)) < 0$ or if $d < 0$ then the algorithm fails. In Step 4 we put

$$f(k) := \sum_{i=0}^{d} f_i\, q^{ki}.$$

Our Maple program implements the $q$-Gosper algorithm for

$$a(k) := \frac{(\alpha_1; q)_k \ldots (\alpha_r; q)_k \, ((-1)^k\, q^{k(k-1)/2})^{s-r+1}\, z^k}{(\beta_1; q)_k \ldots (\beta_s; q)_k \, (q; q)_k}.$$

Here $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ and $z$ are elements of $\mathbb{F}$. In order that $a(k)$ is in $\mathbb{F}$ for each $k = 0, 1, 2, \ldots$ we require that

$$\beta_1, \ldots, \beta_s \neq 1, q^{-1}, q^{-2}, \ldots\ .$$

Also, in order that $a(k) \neq 0$ for $k = 0, 1, 2, \ldots$, we require that

$$\alpha_1, \ldots, \alpha_r \neq 1, q^{-1}, q^{-2}, \ldots \quad \text{and} \quad z \neq 0.$$

Now $a(k)/a(k-1)$ is certainly in $\mathbb{F}(q^k)$:

$$\frac{a(k)}{a(k-1)} = \frac{(1-\alpha_1 q^{k-1}) \ldots (1-\alpha_r q^{k-1})\,(-q^{k-1})^{s-r+1}\, z}{(1-\beta_1 q^{k-1}) \ldots (1-\beta_s q^{k-1})\,(1-q^k)}.$$

If

$$\alpha_1, \ldots, \alpha_r \neq q, q^2, \ldots$$

then $1-q^k$ will be a factor of $r_2(k)$, so $r_2(0) = 0$ and $s(-1) = 0$ by (3.9). We will always make this assumption.

For the $q$-version of Zeilberger's algorithm we make slight adaptations of Zeilberger's algorithm as described in §4. In assumptions (iv) and (v) let $A(n,k)/A(n,k-1)$ and $A(n,k)/A(n-1,k)$ be in $\mathbb{F}(q^n, q^k)$. Throughout replace rational or polynomial dependence on $n, k$ by a similar dependence on $q^n, q^k$. The other adaptations in Steps 1–6 of §4 can be made in a similar way as for the $q$-Gosper algorithm. In connection with Remark 5.2 observe that the substitution $\widetilde{p}(k) := q^{-mk} p(k)$ will be caused by a substitution $\widetilde{p_1}(k) := q^{-mk} p_1(k)$ (cf. (4.12) and (4.13)), while $p_0(k)$ (cf. (4.9)) remains unaffected.

Our Maple program implements the $q$-Zeilberger algorithm for sums (4.1) with

$$A(n,k) := \frac{(q^{-n}; q)_k\, (q^{ni_2}\alpha_2; q)_k \ldots (q^{ni_r}\alpha_r; q)_k}{(q; q)_k\, (q^{nj_1}\beta_1; q)_k \ldots (q^{nj_s}\beta_s; q)_k} \left((-1)^k q^{k(k-1)/2}\right)^{s-r+1} (q^{nv}\zeta)^k,$$

being the coefficients of a $q$-hypergeometric series

$${}_r\phi_s \left[\begin{matrix} q^{-n}, q^{ni_2}\alpha_2, \ldots, q^{ni_r}\alpha_r \\ q^{nj_1}\beta_1, \ldots, q^{nj_s}\beta_s \end{matrix}; q, q^{nv}\zeta\right].$$

We assume that no upper indices coincide with lower indices of the $q$-hypergeometric function, that $\alpha_2, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ and $\zeta$ are elements of $\mathbb{F}$ and that $i_2, \ldots, i_r, j_1, \ldots, j_s, v \in \mathbb{Z}$. As in (4.15), (4.16), (4.17), (4.20) we require that

$$\begin{aligned} &{}^q\log \beta_t \notin \mathbb{Z} \quad \text{if} \quad j_t = -1, -2, \ldots; \qquad {}^q\log \beta_t \neq 0, -1, -2, \ldots \quad \text{if} \quad j_t = 0; \\ &{}^q\log \alpha_t \notin \mathbb{Z} \quad \text{if} \quad i_t = 0; \quad \zeta \neq 0. \end{aligned}$$

We get

$$\frac{A(n,k)}{A(n,k-1)} = \frac{(1-q^{-n+k-1})\,(1-q^{ni_2+k-1}\alpha_2) \ldots (1-q^{ni_r+k-1}\alpha_r)}{(1-q^k)\,(1-q^{nj_1+k-1}\beta_1) \ldots (1-q^{nj_s+k-1}\beta_s)} (-q^{k-1})^{s-r+1}\, q^{nv}\, \zeta$$

and

$$\frac{A(n,k)}{A(n-1,k)} = \frac{1-q^{-n}}{1-q^{-n+k}} \prod_{t=2}^{r} \frac{(q^{(n-1)i_t+k}\alpha_t; q)_{i_t}}{(q^{(n-1)i_t}\alpha_t; q)_{i_t}} \prod_{t=1}^{s} \frac{(q^{(n-1)j_t}\beta_t; q)_{j_t}}{(q^{(n-1)j_t+k}\beta_t; q)_{j_t}}\, q^{vk},$$

where the $q$-shifted factorial $(a; q)_k$ is defined by (5.3), also for negative integer $k$.

## 6. Implementation of the algorithms in Maple 6

The procedures below should work in Maple 6 and in Maple V, Releases 4 and 5.

FUNCTION: `zeilb` — summation of terminating hypergeometric series by Zeilberger's algorithm.

CALLING SEQUENCE:

`zeilb(`$[a1,\ a2,\ \ldots]$`,` $[b1,\ b2,\ \ldots]$`,` $z$`,` $f(n)$`,` $l$`,` $t$`)`

PARAMETERS:

$[a1,\ a2,\ \ldots]$ — list of numerator coefficients
$[b1,\ b2,\ \ldots]$ — list of denominator coefficients
$z$ — argument of hypergeometric function
$n$ — truncate hypergeometric series as a sum from 0 to $n$
$f(n)$ — user proposed name of truncated hypergeometric series as function of $n$
$l$ — required order of recurrence looked for in Zeilberger's algorithm
$t$ — optional, positive integer determining talklevel of output, default `max(1,printlevel)`

DESCRIPTION:

– The function `zeilb`$(a, b, z, f(n), l)$ tries to evaluate (if $l = 0$ or 1) the truncated hypergeometric series

$$f(n) := \sum_{k=0}^{n} \frac{(a1)_k\,(a2)_k\,\ldots}{(b1)_k\,(b2)_k\,\ldots}\,\frac{z^k}{k!}$$

or to find a recurrence relation in $n$ of order $l$ (if $l \geq 2$) for it.

– If $l \geq 1$ then one of the numerator coefficients must equal $-n$.

– If $l = 0$ then Gosper's algorithm is applied.

– If $l = 0$ or 1 and if evaluation of the sum is possible for all $n = 0, 1, 2, \ldots$ as a quotient of products of shifted factorials then the function will return in this form, with the notation `fac`$(c, k)$ being used for the shifted factorial $(c)_k$.

– In all other cases, where the algorithm succeeds, the function will return as a recurrence expressing $f(n)$ in terms of $f(n-1)$, $f(n-2)$, $\ldots$, $f(n-l)$, followed by the inequality $n > n1$ for which the recurrence is valid.

– If the algorithm fails then nothing is returned.

– If $l > 0$, $t > 2$ and the algorithm succeeds then a short proof of the outcome will be printed. There $f(n, k)$ will denote the $k$th term of $f(n)$.

– This function should be defined by inputting the file `zeilb` with the `read` command.

EXAMPLES:

```
> zeilb([-n,b],[c],1,f(n),1);

                          fac(-b + c, n)
                          --------------
                            fac(c, n)

> zeilb([b],[ ],1,f(n),0);

                        (b + n) fac(b, n)
                        -----------------
                          fac(1, n) b
```

```
> zeilb([-n,b,c+1],[b+1,c],1,f(n),1);

                         n f(n - 1)
                         ----------, 1 < n
                           b + n

> zeilb([-n,b],[c],z,f(n),2);

  (z n + z b - c + 2 - 2 n - z) f(n - 1)   (z - 1) (n - 1) f(n - 2)
- -------------------------------------- + ------------------------,  1 < n
               n - 1 + c                          n - 1 + c

> zeilb([-n,n+3,-y],[2,z],1,g(n),2);

  (z + 2 y) (2 n + 1) g(n - 1)   (n - 1) (n - z + 2) g(n - 2)
  ---------------------------- + ----------------------------, 1 < n
      (n + 2) (n + z - 1)            (n + 2) (n + z - 1)

> zeilb([-n,n+3,y+z],[2,z],1,h(n),2);

   (z + 2 y) (2 n + 1) h(n - 1)   (n - 1) (n - z + 2) h(n - 2)
 - ---------------------------- + ----------------------------, 1 < n
       (n + 2) (n + z - 1)            (n + 2) (n + z - 1)
```

Thus for the last two inputs $g(n) = (-1)^n h(n)$. (Check case $n = 1$ by hand.)

FUNCTION: `qzeilb` — summation of terminating $q$-hypergeometric series by the $q$-version of Zeilberger's algorithm.

CALLING SEQUENCE:
`qzeilb(`[$a1$, $a2$, ...], [$b1$, $b2$, ...], $q$, $z$, $f(n)$, $l$, $t$)

PARAMETERS:
[$a1$, $a2$, ...] — list of numerator coefficients
[$b1$, $b2$, ...] — list of denominator coefficients
$q$ — base of $q$-hypergeometric function
$z$ — argument of $q$-hypergeometric function
$n$ — truncate $q$-hypergeometric series as a sum from 0 to $n$
$f(n)$ - user proposed name of truncated $q$-hypergeometric series as function of $n$
$l$ — required order of recurrence looked for in Zeilberger's algorithm
$t$ — optional, positive integer determining talklevel of output, default `max(1,printlevel)`

DESCRIPTION:
– The function `qzeilb`$(a, b, q, z, f(n), l)$ tries to evaluate (if $l = 0$ or 1) the truncated $q$-hypergeometric series

$$f(n) := \sum_{k=0}^{n} \frac{(a1;q)_k\,(a2;q)_k\,\ldots}{(b1;q)_k\,(b2;q)_k\,\ldots}\,\frac{((-1)^k q^{k(k-1)/2})^{r-s+1}\,z^k}{(q;q)_k}$$

or to find a recurrence relation in $n$ of order $l$ (if $l \geq 2$) for it. Here $r$ is the number of terms in the list of numerator coefficients and $s$ the number of terms in the list of denominator coefficients.

– If $l \geq 1$ then one of the numerator coefficients must equal $q^{-n}$.

– If $l = 0$ then the $q$-version of Gosper's algorithm is applied.

– If $l = 0$ or 1 and if evaluation of the sum is possible for all $n = 0, 1, 2, \ldots$ as a quotient of products of $q$-shifted factorials then the function will return in this form, with the notation `qfac`$(c, q, k)$ being used for the $q$-shifted factorial $(c; q)_k$.

– In all other cases, where the algorithm succeeds, the function will return as a recurrence expressing $f(n)$ in terms of $f(n-1)$, $f(n-2)$, …, $f(n-l)$, followed by the inequality $n > n1$ for which the recurrence is valid. Here $f(n)$ denotes the series given by the input, in its dependence on $n$.

– If the algorithm fails then nothing is returned.

– If $l > 0$, $t > 2$ and the algorithm succeeds then a short proof of the outcome will be printed. There $f(n, k)$ denotes the $k$th term of $f(n)$.

– This function should be defined by inputting the file `qzeilb` with the `read` command.

EXAMPLES:

```
> read qzeilb;
> qzeilb([q^(-n),b],[c],q,q,f(n),1);

                                          (-n)
                      qfac(c/b, q, n) (1/b)
                      --------------------------
                            qfac(c, q, n)

> qzeilb([b],[ ],q,q,f(n),0);

                             (n + 1)
                   (-q + b q        ) qfac(b, q, n)
                   --------------------------------
                      qfac(q, q, n) q (b - 1)

> qzeilb([q^(-n),b,q*c],[q*b,c],q,q,f(n),1);

                        n
                      (q  - 1) b f(n - 1)
                      -------------------, 1 < n
                              n
                           b q  - 1

> qzeilb([q^(-n),b],[c],q,z,f(n),2);

      2    2  n      n       n 2        n           n 2
- (z q  - q  q  - c q  q + (q )  q c - q  z b q + (q )  c) f(n - 1)

                                     n         n
       /   n      n           (q - q ) (z b q - q  c) f(n - 2)
      /  (q  q (-q  c + q)) + --------------------------------, 1 < n
     /                                 n      n
                                      q  q (-q  c + q)
```

```
> qzeilb([a^2,q*a,-q*a,b,c,d,a^4*q^(n+1)/b/c/d,q^(-n)],
> [a,-a,a^2*q/b,a^2*q/c,a^2*q/d,b*c*d*a^(-2)*q^(-n),a^2*q^(n+1)],q,q,f(n),1);

        2                 2                 2
     q a               q a               q a                2
qfac(----, q, n) qfac(----, q, n) qfac(----, q, n) qfac(q a , q, n)
     d c               b c               d b

               2              2                2                2
   /         q a              a  q             a  q             a  q
  /  ( qfac(-----, q, n) qfac(----, q, n) qfac(----, q, n) qfac(----, q, n) )
 /          b c d              c                d                b
```

## References


[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, 1935.

[2] H. Böing & W. Koepf, *Algorithms for q-hypergeometric Summation in Computer Algebra*, J. Symb. Comput. 28 (1999), 777–799.

[3] P. Cartier, *Démonstration "automatique" d'identités et fonctions hypergéométriques [d'après D. Zeilberger]*, Séminaire Bourbaki, 1991–92, no.746.

[4] G. Gasper & M. Rahman, *Basic hypergeometric series*, Cambridge University Press, 1990.

[5] R. W. Gosper, Jr., *Decision procedure for indefinite hypergeometric summation*, Proc. Nat. Acad. Sci. USA 75 (1978), 40–42.

[6] W. Koepf, *Hypergeometric summation*, Vieweg, 1998.

[7] W. Koepf, *Maple packages* `hsum6.mpl` *and* `qsum6.mpl`, downloadable from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen/`

[8] T. H. Koornwinder, *Handling hypergeometric series in Maple*, in: *Orthogonal polynomials and their applications*, C. Brezinski, L. Gori & A. Ronveaux (eds.), IMACS Annals on Computing and Applied Mathematics, Vol. 9, Baltzer, 1991, pp. 73–80.

[9] T. H. Koornwinder, *On Zeilberger's algorithm and its q-analogue*, J. Comput. Appl. Math. 48 (1993), 91–111.

[10] B. M. Minton, *Generalized hypergeometric functions of unit argument*, J. Math. Phys. 11 (1970), 1375–1376.

[11] P. Paule & A. Riese, *A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping*, in: *Special Functions, q-Series and Related Topics*, Fields Institute Communications, Vol. 14, Amer. Math. Soc., 1997, pp. 179–210.

[12] P. Paule & M. Schorn, *A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities*, J. Symb. Comput. 20 (1995), 673–698.

[13] M. Petkovšek, *Hypergeometric solutions of difference equations with polynomial coefficients*, J. Symb. Comput. 14 (1992), 243–264.

[14] M. Petkovšek, H. S. Wilf & D. Zeilberger, *A=B*, A. K. Peters, 1996.

[15] H. S. Wilf & D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. 108 (1992), 575–633.

[16] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comput. Appl. Math. 32 (1990), 321–368.

[17] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. 80 (1990), 207–211.

[18] D. Zeilberger, *The method of creative telescoping*, J. Symb. Comput. 11 (1991), 195–204.

[19] D. Zeilberger, *Maple procedures* `EKHAD` *and* `qEKHAD`, obtainable by anonymous ftp from `math.temple.edu`, directory `pub/people/zeilberg/programs`; see also `http://www.math.temple.edu/~zeilberg/programs.html`