

PRODUCT FORMULAS AND ASSOCIATED HYPERGROUPS FOR ORTHOGONAL POLYNOMIALS ON THE SIMPLEX AND ON A PARABOLIC BIANGLE

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ABSTRACT. Explicit product formulas are obtained for families of multivariate polynomials which are orthogonal on simplices and on a parabolic biangle in \mathbb{R}^2 . These product formulas are shown to give rise to measure algebras which are hypergroups. The article also includes an elementary proof that the Michael topology for the space of compact subsets of a topological space (which is used in the definition of a hypergroup) is equivalent to the Hausdorff metric topology when the underlying space has a metric.

1. INTRODUCTION

1.1. **Orthogonal polynomials, product formulas, and hypergroups.** Let k be a positive integer, let H be a compact subset of \mathbb{R}^k , and let

$$\mathbf{e} = \mathbf{e}^{(k)} = (\overbrace{1, \dots, 1}^k) \in H.$$

We use the following notations: $C(H)$ denotes the complex-valued continuous functions on H , $M(H)$ is the Banach space of complex-valued Borel measures supported in H with total variation norm, $M_1(H)$ is the class of probability measures in $M(H)$, $\text{supp}(\nu)$ is the support of $\nu \in M(H)$, and $\delta_{\mathbf{x}}$ is the unit point mass concentrated at \mathbf{x} . A family \mathcal{P} of k -variable polynomials is said to be a set of *orthogonal polynomials* with respect to $\mu \in M_1(H)$ if

$$\int P\bar{Q} d\mu = 0$$

whenever $P, Q \in \mathcal{P}$ with $P \neq Q$, and for $d = 0, 1, \dots$ the set $\{P \in \mathcal{P} : \text{degree } P \leq d\}$ spans the space of all k -variable polynomials with degree $\leq d$.

We assume further that

$$P(\mathbf{e}) = 1 \quad (P \in \mathcal{P}).$$

1991 *Mathematics Subject Classification.* Primary 33C50, 43A62.

Key words and phrases. Product formulas, orthogonal polynomials on the parabolic biangle, orthogonal polynomials on the simplex, hypergroups, Michael topology.

The first author did this research partially at CWI, Amsterdam.

The second author was supported by the National Science Foundation grant DMS-9404316.

The present version of February 25, 1997 will appear in the journal *Constructive Approximation*.

\mathcal{P} has a *product formula* if for each pair $\mathbf{x}, \mathbf{y} \in H$ there is $\mu_{\mathbf{x},\mathbf{y}} \in M(H)$ such that

$$(1.1) \quad \int P d\mu_{\mathbf{x},\mathbf{y}} = P(\mathbf{x}) \cdot P(\mathbf{y}) \quad (P \in \mathcal{P}).$$

(Thus $(x, y) \mapsto \int f d\mu_{\mathbf{x},\mathbf{y}}$ is continuous on $H \times H$ for each $f \in C(H)$.)

Under the additional assumption that each $\mu_{\mathbf{x},\mathbf{y}} \in M_1(H)$ it is possible to define a product (called a *convolution*) on $M(H)$ as follows: if ν and λ belong to $M(H)$ define $\nu * \lambda$ by its action on $f \in C(H)$ by

$$\int f d(\nu * \lambda) = \iint \left(\int f d\mu_{\mathbf{x},\mathbf{y}} \right) d\nu(\mathbf{x}) d\lambda(\mathbf{y}).$$

whence

$$\delta_{\mathbf{x}} * \delta_{\mathbf{y}} = \mu_{\mathbf{x},\mathbf{y}}.$$

It follows that $(M(H), *)$ is a commutative Banach algebra with identity $\delta_{\mathbf{e}}$, and that for each $P \in \mathcal{P}$ the mapping

$$\nu \mapsto \int P d\nu$$

is a complex homomorphism of $(M(H), *)$. Moreover, $(M(H), *)$ satisfies the following properties:

- (H1) A convolution of probability measures is a probability measure.
- (H2) The mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M(H) \times M(H)$ into $M(H)$ with the weak-* topology.
- (H3) $\delta_{\mathbf{e}} * \nu = \nu$ for every $\nu \in M(H)$.

If $(M(H), *)$ also has the following properties:

- (H4) $\mathbf{e} \in \text{supp}(\delta_{\mathbf{x}} * \delta_{\mathbf{y}})$ if and only if $\mathbf{x} = \mathbf{y}$.
- (H5) The mapping $(\mathbf{x}, \mathbf{y}) \mapsto \text{supp}(\delta_{\mathbf{x}} * \delta_{\mathbf{y}})$ is continuous from $H \times H$ into the space of compact subsets of H topologized by the Hausdorff metric.

then $(M(H), *)$ is a hypergroup measure algebra (often called simply a hypergroup or DJS-hypergroup) and is referred to as the *the hypergroup associated with \mathcal{P}* , and (1.1) is said to be a *hypergroup type product formula*. A hypergroup which arises in this way is referred to as a *k-variable continuous polynomial hypergroup*. The founding articles on the subject of hypergroups are by Dunkl [Dun73], Jewett [Jew75] and Spector [Spe78]. The axioms for a DJS-hypergroup are somewhat more general than (H1)–(H5) and are given in full detail in [Jew75] where H is only required to be a locally compact Hausdorff space (although Jewett uses the term “convo” instead of “hypergroup”). See [BH95] and many of the articles in [CGS95] for additional discussion.

The existence of a product formula for polynomials is an unusual situation, and only a few examples are known. That the product formula gives rise to a hypergroup is rarer yet.

The purpose of this article is to establish new product formulas and associated hypergroups for polynomials orthogonal on the parabolic biangle

$$B = \{(x_1, x_2): 0 \leq x_2^2 \leq x_1 \leq 1\}$$

and the k -simplex for $k \geq 2$:

$$\Delta^{(k)} = \{(x_1, \dots, x_k): 0 \leq x_k \leq \dots \leq x_1 \leq 1\}.$$

We have chosen to obtain the product formulas in an explicit form; this is required in the proofs of (H4) and (H5). A proof of product formulas in the form (1.1) with probability measures $\mu_{\mathbf{x}, \mathbf{y}}$ can be had with somewhat less effort.

The balance of the paper is organized as follows: the next section contains a discussion of Axiom (H5), and the rest of the section is used to describe the classical example of the Jacobi polynomials and define several classes of multivariate orthogonal polynomials: the parabolic biangle polynomials, the triangle polynomials, and the simplex polynomials. Section 2 is devoted to the statement of the main results. The product formulas are proved in Section 3, and in Section 4, these product formulas are shown to be of hypergroup type.

1.2. Axiom (H5) and topologies for the space of compact subsets. In the definition of a general hypergroup, Jewett requires that the space of compact subsets which appears in Axiom (H5) have the Michael topology (see [Jew75, §2.5] and [Mic51]). Two of the objections that (H5) has aroused are: first, the Michael topology is very hard to grasp, and second, it is often very difficult to check that the axiom holds.

Regarding the first objection, it is often asked why the simpler Hausdorff metric topology for the compact subsets is not used whenever H is a metric space. (Both topologies are described in Subection 4.1 below.) In fact, in Lemma 4.1 we give an elementary proof that the two topologies are equivalent when both are defined.

An additional reason for the inclusion of this lemma is the curious history of this fact. In 1951, Michael actually proved [Mic51, (4.9.13)] that H is metrizable if and only if $\mathcal{C}(H)$ is; the proof is quite technical. Michael's result is, strictly speaking, weaker than the lemma; but the lemma is probably a corollary of Michael's proof. In 1987, Gallardo and Gebuhrer [GG87, Prop. 1.1] show that every Michael-open subset of $\mathcal{C}(H)$ is also a Hausdorff-open set; their method might also yield the converse. In 1972, Dellacherie [Del72, pp. 41–42] proved the lemma when H is compact. In 1989, Zeuner states [Zeu89, §2.1.1] that the two topologies are the same. Finally, in 1995, Bloom and Heyer [BH95, §1.1.1] say that the Michael topology is stronger than the Hausdorff topology. (Neither proofs nor references are provided in the last two works.)

Regarding the difficulty of checking (H5), we note that even with the Hausdorff topology rather than the Michael topology, our arguments will not assuage the second objection.

There are some alternate hypergroup definitions which avoid the difficulties raised by Axiom (H5) (e.g., [Geb95]), but our use of (H1)–(H5) guarantees that the measure algebras obtained here will be hypergroups in the strictest sense of that term.

1.3. Example. Jacobi polynomials. One of the best-known examples of a product formula of hypergroup type for orthogonal polynomials is the one which arises in connection with the Jacobi polynomials. To define these polynomials recall the Gaussian hypergeometric function

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!},$$

where $a, b, c, z \in \mathbb{C}$; $c \neq 0, -1, -2, \dots$; $|z| < 1$. Considered as a function of z there is a unique analytic continuation to $\{z \in \mathbb{C} : z \notin [1, \infty)\}$.

The Jacobi polynomials are defined by

$$P_n^{(\alpha, \beta)}(x) = \binom{n + \alpha}{n} F\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right),$$

where $\alpha, \beta > -1$; $n = 0, 1, 2, \dots$. For fixed α, β the polynomials $P_n^{(\alpha, \beta)}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function $(1 - x)^\alpha (1 + x)^\beta$. We use the normalized Jacobi polynomials

$$(1.2) \quad R_n^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} = F\left(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right)$$

so that

$$R_n^{(\alpha, \beta)}(1) = 1.$$

The product formula for these polynomials can be given in the form

$$\begin{aligned} R_n^{(\alpha, \beta)}(2x^2 - 1) \cdot R_n^{(\alpha, \beta)}(2y^2 - 1) &= \frac{2\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})\Gamma(\frac{1}{2})} \\ &\cdot \int_0^\pi \int_0^1 R_n^{(\alpha, \beta)}\left(2x^2y^2 + 4xyr \cos \psi (1 - x^2)^{1/2}(1 - y^2)^{1/2} + 2(1 - x^2)(1 - y^2)r^2 - 1\right) \\ &\quad \cdot (1 - r^2)^{\alpha - \beta - 1} r^{2\beta + 1} (\sin \psi)^{2\beta} dr d\psi \end{aligned}$$

see([Koo72]). This is valid for $0 \leq x, y \leq 1$ when $\alpha > \beta > -\frac{1}{2}$.

This formula gives rise to a product $*$ on $M(K)$, where $K = [-1, 1]$, with an interesting harmonic analysis. To see this we first make a change of variables replacing $2x^2 - 1$ by x and $2y^2 - 1$ by y . This has the effect of transforming the argument of $R_n^{(\alpha, \beta)}$ in the integrand into another function which we denote $Z(x, y; r, \psi)$. Now define a measure on $[0, 1] \times [0, \pi]$ by

$$dm^{\alpha, \beta}(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})\Gamma(\frac{1}{2})} (1 - r^2)^{\alpha - \beta - 1} r^{2\beta + 1} (\sin \psi)^{2\beta} dr d\psi.$$

With these changes the product formula becomes

$$R_n^{(\alpha,\beta)}(x) \cdot R_n^{(\alpha,\beta)}(y) = \int_0^\pi \int_0^1 R_n^{(\alpha,\beta)}(Z(x, y; r, \psi)) dm^{\alpha,\beta}(r, \psi) \quad (x, y \in K).$$

Now for any $x, y \in K$ the Riesz representation theorem guarantees the existence of a probability measure $\mu_{x,y} \in M_1(K)$ such that for every f which is continuous on K

$$\int_K f d\mu_{x,y} = \int_0^\pi \int_0^1 f(Z(x, y; r, \psi)) dm^{\alpha,\beta}(r, \psi).$$

so that

$$(1.3) \quad R_n^{(\alpha,\beta)}(x) \cdot R_n^{(\alpha,\beta)}(y) = \int R_n^{(\alpha,\beta)} d\mu_{x,y}(r, \psi).$$

$(M(K), *)$ is a hypergroup. Indeed every 1-variable continuous polynomial hypergroup is associated with the Jacobi polynomials [CS90, CMS91, CS95b].

1.4. Multivariate polynomial families. The story becomes much more complicated for multivariate polynomials. In that case, there is a more diverse collection of examples of 2-variable continuous polynomial hypergroups (see [CS95a] for three types). The issue seems to depend in a strong way on the geometry of the set which supports the orthogonality measure. We introduce the following classes of polynomials:

1.4.1. *Parabolic biangle polynomials.* Let

$$R_{n,k}^{\alpha,\beta}(x_1, x_2) = R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x_1 - 1) \cdot x_1^{\frac{1}{2}k} R_k^{(\beta,\beta)}(x_1^{-\frac{1}{2}}x_2),$$

where $\alpha, \beta > -1$ and n, k are integers such that $n \geq k \geq 0$. These functions are polynomials in x_1 and x_2 of degree n and for fixed α, β they are orthogonal on the region B with respect to the measure $(1 - x_1)^\alpha (x_1 - x_2^2)^\beta dx_1 dx_2$.

For certain values of the parameters α, β the parabolic biangle polynomials have an interpretation as spherical functions for a Gelfand pair (K, M) , where K is a compact group and M is a closed subgroup. These values are: $(\alpha, \beta) = (2n - 3, 1/2)$, $K = \text{Sp}(n) \times \text{Sp}(1)$, $M = \text{Sp}(n - 1) \times \text{diag}(\text{Sp}(1) \times \text{Sp}(1))$, $K/M = S^{4n-1}$, $n = 2, 3, \dots$, and $(\alpha, \beta) = (3, 5/2)$, $K = \text{Spin}(9)$, $M = \text{Spin}(7)$, $K/M = S^{15}$. See [FK79] and the references given there. For these values of the parameters, the general theory of spherical functions on Gelfand pairs yields the existence of suitable product formulas and related hypergroup structures. See also [Koo75].

1.4.2. *Triangle polynomials.* Let

$$R_{n,k}^{\alpha,\beta,\gamma}(x_1, x_2) = R_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2x_1 - 1) \cdot x_1^k R_k^{(\beta,\gamma)}(2x_1^{-1}x_2 - 1),$$

where $\alpha, \beta, \gamma > -1$ and n, k are integers such that $n \geq k \geq 0$.

These functions are polynomials in x_1 and x_2 of degree n , and for fixed α, β, γ they are orthogonal on the triangular region $\Delta^{(2)}$ with respect to the measure $(1-x_1)^\alpha(x_1-x_2)^\beta x_2^\gamma dx_1 dx_2$. See [Koo75] for further details.

1.4.3. Simplex polynomials. These are the k -variable analogues of the triangle polynomials and they are defined by recursion. Let $k = 2, 3, \dots$; $\alpha_1, \dots, \alpha_{k+1} > -1$; n_1, \dots, n_k integers such that $n_1 \geq n_2 \geq \dots \geq n_k \geq 0$. For $k = 2$, let $R_{n_1, n_2}^{\alpha_1, \alpha_2, \alpha_3}(x_1, x_2)$ be the triangle polynomial as in Subsection 1.4.2, and for $k \geq 3$ define

$$(1.4) \quad R_{n_1, \dots, n_k}^{\alpha_1, \dots, \alpha_{k+1}}(x_1, \dots, x_k) \\ = R_{n_1 - n_2}^{(\alpha_1, \alpha_2 + \alpha_3 + \dots + \alpha_{k+1} + 2n_2 + k - 1)}(2x_1 - 1) \cdot x_1^{n_2} R_{n_2, \dots, n_k}^{\alpha_2, \dots, \alpha_{k+1}}\left(\frac{x_2}{x_1}, \dots, \frac{x_k}{x_1}\right).$$

For fixed $\alpha_1, \dots, \alpha_{k+1}$ the functions on the left-hand side of (1.4) are polynomials of degree n_1 in x_1, \dots, x_k which are orthogonal on the simplex $\Delta^{(k)}$ with respect to the measure

$$(1-x_1)^{\alpha_1}(x_1-x_2)^{\alpha_2} \dots (x_{k-1}-x_k)^{\alpha_k}(x_k)^{\alpha_{k+1}} dx_1 dx_2 \dots dx_k.$$

These polynomials were studied before by Kalnins, Miller, and Tratnik [KMT91, (2.8)].

No interpretations of triangle or simplex polynomials as spherical functions are known. For integer or half-integer values of the parameter a weaker group-theoretic interpretation of triangle polynomials can be given as certain $O(p) \times O(q) \times O(r)$ invariant spherical harmonics on the unit sphere in R^{p+q+r} , see [Koo86]. This can be easily iterated to an interpretation of simplex polynomials as special spherical harmonics.

2. HYPERGROUP TYPE PRODUCT FORMULA FOR MULTIVARIATE ORTHOGONAL POLYNOMIALS

2.1. Parabolic biangle polynomials and triangle polynomials. Let

$$I = [0, 1] \quad \text{and} \quad J = [0, \pi].$$

We introduce the following functions and measures (in all cases $|x|, |y|, r \in I$ and ψ and $\psi_j (j = 1, 2, 3) \in J$):

$$D(x, y; r, \psi) = xy + (1-x^2)^{1/2}(1-y^2)^{1/2}r \cos \psi,$$

$$E(x, y; r, \psi) = (x^2 y^2 + (1-x^2)(1-y^2)r^2 + 2xy(1-x^2)^{1/2}(1-y^2)^{1/2}r \cos \psi)^{1/2}.$$

One of our goals in this article is to give the product formulas in an explicit form. This requires that some technical definitions be formulated so that the product formulas may be stated precisely. The definitions will become motivated when the proofs are read. The parameters in the following measures are restricted to those values for

which the measures have finite total variation; the constants are chosen so that all the measures have unit total variation.

$$\begin{aligned}
 dm^\beta(\psi) &= \frac{\Gamma(\beta + \frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\beta + 1)} (\sin \psi)^{2\beta+1} d\psi, \\
 dm^{-1}(\psi) &= d[\frac{1}{2}\delta_0(\psi) + \frac{1}{2}\delta_\pi(\psi)], \\
 dm^{\alpha,\beta}(r, \psi) &= \frac{2\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\beta + 1)} (1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} dr dm^{\beta-\frac{1}{2}}(\psi), \\
 dm^{\alpha,\alpha}(r, \psi) &= \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} (\sin \psi)^{2\alpha} d(\delta_1)(r) d\psi, \\
 d\nu^{\alpha,\beta}(r) &= \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} (1 - r)^\alpha r^\beta dr, \\
 d\nu^{\alpha,-1}(r) &= d(\delta_0)(r), \\
 d\mu^{\alpha,\beta}(r, \psi_1, \psi_2, \psi_3) &= dm^{\beta-\frac{1}{2}}(\psi_3) \cdot dm^{\beta-\frac{1}{2}}(\psi_2) \cdot dm^{\alpha,\beta+\frac{1}{2}}(r, \psi_1), \\
 d\mu^{\alpha,\beta,\gamma}(r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3) \\
 &= dm^{\beta,\gamma}(r_4, \psi_3) \cdot dm^{\beta,\gamma}(r_3, \psi_2) \cdot d\nu^{\beta,\gamma-\frac{1}{2}}(r_2) \cdot dm^{\alpha,\beta+\gamma+1}(r_1, \psi_1), \\
 d\lambda^{\alpha,\beta,\gamma}(r_1, r_2, \psi) &= d\nu^{\beta,\gamma-\frac{1}{2}}(r_2) dm^{\alpha,\beta+\gamma+1}(r_1, \psi).
 \end{aligned}$$

Some of these measures are limiting cases; for instance, the formula for $dm^{\alpha,\beta}$ does not make sense if $\alpha = \beta$, but $dm^{\alpha,\alpha}$ is defined so that for every f which is continuous on $I \times J$ we have

$$\lim_{\beta \rightarrow \alpha^-} \int_{I \times J} f dm^{\alpha,\beta} = \int_{I \times J} f dm^{\alpha,\alpha}.$$

Similar relations hold for the pairs dm^β, dm^{-1} and $d\nu^{\alpha,\beta}, d\nu^{\alpha,-1}$. We use the convention that a single integral sign indicates integration over the full range of the variables indicated in the measure. Thus, the product formula for Jacobi polynomials can be given as

$$\begin{aligned}
 (2.1) \quad R_n^{(\alpha,\beta)}(2x^2 - 1) \cdot R_n^{(\alpha,\beta)}(2y^2 - 1) \\
 = \int_{I \times J} R_n^{(\alpha,\beta)}(2E^2(x, y; r, \psi) - 1) dm^{\alpha,\beta}(r, \psi).
 \end{aligned}$$

A product formula for ultraspherical polynomials,

$$(2.2) \quad R_n^{(\beta,\beta)}(x) \cdot R_n^{(\beta,\beta)}(y) = \int_J R_n^{(\beta,\beta)}(D(x, y; 1, \phi)) dm^{\beta-\frac{1}{2}}(\phi) \quad (\beta \geq -\frac{1}{2}),$$

can be obtained for $\beta > -1/2$ by first letting $\alpha \rightarrow \beta$ in (2.1) with the help of

$$(2.3) \quad \lim_{\gamma \rightarrow 0} \frac{1}{\Gamma(\gamma)} \int_0^1 f(s) s^{\gamma-1} ds = f(0), \quad f \in C([0, 1]),$$

and some algebraic manipulation. The case $\beta = -1/2$ follows by another application of (2.3) by letting $\beta \rightarrow -1/2$.

We introduce some notation that will be used in connection with the parabolic biangle polynomials and the triangle polynomials. When used without explicit arguments, D and E are interpreted as:

$$(2.4) \quad D = D(x_1, y_1; r_1, \psi_1) \quad \text{and} \quad E = E(x_1, y_1; r_1, \psi_1).$$

We also introduce the following functions with the same convention on their use without explicitly indicated arguments:

$$(2.5) \quad C = C(x_1, y_1; r_1, \psi_1) = \frac{D(x_1, y_1; r_1, \psi_1)}{E(x_1, y_1; r_1, \psi_1)},$$

$$(2.6) \quad G = G(x_1, x_2, y_1, y_2; r_1, \psi_1, \psi_2, \psi_3) \\ = D \left(C, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right); 1, \psi_3 \right), \quad (x_1 y_1 \neq 0),$$

$$(2.7) \quad H = H(x_1, x_2, y_1, y_2; r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3) \\ = E \left([(1 - r_2)C^2 + r_2]^{1/2}, E\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; r_3, \psi_2\right); r_4, \psi_3 \right), \quad (x_1 y_1 \neq 0).$$

Theorem 2.1 (Parabolic biangle polynomial product formula). *The parabolic biangle polynomials satisfy the following hypergroup-type product formula: Let $\alpha \geq \beta + \frac{1}{2} \geq 0$. Assume $0 \leq |x_2| \leq x_1 \leq 1$ and $0 \leq |y_2| \leq y_1 \leq 1$, then if $(x_1, y_1), (x_2, y_2) \in B - \{(0, 0)\}$*

$$R_{n,k}^{\alpha,\beta}(x_1^2, x_2) \cdot R_{n,k}^{\alpha,\beta}(y_1^2, y_2) = \int_{I \times J^3} R_{n,k}^{\alpha,\beta}(E^2, EG) d\mu^{\alpha,\beta}(r_1, \psi_1, \psi_2, \psi_3),$$

while if $0 \leq |x_2| \leq x_1 \leq 1$ and $(y_1, y_2) = (0, 0)$

$$R_{n,k}^{\alpha,\beta}(x_1^2, x_2) \cdot R_{n,k}^{\alpha,\beta}(0, 0) = \int_{I \times J} R_{n,k}^{\alpha,\beta}(E^2, D) dm^{\alpha,\beta+\frac{1}{2}}(r_1, \psi_1);$$

note that if $y_1 = 0$, $D = r_1(1 - x_1^2)^{1/2} \cos \psi_1$ and $E = r_1(1 - x_1^2)^{1/2}$.

2.2. Triangle polynomials.

Theorem 2.2 (Triangle polynomial product formula). *The triangle polynomials satisfy the following hypergroup-type product formula: Let $\alpha \geq \beta + \gamma + 1$ and $\beta \geq \gamma \geq -\frac{1}{2}$.*

If $(x_1, x_2), (y_1, y_2) \in \Delta^{(2)} - \{(0, 0)\}$, then

$$\begin{aligned} R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_2^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(y_1^2, y_2^2) \\ = \int_{I^4 \times J^3} R_{n,k}^{\alpha,\beta,\gamma}(E^2, E^2 H^2) d\mu^{\alpha,\beta,\gamma}(r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3), \end{aligned}$$

while if $(x_1, x_2) \in \Delta^{(2)}$ and $(y_1, y_2) = (0, 0)$

$$\begin{aligned} R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_2^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(0, 0) \\ = \int_{I^2 \times J} R_{n,k}^{\alpha,\beta,\gamma}(E^2, E^2[(1-r_2)C^2 + r_2]) d\lambda^{\alpha,\beta,\gamma}(r_1, r_2, \psi_1); \end{aligned}$$

note that if $y_1 = 0$, then $E = r_1(1-x_1)^{1/2}$ and $C = \cos \psi_1$.

2.3. Simplex polynomials. We introduce a suite of definitions and conventions so that the product formula for simplex polynomials can be stated conveniently. We suggest that the reader postpone digestion of these until he is reading the proof of the simplex polynomial product formula (Theorem 2.3) at which point the definitions become fully motivated. We begin by defining for $u, x \in I$

$$\mathbf{V}^{(2)}(x; u) = (1 - u(1 - x), x).$$

For $k > 2$, $u \in I$, and $\mathbf{x} = (x_1, \dots, x_{k-1}) \in \Delta^{(k-1)}$, let

$$\mathbf{V}^{(k)}(\mathbf{x}; u) = (x_1, \dots, x_{k-2}, (1-u)x_{k-2} + ux_{k-1}, x_{k-1}).$$

For $\mathbf{x} = (x_1, \dots, x_k) \in \Delta^{(k)} - \{\mathbf{0}\}$, let

$$L\mathbf{x} = \left(\frac{x_2}{x_1}, \dots, \frac{x_k}{x_1}\right) \in \Delta^{(k-1)}.$$

Now let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $K^{(2)} = I^4 \times J^3$, so a typical $\boldsymbol{\omega} \in K^{(2)}$ can be written as $\boldsymbol{\omega} = (r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3)$, and in the context of simplex polynomials we use the convention that D , E , and H without arguments specified are interpreted as.

$$\begin{aligned} D &= D(\sqrt{x_1}, \sqrt{y_1}; r_1, \psi_1), \\ E &= E(\sqrt{x_1}, \sqrt{y_1}; r_1, \psi_1), \\ H &= H(\sqrt{x_1}, \sqrt{x_2}, \sqrt{y_1}, \sqrt{y_2}; r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3). \end{aligned}$$

Define

$$(2.8) \quad \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = (E^2, E^2 H^2).$$

For $k > 2$, let

$$K^{(k)} = K^{(k-1)} \times K^{(k-1)} \times K^{(k-1)} \times I.$$

If $\boldsymbol{\omega} \in K^{(k)}$ we can write $\boldsymbol{\omega} = (\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u)$ with $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau} \in K^{(k-1)}$ and $u \in I$. Let

$$\begin{aligned}\mathbf{n} &= n_1, \dots, n_k, \\ \boldsymbol{\alpha} &= \alpha_1, \dots, \alpha_{k+1}, \\ P\boldsymbol{\alpha} &= \alpha_2, \dots, \alpha_{k+1}, \\ C\boldsymbol{\alpha} &= \alpha_1, \dots, \alpha_{k-2}, \alpha_{k-1} + \alpha_k + 1, \alpha_{k+1}, \\ \mathbf{e} &= \mathbf{e}^{(k)} = (\overbrace{1, \dots, 1}^k), \\ \mathbf{f} &= \mathbf{e}^{(k-1)}.\end{aligned}$$

Assume that we have already defined $\mathbf{Z}^{(k-1)} = (Z_1^{(k-1)}, \dots, Z_{k-1}^{(k-1)}) = \mathbf{Z}^{(k-1)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\tau})$ ($\mathbf{x}, \mathbf{y} \in \Delta^{(k-1)}$, $\boldsymbol{\tau} \in K^{(k-1)}$), and $d\mu^\beta$ whenever $\beta = \beta_1, \dots, \beta_k$, so that, in particular, $\mu^{P\boldsymbol{\alpha}}$ and $\mu^{C\boldsymbol{\alpha}}$ are defined. Now define

$$\begin{aligned}d\mu^\alpha(\boldsymbol{\omega}) &= d\mu^\alpha(\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) \\ &= d\mu^{C\boldsymbol{\alpha}}(\boldsymbol{\rho}) \cdot d\mu^{P\boldsymbol{\alpha}}(\boldsymbol{\sigma}) \cdot d\mu^{P\boldsymbol{\alpha}}(\boldsymbol{\tau}) \cdot d\nu^{\alpha_k, \alpha_{k-1}}(u).\end{aligned}$$

Let $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{y} = (y_1, \dots, y_k) \in \Delta^{(k)}$, and

$$\mathbf{W} = (W_1, \dots, W_{k-1}) = \mathbf{Z}^{(k-1)}(x_1 \mathbf{f}, y_1 \mathbf{f}; \boldsymbol{\rho}),$$

and define

$$\mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = W_1 \cdot \left(1, \mathbf{Z}^{(k-1)}(\mathbf{V}^{(k-1)}(L\mathbf{W}; u), \mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma}); \boldsymbol{\tau} \right).$$

For $\boldsymbol{\eta} = (r_1, r_2, \psi)$ define

$$\mathbf{U}^{(2)}(x; \boldsymbol{\eta}) = r_1^2(1-x) \cdot (1, (1-r_2)\cos^2\psi + r_2),$$

and if $k > 2$ and $\boldsymbol{\eta} = (r_1, r_2, \dots, r_k, \psi)$, define

$$\begin{aligned}T\boldsymbol{\eta} &= (r_1, \dots, r_{k-1}, \psi), \\ \mathbf{U}^{(k)}(x; \boldsymbol{\eta}) &= \mathbf{V}^{(k)}(\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k), \\ d\lambda^\alpha(\boldsymbol{\eta}) &= d\nu^{\alpha_k, \alpha_{k-1}}(r_k) \cdot d\lambda^{C\boldsymbol{\alpha}}(T\boldsymbol{\eta}).\end{aligned}$$

Theorem 2.3 (Simplex polynomial product formula). *The simplex polynomials satisfy the following hypergroup-type product formula: Let $k \geq 2$, $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{k+1}$,*

with

$$\begin{aligned} \alpha_1 &\geq \alpha_2 + \cdots + \alpha_{k+1} + k - 1, \\ \alpha_2 &\geq \alpha_3 + \cdots + \alpha_{k+1} + k - 2, \\ &\vdots \\ \alpha_{k-1} &\geq \alpha_k + \alpha_{k+1} + 1, \\ \alpha_k &\geq \alpha_{k+1}, \\ \alpha_{k+1} &\geq -\frac{1}{2}. \end{aligned}$$

Let $\mathbf{n} = n_1, \dots, n_k$, with $n_1 \geq \cdots \geq n_k$, and $\mathbf{x}, \mathbf{y} \in \Delta^{(k)} - \{\mathbf{0}\}$, then

$$R_{\mathbf{n}}^{\alpha}(\mathbf{x}) \cdot R_{\mathbf{n}}^{\alpha}(\mathbf{y}) = \int_{K^{(k)}} R_{\mathbf{n}}^{\alpha}(\mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega})) d\mu^{\alpha}(\boldsymbol{\omega}),$$

while if $\mathbf{y} = \mathbf{0}$, then for any $\mathbf{x} \in \Delta^{(k)}$

$$R_{\mathbf{n}}^{\alpha}(\mathbf{x}) \cdot R_{\mathbf{n}}^{\alpha}(\mathbf{0}) = \int_{I^{k \times J}} R_{\mathbf{n}}^{\alpha}(\mathbf{U}^{(k)}(x_1; \boldsymbol{\eta})) d\lambda^{\alpha}(\boldsymbol{\eta}).$$

The measures in the above product formulas have the advantage that their densities are elementary functions, but the disadvantage is that they live on fairly high-dimensional spaces; the dimension of $K^{(k)}$ increases exponentially with k (as one of the referees observed). An interesting question is whether our product formulas can be realized by integration over lower-dimensional regions, while preserving the elementary nature of the measures. In addition, we wonder whether the measures have some canonical explanation in some group-theoretic interpretation of the product formula. To start with, one should address this problem for the case of the parabolic biangle polynomials using their interpretation as spherical functions.

3. PROOF OF THE PRODUCT FORMULAS

All three product formulas will be proved in this section. That they are of hypergroup type will be demonstrated in Section 4.

3.1. Proof of the biangle polynomial product formula. We first prove the formula under the assumption $\alpha > \beta + \frac{1}{2} \geq 0$. The case $\alpha = \beta + \frac{1}{2} \geq 0$ will be a consequence of (2.3).

The proof will require the Jacobi functions which are an extension of definition (1.2) to the case of complex α, β, n, x with $\alpha \neq -1, -2, \dots$ and $x \neq (-\infty, -1]$. The Jacobi functions are defined by

$$\begin{aligned} (3.1) \quad \phi_{\lambda}^{(\alpha, \beta)}(t) &= R_{\frac{1}{2}(i\lambda - \alpha - \beta - 1)}^{(\alpha, \beta)}(\cosh 2t) \\ &= F\left(\frac{-i\lambda + \alpha + \beta + 1}{2}, \frac{i\lambda + \alpha + \beta + 1}{2}; \alpha + 1; -(\sinh t)^2\right). \end{aligned}$$

For fixed α, β , say with $\alpha > \beta > -\frac{1}{2}$, the functions $\phi_\lambda^{(\alpha, \beta)}$ with $\lambda \geq 0$ form a continuous (generalized) orthogonal system on $(0, \infty)$ with respect to the weight function $(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$, (see, for instance, [Fle72] or [Koo75]).

The starting point of our considerations is the integral formula

$$(3.2) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)} \int_0^1 x^{s-1}(1-x)^{c-s-1}(1-xz)^{r-a-b} \\ \cdot F(r-a, r-b; s; xz) F\left(a+b-r, r-s; c-s; \frac{(1-x)z}{1-xz}\right) dx,$$

where $\Re c > \Re s > 0$, $z \in \mathbb{C} - [1, \infty)$. This formula due to Erdelyi [EMOT53, 2.4(3)] is a generalization of Bateman's integral (put $r = a+b$ in (3.2) and it can be derived from Bateman's integral by using fractional integration by parts). Gasper [Gas75] stressed the importance of (3.2) and he applied it in order to derive a Mehler-Dirichlet-type integral representation for Jacobi polynomials. In this paper, we will give several other applications of (3.2).

Let us rewrite (3.2) in the following way. First apply [EMOT53, 2.1(22)] to the second hypergeometric function occurring in the integrand of (3.2). Next substitute

$$a = \frac{1}{2}(-i\lambda + \alpha + \beta + \gamma + 2k + 2), \quad b = \frac{1}{2}(i\lambda + \alpha + \beta + \gamma + 2k + 2), \\ c = \alpha + 1, \quad r = k + \alpha + \gamma + 1, \quad s = \alpha - \beta,$$

where $\alpha > \beta > -1$, $\gamma > -1$, $k \in \mathbb{N}_0$, and $\lambda \in \mathbb{C}$.

Finally, put $z = -(\sinh y)^2$ ($y \geq 0$) and introduce a new integration variable ζ such that $x = \sinh^2 \zeta / \sinh^2 y$. As a result we obtain

$$F\left(\frac{-i\lambda + \alpha + \beta + \gamma + 2k + 2}{2}, \frac{i\lambda + \alpha + \beta + \gamma + 2k + 2}{2}; \alpha + 1; -\sinh^2 y\right) \\ = \frac{2^{1-\beta}\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\beta + 1)} (\cosh y)^{-2k-2\beta-2\gamma-2} (\sinh y)^{-2\alpha} \\ \cdot \int_0^y F\left(-k, k + \beta + \gamma + 1; \beta + 1; 1 - \frac{\cosh^2 \zeta}{\cosh^2 y}\right) \\ \cdot F\left(\frac{-i\lambda + \alpha - \beta + \gamma}{2}, \frac{i\lambda + \alpha - \beta + \gamma}{2}; \alpha - \beta; -\sinh^2 \zeta\right) \\ \cdot (\sinh \zeta)^{2\alpha-2\beta-1} (\cosh \zeta)^{2\gamma+1} (\cosh 2y - \cosh 2\zeta)^\beta d\zeta.$$

In view of (1.2) and (3.1) this becomes

$$(3.3) \quad (\cosh y)^{2k} \phi_\lambda^{(\alpha, \beta + \gamma + 2k + 1)}(y) = \frac{2^{1-\beta} \Gamma(\alpha + 1)}{\Gamma(\alpha - \beta) \Gamma(\beta + 1)} \\ \cdot (\cosh y)^{-2\beta - 2\gamma - 2} (\sinh y)^{-2\alpha} \int_0^y R_k^{(\beta, \gamma)} \left(2 \frac{\cosh^2 \zeta}{\cosh^2 y} - 1 \right) \cdot \phi_\lambda^{(\alpha - \beta - 1, \gamma)}(\zeta) \\ \cdot (\cosh 2y - \cosh 2\zeta)^\beta (\sinh \zeta)^{2\alpha - 2\beta - 1} (\cosh \zeta)^{2\gamma + 1} d\zeta \quad (\alpha > \beta > -1).$$

We have chosen the Jacobi function notation, since the formulas become most elegant in this form and since they suggest results for the functions

$$(3.4) \quad (y, \theta) \mapsto \phi_\lambda^{(\alpha, \beta + \gamma + 2k + 1)}(y) \cdot (\cosh y)^{2k} R_k^{(\beta, \gamma)}(\cos \theta),$$

which form a continuous orthogonal system analogous to the triangle polynomials.

Let us specialize (3.3) to the case $\gamma = -1/2$. Combination of (1.2) and [EMOT53, 2.11(2)] yields the quadratic transformation

$$(3.5) \quad R_k^{(\beta, -\frac{1}{2})}(2x^2 - 1) = R_{2k}^{(\beta, \beta)}(x),$$

which is valid for general complex k . After substituting (3.5) in (3.3) and replacing $2k$ by k we obtain

$$(3.6) \quad (\cosh y)^k \phi_\lambda^{(\alpha, \beta + k + \frac{1}{2})}(y) = \frac{2^{1-\beta} \Gamma(\alpha + 1)}{\Gamma(\alpha - \beta) \Gamma(\beta + 1)} \\ \cdot (\cosh y)^{-2\beta - 1} (\sinh y)^{-2\alpha} \int_0^y R_k^{(\beta, \beta)} \left(\frac{\cosh \zeta}{\cosh y} \right) \phi_\lambda^{(\alpha - \beta - 1, -\frac{1}{2})}(\zeta) \\ \cdot (\cosh 2y - \cosh 2\zeta)^\beta (\sinh \zeta)^{2\alpha - 2\beta - 1} d\zeta \quad (\alpha > \beta > -1).$$

Next, we need a Mehler-Dirichlet-type formula. The Legendre functions of the first kind P_ν^μ (see [EMOT53, 3.7(8)]) satisfy:

$$P_\nu^\mu(\cosh x) = (2/\pi)^{1/2} \frac{\sinh^\mu x}{\Gamma(1/2 - \mu)} \\ \cdot \int_0^x (\cosh x - \cosh y)^{-\mu - 1/2} \cosh((\nu + 1/2)y) dy \quad (\Re \mu < \frac{1}{2}).$$

Now by [EMOT53, §3.2(20)]

$$P_\nu^\mu(\cosh x) = \frac{2^\mu (\cosh^2 x - 1)^{-\mu/2}}{\Gamma(1 - \mu)} F \left(\frac{1 + \nu - \mu}{2}, \frac{-\nu - \mu}{2}; 1 - \mu; 1 - \cosh^2 x \right).$$

Hence

$$P_{i\lambda - \frac{1}{2}}^{-\alpha}(\cosh t) = \frac{2^{-\alpha} (\sinh t)^\alpha}{\Gamma(1 + \alpha)} \phi_\lambda^{(\alpha, -\frac{1}{2})}(t).$$

Thus we obtain the Mehler-Dirichlet type formula

$$(3.7) \quad \phi_\lambda^{(\alpha-\beta-1, -\frac{1}{2})}(\zeta) = \frac{2^{\alpha-\beta-\frac{1}{2}}\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta-\frac{1}{2})\Gamma(\frac{1}{2})} (\sinh \zeta)^{2(\beta-\alpha+1)} \\ \cdot \int_0^\zeta (\cos \lambda \eta) (\cosh \zeta - \cosh \eta)^{\alpha-\beta-\frac{3}{2}} d\eta \quad (\alpha > \beta + \frac{1}{2}).$$

Now substitute (3.7) in (3.6). This yields a representation of the left hand-side of (3.6) by a double integral with integration variables (η, ζ) . On performing the transformation of integration variables from (η, ζ) to (η, χ) given by $\cos \chi = (\cosh \zeta)/(\cosh y)$ we obtain

$$(3.8) \quad (\cosh y)^k \phi_\lambda^{(\alpha, \beta+k+\frac{1}{2})}(y) = \frac{2^{\alpha-\beta+\frac{1}{2}}\Gamma(\alpha+1)}{\Gamma(\alpha-\beta-\frac{1}{2})\Gamma(\beta+1)\Gamma(\frac{1}{2})} \cdot (\sinh y)^{-2\alpha} \\ \cdot \int_0^y \int_0^{\arccos(\frac{\cosh \eta}{\cosh y})} (\cos \lambda \eta) R_k^{(\beta, \beta)}(\cos \chi) (\cosh y \cos \chi - \cosh \eta)^{\alpha-\beta-\frac{3}{2}} \\ \cdot (\sin \chi)^{2\beta+1} d\chi d\eta \quad (\alpha > \beta + \frac{1}{2} > -\frac{1}{2}).$$

For $k = 0$ this formula reduces to the Mehler-Dirichlet-type formula for Jacobi functions (see Koornwinder [Koo75, (2.16) and (2.18)]).

Note that (3.8) shows that

$$(\cosh y)^k \phi_\lambda^{(\alpha, \beta+k+\frac{1}{2})}(y) = \int (\cos \lambda \eta) R_k^{(\beta, \beta)}(\cos \chi) d\sigma_y^{(\alpha, \beta)}(\eta, \chi),$$

where $\sigma_y^{(\alpha, \beta)}$ is a positive measure if $\alpha \geq \beta + \frac{1}{2} \geq -\frac{1}{2}$.

It was pointed out in [Koo75, Remark 6] that the Laplace-type integral representation

$$\phi_\lambda^{(\alpha, \beta+\frac{1}{2})}(y) = \int (K(\cosh y; r, \psi))^{i\lambda-\alpha-\beta-\frac{3}{2}} dm^{\alpha, \beta+\frac{1}{2}}(r, \psi) \quad (\alpha > \beta + \frac{1}{2} > -\frac{1}{2}),$$

where

$$K(x; r, \psi) = [x^2 + (x^2 - 1)r^2 + 2x(x^2 - 1)^{1/2}r \cos \psi]^{1/2},$$

is equivalent with the case $k = 0$ of (3.8), since both formulas are related to each other by the substitution

$$e^{\eta+i\chi} = \cosh y + (\sinh y)re^{i\psi}$$

for the integration variables. Let us make the same substitution for the general case of (3.8). This is most conveniently done in by three substitutions in sequence: first $\sigma + i\tau = e^{\eta+i\chi}$, then $s = (\sigma - \cosh y)/(\sinh y)$, and $t = \tau/(\sinh y)$, and finally

$re^{i\psi} = s + it$. We obtain

$$\begin{aligned} & (\cosh y)^k \phi_\lambda^{(\alpha, \beta+k+\frac{1}{2})}(y) \\ &= \int R_k^{(\beta, \beta)} \left(\frac{\cosh y + (\sinh y)r \cos \psi}{K(\cosh y; r, \psi)} \right) [K(\cosh y; r, \psi)]^{i\lambda - \alpha - \beta - \frac{3}{2}} dm^{\alpha, \beta + \frac{1}{2}}(r, \psi) \\ & \qquad \qquad \qquad (\alpha > \beta + \frac{1}{2} > -\frac{1}{2}). \end{aligned}$$

By substituting (3.1) this formula can be rewritten as

$$\begin{aligned} (3.9) \quad & x^k R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2x^2 - 1) \\ &= \int R_k^{(\beta, \beta)} \left(\frac{x + (x^2 - 1)^{1/2} r \cos \psi}{K(x; r, \psi)} \right) [K(x; r, \psi)]^{2n-k} dm^{\alpha, \beta + \frac{1}{2}}(r, \psi) \\ & \qquad \qquad \qquad (\alpha > \beta + \frac{1}{2} > -\frac{1}{2}; x \geq 1). \end{aligned}$$

For $k = 0$ this formula reduces to the Laplace-type integral representation for Jacobi polynomials (see [Koo72, (1)] and [Ask74]).

Our next task is to obtain an integral for

$$(3.10) \quad x_1^k R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2x_1^2 - 1) \cdot y_1^k R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2y_1^2 - 1).$$

We require a formula due to Bateman (see [Koo74, (2.19)]). If the coefficients c_m are defined by

$$(3.11) \quad R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2x^2 - 1) = \sum_{m=0}^{n-k} c_m x^{2m}$$

then

$$\begin{aligned} (3.12) \quad & \sum_{m=0}^{n-k} c_m (x^2 + y^2 - 1)^m R_m^{(\alpha, \beta+k+\frac{1}{2})} \left(\frac{2x^2 y^2}{x^2 + y^2 - 1} - 1 \right) \\ &= R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2x^2 - 1) \cdot R_{n-k}^{(\alpha, \beta+k+\frac{1}{2})}(2y^2 - 1). \end{aligned}$$

In (3.9) we make the following substitution on both sides:

$$x = \frac{x_1 y_1}{(x_1^2 + y_1^2 - 1)^{1/2}} \quad (x_1, y_1 \geq 1), \quad n = m + k, \quad r = r_1, \quad \text{and} \quad \psi = \psi_1,$$

then multiply by $c_m (x_1^2 + y_1^2 - 1)^{m+\frac{1}{2}k}$ and sum for $m = 0, \dots, n - k$. The left-hand side is converted to (3.10). On the right-hand side, analytic continuation in x_1 and y_1 and the substitution $\psi_1 = \pi - \psi_1$ yields

$$\int R_k^{(\beta, \beta)}(DE^{-1}) \left[E^k \sum_{m=0}^{n-k} c_m E^{2m} \right] dm^{\alpha, \beta + \frac{1}{2}}(r_1, \psi_1).$$

So finally for $\alpha > \beta + \frac{1}{2} > -\frac{1}{2}$

$$\begin{aligned}
(3.13) \quad R_{n,k}^{\alpha,\beta}(x_1^2, x_1) \cdot R_{n,k}^{\alpha,\beta}(y_1^2, y_1) \\
&= x_1^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x_1^2 - 1) \cdot y_1^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2y_1^2 - 1) \\
&= \int R_k^{(\beta,\beta)}(DE^{-1}) \cdot E^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2E^2 - 1) dm^{\alpha,\beta+\frac{1}{2}}(r_1, \psi_1).
\end{aligned}$$

Equation (3.13) can be rewritten

$$(3.14) \quad R_{n,k}^{\alpha,\beta}(x_1^2, x_1) \cdot R_{n,k}^{\alpha,\beta}(y_1^2, y_1) = \int R_{n,k}^{\alpha,\beta}(E^2, D) dm^{\alpha,\beta+\frac{1}{2}}(r_1, \psi_1).$$

Now

$$(3.15) \quad R_{n,k}^{\alpha,\beta}(0, 0) \cdot R_{n,k}^{\alpha,\beta}(x_1^2, x_2) = R_{n,k}^{\alpha,\beta}(0, 0) \cdot R_{n,k}^{\alpha,\beta}(x_1^2, x_1),$$

since $R_{n,0}^{\alpha,\beta}(x_1^2, x_2) = R_{n,0}^{\alpha,\beta}(x_1^2, x_1)$ while if $k > 0$, $R_{n,k}^{\alpha,\beta}(0, 0) = 0$. Thus the special case of the product formula in Theorem 2.1 follows.

We can now establish the full product formula using (3.13) and the product formula (2.2) for the ultraspherical polynomials twice

$$\begin{aligned}
R_{n,k}^{\alpha,\beta}(x_1^2, x_2) \cdot R_{n,k}^{\alpha,\beta}(y_1^2, y_2) \\
&= x_1^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x_1^2 - 1) \cdot x_2^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2x_2^2 - 1) \cdot R_k^{(\beta,\beta)}\left(\frac{x_2}{x_1}\right) \cdot R_k^{(\beta,\beta)}\left(\frac{y_2}{y_1}\right) \\
&= \int E^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2E^2 - 1) \cdot R_k^{(\beta,\beta)}(DE^{-1}) dm^{\alpha,\beta+\frac{1}{2}}(r_1, \psi_1) \\
&\quad \cdot \int R_k^{(\beta,\beta)}\left(D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right)\right) dm^{\beta-\frac{1}{2}}(\psi_2) \\
&= \int E^k R_{n-k}^{(\alpha,\beta+k+\frac{1}{2})}(2E^2 - 1) \cdot R_k^{(\beta,\beta)}\left(D(DE^{-1}, D\left(\frac{x_2}{x_1}, \frac{y_2}{x_2}; 1, \psi_2\right); 1, \psi_3)\right) \\
&\quad \cdot dm^{\beta-\frac{1}{2}}(\psi_3) \cdot dm^{\beta-\frac{1}{2}}(\psi_2) \cdot dm^{\alpha,\beta+\frac{1}{2}}(r_1, \psi_1) \\
&= \int R_{n,k}^{\alpha,\beta}(E^2, EG) d\mu^{\alpha,\beta}(r_1, \psi_1, \psi_2, \psi_3) \quad (\alpha > \beta + \frac{1}{2} \geq 0).
\end{aligned}$$

3.2. Proof of triangle polynomial product formula. We will first prove the formula under the assumption $\alpha > \beta + \gamma + 1$, $\beta > \gamma > -\frac{1}{2}$. The remaining cases can be obtained using (2.3).

We recall the following result of Askey and Fitch [AF69, (3.3)]:

$$\begin{aligned}
(1-x)^{\alpha+\mu} R_k^{(\alpha+\mu, \beta-\mu)}(x) \\
&= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 (1-y)^\alpha R_k^{(\alpha,\beta)}(y)(y-x)^{\mu-1} dy \quad (\alpha, \beta > -1, \mu > 0).
\end{aligned}$$

Now make the substitutions

$$\alpha = \beta, \quad \beta = \gamma, \quad \mu = \gamma + \frac{1}{2}, \quad x = 2u^2 - 1, \quad y = 2(1-r)u^2 + 2r - 1,$$

to obtain

$$(3.16) \quad R_k^{(\beta+\gamma+\frac{1}{2}, -\frac{1}{2})}(2u^2 - 1) = \int_0^1 R_k^{(\beta, \gamma)}(2(1-r)u^2 + 2r - 1) d\nu^{\beta, \gamma - \frac{1}{2}}(r), \quad (\beta > -1, \gamma > -\frac{1}{2}).$$

We also observe that

$$R_{n,k}^{\alpha, \beta, \gamma}(x, x) = x^k R_{n-k}^{(\alpha, \beta + \gamma + 2k + 1)}(2x - 1) = R_{n,k}^{\alpha, \beta + \gamma + \frac{1}{2}, -\frac{1}{2}}(x, x).$$

Equation (3.5) can be used to obtain

$$R_{n,k}^{\alpha, \beta, -\frac{1}{2}}(x_1, x_2) = R_{n+k, 2k}^{\alpha, \beta}(x_1, x_2^{1/2})$$

Thus using (3.14) we have

$$\begin{aligned} R_{n,k}^{\alpha, \beta, -\frac{1}{2}}(x_1^2, x_1^2) \cdot R_{n,k}^{\alpha, \beta, -\frac{1}{2}}(y_1^2, y_1^2) &= R_{n+k, 2k}^{\alpha, \beta}(x_1^2, x_1) \cdot R_{n+k, 2k}^{\alpha, \beta}(y_1^2, y_1) \\ &= \int R_{n+k, 2k}^{\alpha, \beta}(E^2, D) dm^{\alpha, \beta + \frac{1}{2}}(r_1, \psi_1) \\ &= \int R_{n,k}^{\alpha, \beta, -\frac{1}{2}}(E^2, D^2) dm^{\alpha, \beta + \frac{1}{2}}(r_1, \psi_1) \quad (\alpha > \beta + \frac{1}{2} > -\frac{1}{2}). \end{aligned}$$

Thus with β replaced by $\beta + \gamma + \frac{1}{2}$ and using (3.16)

$$\begin{aligned} R_{n,k}^{\alpha, \beta, \gamma}(x_1^2, x_1^2) \cdot R_{n,k}^{\alpha, \beta, \gamma}(y_1^2, y_1^2) &= R_{n,k}^{\alpha, \beta + \gamma + \frac{1}{2}, -\frac{1}{2}}(x_1^2, x_1^2) \cdot R_{n,k}^{\alpha, \beta + \gamma + \frac{1}{2}, -\frac{1}{2}}(y_1^2, y_1^2) \\ &= \int R_{n,k}^{\alpha, \beta + \gamma + \frac{1}{2}, -\frac{1}{2}}(E^2, D^2) dm^{\alpha, \beta + \gamma + 1}(r_1, \psi_1) \\ &= \int R_{n-k}^{(\alpha, \beta + \gamma + 2k + 1)}(2E^2 - 1) \cdot E^{2k} R_k^{(\beta + \gamma + \frac{1}{2}, -\frac{1}{2})}(2C^2 - 1) dm^{\alpha, \beta + \gamma + 1}(r_1, \psi_1) \\ &= \int R_{n-k}^{(\alpha, \beta + \gamma + 2k + 1)}(2E^2 - 1) \cdot E^{2k} \\ &\quad \cdot R_k^{(\beta, \gamma)}(2(1-r_2)C^2 + 2r_2 - 1) d\nu^{\beta, \gamma - \frac{1}{2}}(r_2) \cdot dm^{\alpha, \beta + \gamma + 1}(r_1, \psi_1) \\ &= \int R_{n,k}^{\alpha, \beta, \gamma} \left(E^2, E^2[(1-r_2)C^2 + r_2] \right) d\nu^{\beta, \gamma - \frac{1}{2}}(r_2) \cdot dm^{\alpha, \beta + \gamma + 1}(r_1, \psi_1), \\ &\quad (\alpha > \beta + \gamma + 1, \quad \beta > -1, \quad \gamma > -\frac{1}{2}). \end{aligned}$$

Now by the same argument as used for (3.15)

$$R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_2^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(0, 0) = R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_1^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(0, 0)$$

and the special case of the product formula in Theorem 2.2 follows from the above.

Finally, using (2.1) twice

$$\begin{aligned} & R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_2^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(x_2^2, y_2^2) \\ &= R_{n,k}^{\alpha,\beta,\gamma}(x_1^2, x_1^2) \cdot R_{n,k}^{\alpha,\beta,\gamma}(x_2^2, x_2^2) \cdot R_k^{(\beta,\gamma)}\left(2\frac{x_2^2}{x_1^2} - 1\right) \cdot R_k^{(\beta,\gamma)}\left(2\frac{y_2^2}{x_2^2} - 1\right) \\ &= \int R_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2E^2 - 1) \cdot E^{2k} \\ &\quad \cdot R_k^{(\beta,\gamma)}\left(2(1-r_2)C^2 + 2r_2 - 1\right) d\nu^{\beta,\gamma-\frac{1}{2}}(r_2) dm^{\alpha,\beta+\gamma+1}(r_1, \psi_1) \\ &\quad \cdot \int R_k^{(\beta,\gamma)}\left(2E^2\left(\frac{x_2}{x_1}, \frac{y_2}{x_2}; r_3, \psi_2\right) - 1\right) dm^{\beta,\gamma}(r_3, \psi_2) \\ &= \int R_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2E^2 - 1) \cdot E^{2k} \\ &\quad \cdot R_k^{(\beta,\gamma)}\left(2E^2\left([(1-r_2)C^2 + r_2]^{1/2}, E\left(\frac{x_2}{x_1}, \frac{y_2}{x_2}; r_3, \psi_2\right); r_4, \psi_3\right) - 1\right) \\ &\quad \cdot dm^{\beta,\gamma}(r_4, \psi_3) \cdot dm^{\beta,\gamma}(r_3, \psi_2) \cdot d\nu^{\beta,\gamma-\frac{1}{2}}(r_2) \cdot dm^{\alpha,\beta+\gamma+1}(r_1, \psi_1) \\ &= \int R_{n,k}^{\alpha,\beta,\gamma}(E^2, E^2H^2) d\mu^{\alpha,\beta,\gamma}(r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3), \\ &\hspace{15em} (\alpha > \beta + \gamma + 1, \quad \beta > \gamma > -\frac{1}{2}). \end{aligned}$$

3.3. Proof of the simplex polynomial product formula. The bivariate simplex polynomials coincide with the triangle polynomials, so the product formulas for the case $k = 2$ is given in Theorem 2.2. Thus we assume that the product formulas in Theorem 2.3 are valid with $k - 1$ in place of k . We recall some notations from Subsection 2.3 and introduce some further notations:

$$\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{k+1},$$

$$\mathbf{n} = n_1, \dots, n_k,$$

$$B(\boldsymbol{\alpha}, \mathbf{n}) = (\alpha_1, \alpha_2 + \dots + \alpha_{k+1} + 2n_2 + k - 1),$$

$$C\boldsymbol{\alpha} = \begin{cases} \alpha_1 + \alpha_2 + 1, \alpha_3 & \text{if } k = 2, \\ \alpha_1, \dots, \alpha_{k-2}, \alpha_{k-1} + \alpha_k + 1, \alpha_{k+1}, & \text{if } k > 2, \end{cases}$$

$$\mathbf{e} = \mathbf{e}^{(k)} = (\overbrace{1, \dots, 1}^k),$$

$$\mathbf{f} = \mathbf{e}^{(k-1)},$$

and if p and q are nonnegative integers with $p < q$ and $\mathbf{s} = s_p, \dots, s_q$, the two truncation operators are defined by

$$P\mathbf{s} = s_{p+1}, \dots, s_q, \quad \text{and} \quad Q\mathbf{s} = s_p, \dots, s_{q-1},$$

so that, for instance,

$$(3.17) \quad CP\boldsymbol{\alpha} = PC\boldsymbol{\alpha} = \alpha_2, \dots, \alpha_{k-2}, \alpha_{k-1} + \alpha_k + 1, \alpha_{k+1},$$

$$(3.18) \quad B(C\boldsymbol{\alpha}, Q\mathbf{n}) = B(\boldsymbol{\alpha}, \mathbf{n}),$$

and (1.4) can be written

$$(3.19) \quad R_{\mathbf{n}}^{\boldsymbol{\alpha}}(\mathbf{x}) = R_{n_1-n_2}^{B(\boldsymbol{\alpha}, \mathbf{n})}(2x_1 - 1) \cdot x_1^{n_2} R_{P\mathbf{n}}^{PC\boldsymbol{\alpha}}(L\mathbf{x}).$$

We will make use of the following:

$$(3.20) \quad \mathbf{V}^{(k+1)}(\mathbf{x}; u) = (x_1, x_1 \mathbf{V}^{(k)}(L\mathbf{x}; u)) = x_1 (1, \mathbf{V}^{(k)}(L\mathbf{x}; u)),$$

We begin with an auxiliary result:

Lemma 3.1. *If $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{k+1}$, $\mathbf{n} = n_1, \dots, n_k$ with $n_1 \geq \dots \geq n_k$, and $\mathbf{x} \in \Delta^{(k-1)}$*

$$(3.21) \quad R_{Q\mathbf{n}}^{C\boldsymbol{\alpha}}(\mathbf{x}) = \int_0^1 R_{\mathbf{n}}^{\boldsymbol{\alpha}}(\mathbf{V}^{(k)}(\mathbf{x}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u).$$

Proof. The proof is by induction on the number k starting with $k = 3$. Formula (3.2) with $a = -n$, $b = n + \alpha + \beta + \gamma + 2$, $c = \alpha + \beta + 2$, $r = \alpha - k + 1$, and $s = \alpha + 1$ followed by [EMOT53, 2.1.4(23)] leads to

$$(3.22) \quad R_n^{(\alpha+\beta+1, \gamma)}(1 - 2z) = \int R_{n,k}^{\alpha, \beta, \gamma}(1 - xz, 1 - z) d\nu^{\beta, \alpha}(x).$$

If $k = 3$ we have (with the help of (3.22))

$$\begin{aligned} R_{n_1, n_2}^{\alpha_1, \alpha_2 + \alpha_3 + 1, \alpha_4}(x_1, x_2) &= R_{n_1 - n_2}^{(\alpha_1, \alpha_2 + \alpha_3 + \alpha_4 + 2n_2 + 2)}(2x_1 - 1) \cdot x_1^{n_2} R_{n_2}^{(\alpha_2 + \alpha_3 + 1, \alpha_4)}(2x_1^{-1}x_2 - 1) \\ &= R_{n_1 - n_2}^{(\alpha_1, \alpha_2 + \alpha_3 + \alpha_4 + 2n_2 + 2)}(2x_1 - 1) \\ &\quad \cdot x_1^{n_2} \int_0^1 R_{n_2, n_3}^{\alpha_2, \alpha_3, \alpha_4}(1 - u + ux_1^{-1}x_2, x_1^{-1}x_2) d\nu^{\alpha_3, \alpha_2}(u) \\ &= \int_0^1 R_{n_1, n_2, n_3}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(x_1, (1 - u)x_1 + ux_2, x_2) d\nu^{\alpha_3, \alpha_2}(u) \end{aligned}$$

which is (3.21) for $k = 3$.

Now assume that that (3.21) is true for some $k \geq 3$. Then if $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_{k+2}$, $\mathbf{n} = n_1, \dots, n_{k+1}$ with $n_1 \geq \dots \geq n_{k+1}$, and $\mathbf{x} \in \Delta^{(k)}$, equations (3.17)–(3.19) yield

$$(3.23) \quad \begin{aligned} R_{Q\mathbf{n}}^{C\boldsymbol{\alpha}}(\mathbf{x}) &= R_{n_1 - n_2}^{B(C\boldsymbol{\alpha}, Q\mathbf{n})}(2x_1 - 1) \cdot x_1^{n_2} R_{PQ\mathbf{n}}^{PC\boldsymbol{\alpha}}(L\mathbf{x}) \\ &= R_{n_1 - n_2}^{B(\boldsymbol{\alpha}, \mathbf{n})}(2x_1 - 1) \cdot x_1^{n_2} R_{QP\mathbf{n}}^{CP\boldsymbol{\alpha}}(L\mathbf{x}) \end{aligned}$$

so, by the inductive assumption and (3.20)

$$\begin{aligned}
R_{Q\mathbf{n}}^C \alpha(\mathbf{x}) &= R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2x_1 - 1) \cdot x_1^{n_2} \int R_{P\mathbf{n}}^P \alpha(\mathbf{V}^{(k)}(L\mathbf{x}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u) \\
&= \int R_{\mathbf{n}}^\alpha(x_1, x_1 \mathbf{V}^{(k)}(L\mathbf{x}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u) \\
&= \int R_{\mathbf{n}}^\alpha(\mathbf{V}^{(k+1)}(\mathbf{x}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u).
\end{aligned}$$

□

We now turn to the inductive step of the proof of Theorem 2.3. Assume $k \geq 3$ and that the product formula is true for simplex polynomials of fewer than k variables. As in the other proofs of product formulas it will suffice to obtain the formula under the assumption that the inequalities for $\alpha_1, \dots, \alpha_{k+1}$ are strict. Let $\alpha = \alpha_1, \dots, \alpha_{k+1}$ and $\mathbf{n} = n_1, \dots, n_k$ satisfy the hypotheses of the theorem with strict inequalities for $\alpha_1, \dots, \alpha_{k+1}$ in place of the weak inequalities. Let $\rho, \sigma, \tau \in K^{(k-1)}$ and $\mathbf{W} = (W_1, \dots, W_{k-1}) = \mathbf{Z}^{(k-1)}(x\mathbf{f}, y\mathbf{f}; \rho)$. We then have with the help of (3.18), (3.19), (3.23), the inductive assumption, and the observation that $C\alpha$ and $Q\mathbf{n}$ satisfy the hypotheses of Theorem 2.3 with $k - 1$ variables.

$$\begin{aligned}
R_{\mathbf{n}}^\alpha(x\mathbf{e}) \cdot R_{\mathbf{n}}^\alpha(y\mathbf{e}) &= x^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2x - 1) \cdot y^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2y - 1) \\
&= R_{Q\mathbf{n}}^C \alpha(x\mathbf{f}) \cdot R_{Q\mathbf{n}}^C \alpha(y\mathbf{f}) \\
&= \int R_{Q\mathbf{n}}^C \alpha(\mathbf{W}) d\mu^{C\alpha}(\rho) \\
&= \int W_1^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2W_1 - 1) \cdot R_{PQ\mathbf{n}}^{PC} \alpha(L\mathbf{W}) d\mu^{C\alpha}(\rho).
\end{aligned}$$

Now, by (3.17) $R_{PQ\mathbf{n}}^{PC} \alpha(L\mathbf{W}) = R_{QP\mathbf{n}}^{CP} \alpha(L\mathbf{W})$, so an application of Lemma 3.1 yields

$$\begin{aligned}
(3.24) \quad R_{\mathbf{n}}^\alpha(x\mathbf{e}) \cdot R_{\mathbf{n}}^\alpha(y\mathbf{e}) &= \int W_1^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2W_1 - 1) \cdot R_{P\mathbf{n}}^P \alpha(\mathbf{V}^{(k-1)}(L\mathbf{W}; u)) \\
&\quad \cdot d\nu^{\alpha_k, \alpha_{k-1}}(u) \cdot d\mu^{C\alpha}(\rho) \\
&= \int R_{\mathbf{n}}^\alpha(W_1, W_1 \mathbf{V}^{(k-1)}(L\mathbf{W}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u) \cdot d\mu^{C\alpha}(\rho) \\
&= \int R_{\mathbf{n}}^\alpha(\mathbf{V}^{(k)}(\mathbf{W}; u)) d\nu^{\alpha_k, \alpha_{k-1}}(u) \cdot d\mu^{C\alpha}(\rho)
\end{aligned}$$

by (3.20). Using (3.24) and the product formula for simplex polynomials of $k - 1$ variables twice we have for $\mathbf{x}, \mathbf{y} \in \Delta^{(k)} - \{\mathbf{0}\}$

$$\begin{aligned}
 R_{\mathbf{n}}^{\alpha}(\mathbf{x}) \cdot R_{\mathbf{n}}^{\alpha}(\mathbf{y}) &= R_{\mathbf{n}}^{\alpha}(x_1 \mathbf{e}) \cdot R_{\mathbf{n}}^{\alpha}(y_1 \mathbf{e}) \cdot R_{P_{\mathbf{n}}}^{P\alpha}(L\mathbf{x}) \cdot R_{P_{\mathbf{n}}}^{P\alpha}(L\mathbf{y}) \\
 &= \int W_1^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2W_1 - 1) \\
 &\quad \cdot R_{P_{\mathbf{n}}}^{P\alpha}(\mathbf{V}^{(k-1)}(L\mathbf{W}; u)) \, d\nu^{\alpha_k, \alpha_{k-1}}(u) \, d\mu^{C\alpha}(\boldsymbol{\rho}) \\
 &\quad \cdot \int R_{P_{\mathbf{n}}}^{P\alpha}(\mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma})) \, d\mu^{P\alpha}(\boldsymbol{\sigma}) \\
 &= \int W_1^{n_2} R_{n_1-n_2}^{B\alpha}(2W_1 - 1) \\
 &\quad \cdot R_{P_{\mathbf{n}}}^{P\alpha}\left(\mathbf{Z}^{(k-1)}(\mathbf{V}^{(k-1)}(L\mathbf{W}; u), \mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma}); \boldsymbol{\tau}\right) \\
 &\quad \cdot d\nu^{\alpha_k, \alpha_{k-1}}(u) \cdot d\mu^{C\alpha}(\boldsymbol{\rho}) \cdot d\mu^{P\alpha}(\boldsymbol{\sigma}) \cdot d\mu^{P\alpha}(\boldsymbol{\tau}) \\
 &= \int R_{\mathbf{n}}^{\alpha}(\mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega})) \, d\mu^{\alpha}(\boldsymbol{\omega}).
 \end{aligned}$$

The case where $\mathbf{y} = \mathbf{0}$ is handled separately; for $x_1 \in I$ and $\boldsymbol{\eta} = (r_1, \dots, r_k, \psi)$ let

$$\mathbf{U} = (U_1, \dots, U_{k-1}) = \mathbf{U}^{(k-1)}(x_1; T\boldsymbol{\eta}).$$

Since $R_{\mathbf{n}}^{\alpha}(\mathbf{x}) = R_{\mathbf{n}}^{\alpha}(x_1 \mathbf{e})$ when $n_2 = 0$ and $R_{\mathbf{n}}^{\alpha}(\mathbf{0}) = 0$ otherwise, we have from (3.19), (3.23), and the inductive assumption that

$$\begin{aligned}
 R_{\mathbf{n}}^{\alpha}(\mathbf{0})R_{\mathbf{n}}^{\alpha}(\mathbf{x}) &= R_{\mathbf{n}}^{\alpha}(\mathbf{0})R_{\mathbf{n}}^{\alpha}(x_1 \mathbf{e}) \\
 &= R_{Q_{\mathbf{n}}}^{C\alpha}(\mathbf{0}\mathbf{f})R_{Q_{\mathbf{n}}}^{C\alpha}(x_1 \mathbf{f}) \\
 &= \int R_{Q_{\mathbf{n}}}^{C\alpha}(\mathbf{U}) \, d\lambda^{C\alpha}(T\boldsymbol{\eta}) \\
 &= \int R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2U_1 - 1) \cdot U_1^{n_2} R_{Q_{P_{\mathbf{n}}}}^{CP\alpha}(L\mathbf{U}) \, d\lambda^{C\alpha}(T\boldsymbol{\eta}).
 \end{aligned}$$

Now an application of Lemma 3.1 and (3.20) yields

$$\begin{aligned}
R_{\mathbf{n}}^{\alpha}(\mathbf{0})R_{\mathbf{n}}^{\alpha}(\mathbf{x}) &= \int U_1^{n_2} R_{n_1-n_2}^{B(\alpha, \mathbf{n})}(2U_1 - 1) \cdot R_{P_{\mathbf{n}}}^{P\alpha}(\mathbf{V}^{(k-1)}(L\mathbf{U}; r_k)) \\
&\quad \cdot d\nu^{\alpha_k, \alpha_{k-1}}(r_k) \cdot d\lambda^{C\alpha}(T\boldsymbol{\eta}) \\
&= \int R_{\mathbf{n}}^{\alpha}(U_1, U_1\mathbf{V}^{(k-1)}(L\mathbf{U}; r_k)) d\lambda^{\alpha}(\boldsymbol{\eta}) \\
&= \int R_{\mathbf{n}}^{\alpha}(\mathbf{V}^{(k)}(\mathbf{U}; r_k)) d\lambda^{\alpha}(\boldsymbol{\eta}) \\
&= \int R_{\mathbf{n}}^{\alpha}(\mathbf{U}^{(k)}(x_1; \boldsymbol{\eta})) d\lambda^{\alpha}(\boldsymbol{\eta}).
\end{aligned}$$

4. PROOFS THAT THE PRODUCT FORMULAS ARE OF HYPERGROUP TYPE

We use the notation of Subsection 1.1. The existence of a product formula (1.1) with $\delta_{\mathbf{x}} * \delta_{\mathbf{y}} = \mu_{\mathbf{x}, \mathbf{y}} \in M_1(H)$ for each $\mathbf{x}, \mathbf{y} \in H$ leads immediately to (H1), (H2), and (H3). We give an overview of the proofs of (H4) and (H5) before we deal with the specific cases. When H is either B or $\Delta^{(k)}$ for some $k \geq 2$ let $H' = H - \{\mathbf{0}\}$. The product formulas in Theorems 2.1, 2.2, and 2.3 all have the general form

$$\begin{aligned}
P(\mathbf{x}) \cdot P(\mathbf{y}) &= \int_K P(\mathbf{Z}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega})) dm(\boldsymbol{\omega}) \quad (\mathbf{x}, \mathbf{y} \in H'), \\
P(\mathbf{x}) \cdot P(\mathbf{0}) &= \int_{K_0} P(\mathbf{Z}_0(\mathbf{x}; \boldsymbol{\omega})) dm_0(\boldsymbol{\omega}) \quad (\mathbf{x} \in H),
\end{aligned}$$

so that

$$\begin{aligned}
\text{supp}(\delta_{\mathbf{x}} * \delta_{\mathbf{y}}) &= \text{supp}(\mu_{\mathbf{x}, \mathbf{y}}) = \mathbf{Z}(\mathbf{x}, \mathbf{y}; K) \quad (\mathbf{x}, \mathbf{y} \in H') \\
\text{supp}(\delta_{\mathbf{x}} * \delta_{\mathbf{0}}) &= \text{supp}(\mu_{\mathbf{x}, \mathbf{0}}) = \mathbf{Z}_0(\mathbf{x}; K_0) \quad (\mathbf{x} \in H).
\end{aligned}$$

Thus (H4) will be established by showing

$$(4.1) \quad \mathbf{e} \in \mathbf{Z}(\mathbf{x}, \mathbf{y}; K) \quad \text{if and only if} \quad \mathbf{x} = \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in H')$$

$$(4.2) \quad \mathbf{e} \in \mathbf{Z}_0(\mathbf{x}; K_0) \quad \text{if and only if} \quad \mathbf{x} = \mathbf{0} \quad (\mathbf{x} \in H).$$

To prove (H5) we note that \mathbf{Z} is a continuous function on $(H')^2 \times K$ and \mathbf{Z}_0 is continuous on $H \times K_0$, thus, Lemma 4.2 (below) shows that $(\mathbf{x}, \mathbf{y}) \mapsto \text{supp}(\delta_{\mathbf{x}} * \delta_{\mathbf{y}})$ is continuous on the domains $(H')^2$, $H \times \{\mathbf{0}\}$, and $\{\mathbf{0}\} \times H$. Thus (H5) follows when we show that

$$(4.3) \quad \lim_{\substack{(\mathbf{x}, \mathbf{y}) \rightarrow (\mathbf{z}, \mathbf{0}) \\ \mathbf{x}, \mathbf{y} \in H'}} \mathbf{Z}(\mathbf{x}, \mathbf{y}; K) = \mathbf{Z}_0(\mathbf{z}; K_0).$$

We also observe for later use that $\delta_{\mathbf{x}} * \delta_{\mathbf{e}} = \delta_{\mathbf{x}}$ is equivalent to

$$\mathbf{Z}(\mathbf{x}, \mathbf{e}; \boldsymbol{\omega}) = \mathbf{x} \quad (\boldsymbol{\omega} \in K).$$

Thus each proof in this section consists of establishing equations (4.1), (4.2), and (4.3); we do this for the formulas as stated in Theorems 2.1–2.3. That the formulas for the polynomials are of hypergroup type follows by a simple change of variables in each case. For example, in the case of the parabolic biangle polynomials we use the change of variables $(x_1, x_2) \mapsto (\sqrt{x_1}, x_2)$.

There is a much simpler (but less elementary) argument than ours for the continuity of support in the case of the parabolic biangle polynomials. For certain values of the parameters the parabolic biangle polynomials are the spherical functions for a Gelfand pair, and the algebra of bi-invariant measures forms a hypergroup. In that case it follows that $\text{supp } \delta_x * \delta_y$ is a continuous function of (x, y) . But $\text{supp } \delta_x * \delta_y$ is the same for all values of the parameters so the continuity of support follows.

4.1. The topology of the space of compact subsets. In order to prove (4.3) we need to understand the topology of $\mathcal{C}(H) = \{A : A \text{ is a compact subset of } H\}$. The topology referred to in Axiom (H5), which we refer to as the *Michael topology*, has a subbasis consisting of all sets of the form

$$\mathcal{C}_{\mathcal{U}}(\mathcal{V}) = \{A \in \mathcal{C}(H) : A \cap \mathcal{U} \neq \emptyset \text{ and } A \subset \mathcal{V}\}$$

where \mathcal{U} and \mathcal{V} are open subsets of H . (This is the definition as given in [Jew75, §2.5] which is equivalent to Michael's definition of the *finite topology* [Mic51, Def. 1.7] and what Dellacherie [Del72] calls the *Hausdorff topology*.)

If H is a metric space with metric d , there is induced a metric on $\mathcal{C}(H)$ as follows: first define for $A \in \mathcal{C}(H)$ and $r > 0$

$$V_r(A) = \{y : d(x, y) < r \text{ for some } x \in A\},$$

and for $A, B \in \mathcal{C}(H)$ let

$$d(A, B) = \inf\{r : A \subset V_r(B) \text{ and } B \subset V_r(A)\}.$$

d is called the *Hausdorff metric* and corresponding topology with basis consisting of the sets

$$N_r(A) = \{B \in \mathcal{C}(H) : d(A, B) < r\} \quad (A \in \mathcal{C}(H), r > 0),$$

is called the *Hausdorff topology*. The Hausdorff and Michael topologies are identical when both are defined. The proof of this fact is elementary, but we supply it as a convenience to the reader.

Lemma 4.1. *Let H be a metric space. The Hausdorff topology and the Michael topology for $\mathcal{C}(H)$ coincide.*

Proof. Let $A \in \mathcal{C}(H)$ and $r > 0$. Let $\mathcal{V} = V_r(A)$. Now, since A is compact, there is a finite sequence $a_1, \dots, a_n \in A$ such that the sets $\mathcal{U}_k = V_{r/2}(\{a_k\})$ form an open cover of A . We will show that $\bigcap_{k=1}^n \mathcal{C}_{\mathcal{U}_k}(\mathcal{V}) \subset N_r(A)$. Suppose $B \in \bigcap_{k=1}^n \mathcal{C}_{\mathcal{U}_k}(\mathcal{V})$, then on the one hand $B \subset \mathcal{V} = V_r(A)$. On the other hand, if $x \in A$, then $x \in \mathcal{U}_k$ for some k , but $\mathcal{U}_k \cap B \neq \emptyset$, so $x \in V_r(B)$. So $A \subset V_r(B)$. Hence every Hausdorff-open subset of $\mathcal{C}(H)$ is also Michael-open.

Now suppose that \mathcal{U} and \mathcal{V} are open subsets of H . Let $A \in \mathcal{C}_{\mathcal{U}}(\mathcal{V})$. It will suffice to produce an $r > 0$ so that $N_r(A) \subset \mathcal{C}_{\mathcal{U}}(\mathcal{V})$. Since $A \in \mathcal{C}_{\mathcal{U}}(\mathcal{V})$, $A \cap \mathcal{U}$ must contain a point x , and since \mathcal{U} is open, there is $r > 0$ such that $V_r(\{x\}) \subset \mathcal{U}$ and $V_r(A) \subset \mathcal{V}$. Now suppose $B \in \mathcal{C}(H)$ with $d(A, B) < r$. Then, $A \subset V_r(B)$ so there is $y \in B$ such that $d(x, y) < r$, so that $y \in \mathcal{U}$; thus $B \cap \mathcal{U} \neq \emptyset$. Moreover, $B \subset V_r(A) \subset \mathcal{V}$. Thus $B \in \mathcal{C}_{\mathcal{U}}(\mathcal{V})$, and the two topologies coincide. \square

Lemma 4.2. *Let H and W be metric spaces, and let K be a compact space. Let $f: H \times K \rightarrow W$ be continuous and define $F: H \rightarrow \mathcal{C}(W)$ by $F(x) = f(x, K) = \{f(x, \xi): \xi \in K\}$. Then $F: H \rightarrow \mathcal{C}(W)$ is continuous.*

Proof. Let d_H and d_W be the metrics of H and W . Let $\epsilon > 0$, then since K is compact, there is $\delta > 0$ such that if $d_H(x_1, x_0) < \delta$, then

$$d_W(f(x_1, \xi), f(x_0, \xi)) < \epsilon \quad (\xi \in K),$$

whence

$$d_W(F(x_1), F(x_0)) < \epsilon.$$

\square

Thus (4.3) suffices to establish (H5).

The following lemma is even more elementary, but we state it here for convenient reference.

Lemma 4.3. *Suppose X is a compact space and H is a metric space with metric d . Let f and g be two continuous functions from X to H such that for every $x \in X$ $d(f(x), g(x)) < \epsilon$, then $d(f(X), g(X)) < \epsilon$.*

4.2. Proof that the parabolic biangle polynomial product formula is of hypergroup type. We first gather certain identities about the functions D and E (first introduced in Subsection 2.1) into a lemma which we will use in the ensuing discussion. Let

$$I = [0, 1], \quad J = [0, \pi], \quad \text{and} \quad K = [-1, 1].$$

Lemma 4.4.

$$(4.4) \quad D(x, 1; r, \psi) = E(x, 1; r, \psi) = x \quad (x \in K),$$

$$D(x, I; I \times J) = K \quad (x \in I),$$

$$(4.5) \quad E(x, I; I \times J) = I \quad (x \in I),$$

$$(4.6) \quad D(K, x; 1, J) = K \quad (x \in K).$$

Moreover, for $x, y \in I$, the following statements are all equivalent to each other:

- (1) $D(x, y; r, \psi) = 1$;
- (2) $E(x, y; r, \psi) = 1$; and
- (3) $x = y$ and either $x = 1$, or $r = 1$ and $\psi = 0$.

Proof. The first equation is obvious and the rest of the lemma is easily proved by setting $x = \cos \phi$ and $y = \cos \theta$ so that, for instance,

$$D(\cos \phi, \cos \theta; I \times J) = [\cos(\phi + \theta), \cos(\phi - \theta)].$$

□

Recall the notation of (2.4)–(2.6) then, as in Theorem 2.1,

$$\mathbf{Z} = \mathbf{Z}(x_1, x_2, y_1, y_2; r_1, \psi_1, \psi_2, \psi_3) = (E^2, EG)$$

and

$$\mathbf{Z}_0 = \mathbf{Z}_0(x_1, x_2, ; r_1, \psi_1) = (r_1^2(1 - x_1^2), r_1(1 - x_1^2)^{1/2} \cos \psi_1).$$

So (4.2) is immediate. To establish (4.1), first it is a straightforward computation that if $(x_1, x_2) = (y_1, y_2) \in B'$ then $\mathbf{Z} = (1, 1)$ for $r = 1$ and $\psi_1 = \psi_2 = \psi_3 = 0$. Conversely, suppose (x_1, x_2) and $(y_1, y_2) \in B'$ and $\mathbf{Z} = (1, 1)$ for some $r, \psi_1, \psi_2, \psi_3$, then $E = D = 1$, $x_1 = y_1$, and

$$1 = G(x_1, x_2, y_1, y_2; 1, 0, \psi_2, \psi_3) = D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right),$$

whence $x_2 = y_2$.

Now turning to the proof of (4.3) define

$$S(x_1, x_2, y_1, y_2) = \begin{cases} \mathbf{Z}(x_1, x_2, y_1, y_2; I \times J^3), & \text{if } (x_1, x_2), (y_1, y_2) \in B'; \\ \mathbf{Z}_0(x_1, x_2; I \times J), & \text{if } (x_1, x_2) \in B \text{ and } (y_1, y_2) = (0, 0), \end{cases}$$

and define

$$\mathbf{Y}(v_1, v_2) = (v_1^2, v_1 v_2).$$

Now, if $(z_1, z_2) \in B$, then

$$\begin{aligned} S(z_1, z_2, 0, 0) &= \{(s_1^2, s_1 s_2) : s_1 \in [0, (1 - z_1^2)^{1/2}], \text{ and } s_2 \in K\} \\ &= \mathbf{Y}([0, (1 - z_1^2)^{1/2}] \times K). \end{aligned}$$

Define for $x_1 \neq 0$ and $x_2 \neq 0$

$$\mathbf{X}(x_1, x_2, y_1, y_2; r_1, \psi_1, \psi_2, \psi_3) = (E, G),$$

so that

$$S(x_1, x_2, y_1, y_2) = \mathbf{Y} \circ \mathbf{X}(x_1, x_2, y_1, y_2; I \times J^3).$$

Thus it suffices to show

$$(4.7) \quad \lim_{(x_1, x_2, y_1, y_2) \rightarrow (z_1, z_2, 0, 0)} \mathbf{X}(x_1, y_1, x_2, y_2; I \times J^3) = [0, (1 - z_1^2)^{1/2}] \times K.$$

Now

$$G = D\left(C, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right); 1, \psi_3\right).$$

Let $\epsilon > 0$, then for sufficiently small $\eta > 0$, if $|x_1 - z_1| < \eta$, $y_1 < \eta$, $r_1 \in I$, and $\psi_1 \in J$ then

$$\begin{aligned} |E(x_1, y_1; r_1, \psi_1) - r_1(1 - z_1^2)^{1/2}| &< \epsilon, \\ |C(x_1, y_1; r_1, \psi_1) - \cos \psi_1| &< \epsilon. \end{aligned}$$

Thus η may be decreased (but still remain positive) so that if $|x_1 - z_1| < \eta$, $y_1 < \eta$, $r_1 \in I$, and $\psi_1, \psi_2, \psi_3 \in J$, then

$$\left| G(x_1, x_2, y_1, y_2; r_1, \psi_1, \psi_2, \psi_3) - D\left(\cos \psi_1, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right); 1, \psi_3\right) \right| < \epsilon$$

thus

$$\begin{aligned} &\left| \mathbf{X}(x_1, x_2, y_1, y_2; r_1, \psi_1, \psi_2, \psi_3) \right. \\ &\quad \left. - \left(r_1(1 - z_1^2)^{1/2}, D\left(\cos \psi_1, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right); 1, \psi_3\right) \right) \right| < 2\epsilon. \end{aligned}$$

Thus by Lemma 4.3 (we use d for the Hausdorff metric)

$$\begin{aligned} d\left[\mathbf{X}(x_1, x_2, y_1, y_2; I \times J, \psi_2, \psi_3), \right. \\ \left. \left([0, (1 - z_1^2)^{1/2}] \times D\left(K, D\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; 1, \psi_2\right); 1, \psi_3\right) \right) \right] < 2\epsilon. \end{aligned}$$

One more application of Lemma 4.3 and (4.6) yields

$$d\left[\mathbf{X}(x_1, x_2, y_1, y_2; I \times J, \psi_2, J), ([0, (1 - z_1^2)^{1/2}] \times K) \right] < 2\epsilon.$$

so finally

$$d\left[\mathbf{X}(x_1, x_2, y_1, y_2; I \times J^3), ([0, (1 - z_1^2)^{1/2}] \times K) \right] < 2\epsilon$$

whence (4.7) holds.

4.3. Proof that the triangle polynomial product formula is of hypergroup type. The arguments for (4.1) and (4.2) are similar to the ones in the previous section and we can turn our attention immediately to (H5). Recalling the notation of (2.4)–(2.7) we define

$$\begin{aligned} \mathbf{Z}(x_1, x_2, y_1, y_2; r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3) &= (E^2, E^2 H^2), \\ \mathbf{Z}_0(x_1, x_2; r_1, r_2, \psi) &= r_1^2(1 - x_1^2)(1, (1 - r_2) \cos^2 \psi + r_2), \\ S(x_1, x_2, y_1, y_2) &= \begin{cases} \mathbf{Z}(x_1, x_2, y_1, y_2; I^4 \times J^3) & \text{if } (x_1, x_2), (y_1, y_2) \in \Delta^{(2)} - \{(0, 0)\}, \\ \mathbf{Z}_0(x_1, x_2; I^2 \times J) & \text{if } (x_1, x_2) \in \Delta^{(2)} \text{ and } (y_1, y_2) = (0, 0). \end{cases} \end{aligned}$$

Let

$$\mathbf{Y}(v_1, v_2) = (v_1^2, v_1^2 v_2^2).$$

Then if $(z_1, z_2) \in \Delta^{(2)}$

$$\begin{aligned} S(z_1, z_2, 0, 0) &= \{(s_1^2, s_2^2) : 0 \leq s_2 \leq s_1 \leq (1 - z_1^2)^{1/2}\} \\ &= \mathbf{Y}([0, (1 - z_1^2)^{1/2}] \times I). \end{aligned}$$

Define for $x_1 \neq 0$ and $y_1 \neq 0$

$$\mathbf{X}(x_1, x_2, y_1, y_2; r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3) = (E, H).$$

If η is sufficiently small, then $|x_1 - z_1| < \eta$ and $y_1 < \eta$ implies

$$\begin{aligned} |D(x_1, y_1; r_1, \psi_1) - r_1(1 - z_1^2)^{1/2} \cos \psi_1| &< \epsilon, \\ |E(x_1, y_1; r_1, \psi_1) - r_1(1 - z_1^2)^{1/2}| &< \epsilon, \\ |C(x_1, y_1; r_1, \psi_1) - \cos \psi_1| &< \epsilon, \end{aligned}$$

thus

$$\left| \mathbf{X}(x_1, x_2, y_1, y_2; r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3) - \left(r_1(1 - z_1^2)^{1/2}, E \left([(1 - r_2) \cos^2 \psi_1 + r_2]^{1/2}, E \left(\frac{x_2}{x_1}, \frac{y_2}{y_1}; r_3, \psi_2 \right); r_4, \psi_3 \right) \right) \right| < 2\epsilon$$

whence by Lemma 4.3 and (4.5)

$$d[\mathbf{X}(x_1, x_2, y_1, y_2; r_1, I^3 \times J^3), ((r_1^2(1 - z_1^2)^{1/2}) \times I)] < 2\epsilon.$$

So finally

$$d[\mathbf{X}(x_1, x_2, y_1, y_2; I^4 \times J^3), ([0, (1 - z_1^2)^{1/2}] \times I)] < 2\epsilon,$$

thus

$$\lim_{(x_1, x_2, y_1, y_2) \rightarrow (z_1, z_2, 0, 0)} \mathbf{X}(x_1, x_2, y_1, y_2; I^4 \times J^3) = [0, (1 - z_1^2)^{1/2}] \times I,$$

and so S is continuous at $(z_1, z_2, 0, 0)$.

4.4. Proof that the simplex polynomial product formula is of hypergroup type. Recall the notations of Subsections 2.3 and 3.3. Equations (4.1) and (4.2) are contained in the following lemma: we write

$$\mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = (Z_1^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}), \dots, Z_k^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}))$$

Lemma 4.5. *For $k \geq 2$ if $x, y \in I$, $\mathbf{x}, \mathbf{y} \in \Delta^{(k)}$, and $\mathbf{z} \in \Delta^{(k-1)}$, then:*

- (i) $\mathbf{V}^{(k)}(\mathbf{z}; u) = \mathbf{e}$ for all $u \in I$ if and only if $\mathbf{z} = \mathbf{f}$.
- (ii) $\mathbf{e} \in \mathbf{U}^{(k)}(x; I^k \times J)$ if and only if $x = 0$.
- (iii) $\mathbf{e} \in \mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; K^{(k)})$ if and only if $\mathbf{x} = \mathbf{y}$.
- (iv) If for some $\boldsymbol{\rho} \in K^{(k)}$, $Z_1^{(k)}(x\mathbf{e}, y\mathbf{e}; \boldsymbol{\rho}) = 1$, then $x = y$ and $\mathbf{Z}^{(k)}(x\mathbf{e}, y\mathbf{e}; \boldsymbol{\rho}) = \mathbf{e}$.

Proof. (i) is immediate from the definition of $\mathbf{V}^{(k)}(\mathbf{x}; u)$. Concerning (ii)–(iv), these are true when $k = 2$ by definition of the functions and the fact that the bivariate simplex polynomials coincide with the triangle polynomials. We proceed by induction assuming (ii)–(iv) are true, and we prove the corresponding statements with $k + 1$ in place of k and $\mathbf{d} = \mathbf{e}^{(k+1)}$ in place of \mathbf{e} .

(ii) Now $\mathbf{d} \in \mathbf{U}^{(k+1)}(x; I^{k+1} \times J)$ if and only if $\mathbf{d} \in \mathbf{V}^{(k+1)}(\mathbf{U}^{(k)}(x; I^k \times J); I)$ if and only if $\mathbf{e} \in \mathbf{U}^{(k)}(x; I^k \times J)$ if and only if $x = 0$ by the inductive assumption.

(iii) If $\mathbf{x} \in \Delta^{(k+1)}$, then there are $\boldsymbol{\rho}$ and $\boldsymbol{\sigma} \in K^{(k)}$ such that $\mathbf{W} = \mathbf{Z}^{(k)}(x_1\mathbf{e}, x_1\mathbf{e}; \boldsymbol{\rho}) = \mathbf{e}$ and $\mathbf{Z}^{(k)}(L\mathbf{x}, L\mathbf{x}; \boldsymbol{\sigma}) = \mathbf{e}$ by the inductive assumption. Now for any $\boldsymbol{\tau} \in K^{(k)}$ and $u \in I$

$$Z_1^{(k+1)}(\mathbf{x}, \mathbf{x}; \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) = Z_1^{(k)}(x_1\mathbf{e}, x_1\mathbf{e}; \boldsymbol{\rho}) = 1$$

and using (iii), then (i), and then (iii) again we have

$$\begin{aligned} LZ^{(k+1)}(\mathbf{x}, \mathbf{x}; \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) &= \mathbf{Z}^{(k)}(\mathbf{V}^{(k)}(L\mathbf{W}; u), \mathbf{Z}^{(k)}(L\mathbf{x}, L\mathbf{x}; \boldsymbol{\sigma}); \boldsymbol{\tau}) \\ &= \mathbf{Z}^{(k)}(\mathbf{V}^{(k)}(\mathbf{f}; u), \mathbf{e}; \boldsymbol{\tau}) \\ &= \mathbf{Z}^{(k)}(\mathbf{e}, \mathbf{e}; \boldsymbol{\tau}) = \mathbf{e}, \end{aligned}$$

thus $\mathbf{d} \in \mathbf{Z}^{(k+1)}(\mathbf{x}, \mathbf{x}; K^{(k+1)})$.

Now suppose there are $\mathbf{x}, \mathbf{y} \in \Delta^{(k+1)}$ and $\boldsymbol{\omega} = (\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) \in K^{(k+1)}$ such that $\mathbf{Z}^{(k+1)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = \mathbf{d}$. Then $Z_1^{(k)}(x_1\mathbf{e}, y_1\mathbf{e}; \boldsymbol{\rho}) = 1$, whence by (iv) $x_1 = y_1$ and $\mathbf{W} = \mathbf{Z}^{(k)}(x_1\mathbf{e}, y_1\mathbf{e}; \boldsymbol{\rho}) = \mathbf{e}$. So $\mathbf{V}^{(k)}(L\mathbf{W}; u) = \mathbf{e}$, and $\mathbf{e} = LZ^{(k+1)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = \mathbf{Z}^{(k)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma})$, so $L\mathbf{x} = L\mathbf{y}$ by (iii). Thus $\mathbf{x} = \mathbf{y}$.

(iv) Suppose there are $x, y \in I$, $\boldsymbol{\omega} = (\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) \in K^{(k+1)}$ such that $Z_1(x\mathbf{d}, y\mathbf{d}; \boldsymbol{\omega}) = 1$. Now $Z_1^{(k+1)}(x\mathbf{d}, y\mathbf{d}; \boldsymbol{\omega}) = Z_1^{(k)}(x\mathbf{e}, y\mathbf{e}; \boldsymbol{\rho})$, so it follows by (iv) that $x = y$ and $\mathbf{W} = \mathbf{Z}^{(k)}(x\mathbf{e}, y\mathbf{e}; \boldsymbol{\rho}) = \mathbf{e}$, whence

$$LZ^{(k+1)}(x\mathbf{d}, y\mathbf{d}; \boldsymbol{\omega}) = \mathbf{Z}^{(k)}(\mathbf{V}^{(k)}(L\mathbf{W}; u), \mathbf{Z}^{(k)}(\mathbf{e}, \mathbf{e}; \boldsymbol{\sigma}); \boldsymbol{\tau}) = \mathbf{Z}^{(k)}(\mathbf{e}, \mathbf{e}; \boldsymbol{\tau}) = \mathbf{e}.$$

Thus $\mathbf{Z}(x\mathbf{d}, y\mathbf{d}; \boldsymbol{\omega}) = \mathbf{d}$. □

The k -simplices satisfy the following identities:

$$(4.8) \quad \begin{aligned} \Delta^{(1)} &= I, \\ \Delta^{(2)} &= \bigcup_{x \in I} ([x, 1] \times \{x\}), \end{aligned}$$

$$(4.9) \quad \Delta^{(k)} = \bigcup_{x \in I} x(1, \Delta^{(k-1)}), \quad (k \geq 2),$$

$$(4.10) \quad L(t\Delta^{(k)}) = \Delta^{(k-1)}, \quad (t > 0).$$

Define

$$S(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; K^{(k)}) & \text{if } \mathbf{x}, \mathbf{y} \in \Delta^{(k)} - \{\mathbf{0}\}, \\ \mathbf{U}^{(k)}(x_1; I^k \times J) & \text{if } \mathbf{x} \in \Delta^{(k)} \text{ and } \mathbf{y} = \mathbf{0}. \end{cases}$$

We first find $S(\mathbf{z}, \mathbf{0})$. This requires a series of computations which we organize as lemmas.

Lemma 4.6. *If $\boldsymbol{\eta} = (r_1, \dots, r_k, \psi) \in I^k \times J$ and $x \in I$, then $U_1^{(k)}(x; \boldsymbol{\eta}) = r_1^2(1 - x)$.*

Proof. From the definition of $\mathbf{U}^{(k)}$ we see that

$$U_1^{(k)}(x; \boldsymbol{\eta}) = U_1^{(k-1)}(x; T\boldsymbol{\eta}) = U_1^{(2)}(x; r_1, r_2, \psi) = r_1^2(1 - x).$$

□

Lemma 4.7. *For $k \geq 2$, $\mathbf{V}^{(k)}(\Delta^{(k-1)}; I) = \Delta^{(k)}$.*

Proof. This is immediate from the definition of $\mathbf{V}^{(k)}(\mathbf{x}; u)$.

□

Lemma 4.8. *For $k \geq 2$, $L\mathbf{U}^{(k)}(x; r_1, I^{k-1} \times J) = \Delta^{(k-1)}$.*

Proof.

$$L\mathbf{U}^{(2)}(x; r_1, r_2, \psi) = (1 - r_2) \cos^2 \psi + r_2$$

so

$$L\mathbf{U}^{(2)}(x; r_1, I \times J) = I = \Delta^{(1)}.$$

Now assume $L\mathbf{U}^{(k-1)}(x; r_1, I^{k-2} \times J) = \Delta^{(k-2)}$. We have (see (3.20))

$$\begin{aligned} L\mathbf{U}^{(k)}(x; \boldsymbol{\eta}) &= L\mathbf{V}^{(k)}(\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k) \\ &= \mathbf{V}^{(k-1)}(L\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k), \end{aligned}$$

therefore

$$\begin{aligned} L\mathbf{U}^{(k)}(x; r_1, I^{k-1} \times J) &= \mathbf{V}^{(k-1)}(L\mathbf{U}^{(k-1)}(x; r_1, I^{(k-2)} \times J); I) \\ &= \mathbf{V}^{(k-1)}(\Delta^{(k-2)}; I) = \Delta^{(k-1)} \end{aligned}$$

by Lemma 4.7.

□

Lemma 4.9. $\mathbf{U}^{(k)}(x; I^k \times J) = (1 - x)\Delta^{(k)}$.

Proof.

$$\begin{aligned} \mathbf{U}^{(k)}(x; \boldsymbol{\eta}) &= \mathbf{V}^{(k)}(\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k) \\ &= U_1^{(k-1)}(x; T\boldsymbol{\eta}) \left(1, \mathbf{V}^{(k-1)}(L\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k) \right) \\ &= r_1^2(1 - x) \left(1, \mathbf{V}^{(k-1)}(L\mathbf{U}^{(k-1)}(x; T\boldsymbol{\eta}); r_k) \right) \end{aligned}$$

by Lemma 4.6. Therefore by using first Lemma 4.8 and then Lemma 4.7

$$\begin{aligned} \mathbf{U}^{(k)}(x; r_1, I^{k-1} \times J) &= r_1^2(1-x) \left(1, \mathbf{V}^{(k-1)}(L\mathbf{U}^{(k-1)}(x; r_1, I^{k-2} \times J); I) \right) \\ &= r_1^2(1-x) \left(1, \mathbf{V}^{(k-1)}(\Delta^{(k-2)}; I) \right) \\ &= r_1^2(1-x) (1, \Delta^{(k-1)}) \end{aligned}$$

so the lemma follows by (4.9). \square

An immediate consequence of Lemma 4.9 is

$$S(\mathbf{z}, \mathbf{0}) = (1 - z_1)\Delta^{(k)}.$$

We now establish the continuity of S at $(\mathbf{z}, \mathbf{0})$, but first we introduce a convenient limit notation that will be especially useful for continuous compact set-valued functions.

Let $F(\mathbf{x}, \mathbf{y})$ and $G(\mathbf{x}, \mathbf{y})$ be functions defined for $\mathbf{x}, \mathbf{y} \in \Delta^{(k)} - \{\mathbf{0}\}$ and taking values either in $\Delta^{(k)}$ or in $\mathcal{C}(\Delta^{(k)})$ and let \mathbf{z} be a fixed point in $\Delta^{(k)}$; we will write

$$F(\mathbf{x}, \mathbf{y}) \sim G(\mathbf{x}, \mathbf{y})$$

as shorthand for

$$\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (\mathbf{z}, \mathbf{0})} F(\mathbf{x}, \mathbf{y}) = \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (\mathbf{z}, \mathbf{0})} G(\mathbf{x}, \mathbf{y}).$$

Thus the continuity of S at $(\mathbf{z}, \mathbf{0})$ is equivalent to $S(\mathbf{x}, \mathbf{y}) \sim S(\mathbf{z}, \mathbf{0})$ or

$$(4.11) \quad \mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; K^{(k)}) \sim (1 - z_1)\Delta^{(k)}.$$

This is identical to the last relation in Lemma 4.11 which will be established by mathematical induction. The first step in the induction is contained in Lemma 4.10 which is in large part a reformulation of the contents of Subsection 4.3.

Lemma 4.10. *Let $\boldsymbol{\rho} = (r_1, r_2, r_3, r_4, \psi_1, \psi_2, \psi_3)$, $v \in \mathbb{R}$, $\mathbf{x}, \mathbf{y}, \mathbf{v} \in \Delta^{(2)} - \{\mathbf{0}\}$, and $\mathbf{z} \in \Delta^{(2)}$, and $\mathbf{e} = (1, 1)$. Then*

$$(4.12) \quad Z_1^{(2)}(v\mathbf{e}, I\mathbf{e}; K^{(2)}) = I,$$

$$(4.13) \quad Z_1^{(2)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) \sim r_1^2(1 - z_1),$$

$$(4.14) \quad LZ^{(2)}(\mathbf{x}, \mathbf{y}; r_1, I^3 \times J^3) \sim I,$$

$$(4.15) \quad \mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{y}; K^{(2)}) \sim (1 - z_1)\Delta^{(2)},$$

$$(4.16) \quad \mathbf{Z}^{(2)}(x_1\mathbf{e}, y_1\mathbf{e}; K^{(2)}) \sim (1 - z_1)\Delta^{(2)},$$

$$(4.17) \quad \mathbf{Z}^{(2)}(\mathbf{v}, \Delta^{(2)}; K^{(2)}) = \Delta^{(2)}.$$

Proof. Equation (4.12) follows from (4.5), equation (4.13) from the definition of $\mathbf{Z}^{(2)}$ (eq. (2.8)), and (4.14) follows from Lemma 4.3 and (4.5). Equation (4.15) follows from (4.13) and (4.14). Equation (4.16) is a special case of (4.15).

To obtain (4.17), let $\mathbf{v}, \mathbf{w} \in \Delta^{(2)}$, so we can write $\mathbf{v} = v(1, s)$ and $\mathbf{w} = w(1, t)$ with $v, w, s, t \in I$; then $L\mathbf{v} = s$ and $L\mathbf{w} = t$ are independent of v and w . Hence we have

$$\mathbf{Z}^{(2)}(\mathbf{v}, \mathbf{w}; \boldsymbol{\rho}) = E^2(\sqrt{v}, \sqrt{w}; r_1, \psi_1) \cdot \left(1, E \left([(1 - r_2)C^2(\sqrt{v}, \sqrt{w}; r_1, \psi_1) + r_2]^{1/2}, E(s, t; r_3, \psi_2); r_4; \psi_3 \right) \right).$$

Letting t take all values in I we get from two applications of (4.5)

$$\begin{aligned} \mathbf{Z}^{(2)}(\mathbf{v}, w(1, I); r_1, r_2, I^2, \psi_1, J^2) \\ = E^2(\sqrt{v}, \sqrt{w}; r_1, \psi_1) \left(1, E \left([(1 - r_2)C^2(\sqrt{v}, \sqrt{w}; r_1, \psi_1) + r_2]^{1/2}, I; I \times J \right) \right) \\ = E^2(\sqrt{v}, \sqrt{w}; r_1, \psi_1)(1, I) \end{aligned}$$

whence letting w take all values in I , and using (4.9) and (4.5) we obtain (4.17). \square

We introduce some additional notation. Let $K_1^{(2)} = I^3 \times J^3$, thus $K^{(2)} = I \times K_1^{(2)}$, and if $\boldsymbol{\rho} \in K^{(2)}$, we can write $\boldsymbol{\rho} = (r_1, \boldsymbol{\nu})$ with $r_1 \in I$ and $\boldsymbol{\nu} \in K_1^{(2)}$. Similarly for $k > 2$, define $K_1^{(k)} = K_1^{(k-1)} \times K^{(k-1)} \times K^{(k-1)} \times I$, so $K^{(k)} = I \times K_1^{(k)}$ and if $\boldsymbol{\omega} = (\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}, u) \in K^{(k)}$ with $\boldsymbol{\rho}, \boldsymbol{\sigma}$, and $\boldsymbol{\tau} \in K^{(k-1)}$, then $\boldsymbol{\rho} = (r_1, \boldsymbol{\nu})$ with $r_1 \in I$ and $\boldsymbol{\nu} \in K_1^{(k-1)}$.

Lemma 4.11. *If $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \Delta^{(k)} - \{\mathbf{0}\}$, and $\mathbf{z} \in \Delta^{(k)}$, then:*

$$(4.18) \quad Z_1^{(k-1)}(\mathbf{v}\mathbf{f}, I\mathbf{f}; K^{(k-1)}) = I,$$

$$(4.19) \quad Z_1^{(k-1)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) \sim r_1^2(1 - z_1),$$

$$(4.20) \quad LZ^{(k-1)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(k-1)}) \sim \Delta^{(k-2)},$$

$$(4.21) \quad \mathbf{Z}^{(k-1)}(x_1\mathbf{f}, y_1\mathbf{f}; K^{(k-1)}) \sim (1 - z_1)\Delta^{(k-1)},$$

$$(4.22) \quad \mathbf{Z}^{(k-1)}(\mathbf{v}, \Delta^{(k-1)}; K^{(k-1)}) \sim \Delta^{(k-1)},$$

$$(4.23) \quad \mathbf{Z}^{(k)}(\mathbf{x}, \mathbf{y}; K^{(k)}) \sim (1 - z_1)\Delta^{(k)}.$$

Proof. The proof of the lemma is by mathematical induction beginning with $k = 3$. For $k = 3$, equations (4.18)–(4.22) coincide with equations (4.12)–(4.14) and (4.16)–(4.17), so we need only to obtain (4.23). Now from the definition of $\mathbf{Z}^{(k)}$ and (4.13)

$$Z_1^{(3)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = Z_1^{(2)}(x_1, x_1, y_1, y_1; \boldsymbol{\rho}) \sim r_1^2(1 - z_1),$$

and so (see (2.7))

$$\begin{aligned} LZ^{(3)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) &= \mathbf{Z}^{(2)}(\mathbf{V}^{(2)}[(1 - r_2)C^2(x_1, y_1; r_1, \psi_1) + r_2; u], \mathbf{Z}^{(2)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma}); \boldsymbol{\tau}) \\ &\sim \mathbf{Z}^{(2)}\left(\mathbf{V}^{(2)}[(1 - r_2)\cos^2\psi_1 + r_2; u], \mathbf{Z}^{(2)}(L\mathbf{x}, L\mathbf{y}; \boldsymbol{\sigma}); \boldsymbol{\tau}\right) \end{aligned}$$

so by Lemma 4.7 and (4.17)

$$\begin{aligned} LZ^{(3)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(3)}) &\sim \mathbf{Z}^{(2)}(\mathbf{V}^{(2)}(I; I), \mathbf{Z}^{(2)}(L\mathbf{x}, L\mathbf{y}; K^{(2)}); K^{(2)}) \\ &= \mathbf{Z}^{(2)}(\Delta^{(2)}, \mathbf{Z}^{(2)}(L\mathbf{x}, L\mathbf{y}; K^{(2)}); K^{(2)}) \\ &= \Delta^{(2)}. \end{aligned}$$

Thus

$$\mathbf{Z}^{(3)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(3)}) \sim r_1^2(1 - z_1)(1, \Delta^{(2)})$$

hence (4.23) holds with $k = 3$.

Now assume (4.18)–(4.23) all hold. We establish these with k replaced by $k + 1$. By definition of $\mathbf{Z}^{(k)}$ we have

$$Z_1^{(k)}(\mathbf{v}, \mathbf{w}; \boldsymbol{\omega}) = Z_1^{(k-1)}(v_1\mathbf{f}, w_1\mathbf{f}; \boldsymbol{\rho})$$

so (4.18) implies

$$Z_1^{(k)}(v\mathbf{e}, I\mathbf{e}; K^{(k)}) = I.$$

and a special case of (4.19) yields

$$(4.24) \quad Z_1^{(k)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) \sim r_1^2(1 - z_1).$$

Now using (4.20) and Lemma 4.7

$$\begin{aligned} LZ^{(k)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(k)}) &= \mathbf{Z}^{(k-1)}\left(\mathbf{V}^{(k-1)}(LZ^{(k-1)}(x_1\mathbf{f}, y_1\mathbf{f}; r_1, K_1^{(k-1)}); I), \mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; K^{(k-1)}); K^{(k-1)}\right) \\ &\sim \mathbf{Z}^{(k-1)}\left(\mathbf{V}^{(k-1)}(\Delta^{(k-2)}; I), \mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; K^{(k-1)}); K^{(k-1)}\right) \\ &= \mathbf{Z}^{(k-1)}(\Delta^{(k-1)}, \mathbf{Z}^{(k-1)}(L\mathbf{x}, L\mathbf{y}; K^{(k-1)}); K^{(k-1)}). \end{aligned}$$

Thus (4.22) implies

$$(4.25) \quad LZ^{(k)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(k)}) \sim \Delta^{(k-1)}.$$

As a special case of (4.23) we have

$$\mathbf{Z}^{(k)}(x_1\mathbf{e}, y_1\mathbf{e}; K^{(k)}) \sim (1 - z_1)\Delta^{(k)}.$$

If $\mathbf{v}, \mathbf{w} \in \Delta^{(k)}$, we can write $\mathbf{v} = v(1, \mathbf{s})$ and $\mathbf{w} = w(1, \mathbf{t})$ with $v, w \in I$ and $\mathbf{s}, \mathbf{t} \in \Delta^{(k-1)}$ (see (4.9)), so that $L\mathbf{v} = \mathbf{s}$ and $L\mathbf{w} = \mathbf{t}$. Thus

$$\begin{aligned} \mathbf{Z}^{(k)}(\mathbf{v}, \mathbf{w}; \boldsymbol{\omega}) &= \mathbf{Z}^{(k)}\left(v(1, \mathbf{s}), w(1, \mathbf{t}); \boldsymbol{\omega}\right) \\ &= Z_1^{(k-1)}(vf, wf; \boldsymbol{\rho})\left(1, \mathbf{Z}^{(k-1)}\left(\mathbf{V}^{(k-1)}(LZ^{(k-1)}(vf, wf; \boldsymbol{\rho}); u), \mathbf{Z}^{(k-1)}(\mathbf{s}, \mathbf{t}; \boldsymbol{\sigma}); \boldsymbol{\tau}\right)\right), \end{aligned}$$

so by two applications of (4.22)

$$\begin{aligned} & \mathbf{Z}^{(k)}(v(1, \mathbf{s}), w(1, \Delta^{(k-1)}); \boldsymbol{\rho}, K^{(k-1)} \times K^{(k-1)} \times I) \\ &= Z_1^{(k-1)}(v\mathbf{f}, w\mathbf{f}; \boldsymbol{\rho}) \left(1, \mathbf{Z}^{(k-1)} \left[\mathbf{V}^{(k-1)}(L\mathbf{Z}^{(k-1)}(v\mathbf{f}, w\mathbf{f}; \boldsymbol{\rho}); u), \Delta^{(k-1)}; K^{(k-1)} \right] \right) \\ &= Z_1^{(k-1)}(v\mathbf{f}, w\mathbf{f}; \boldsymbol{\rho})(1, \Delta^{(k-1)}). \end{aligned}$$

Now by (4.18) and (4.9)

$$(4.26) \quad \mathbf{Z}^{(k)}(\mathbf{v}, \Delta^{(k)}; K^{(k)}) \sim \Delta^{(k)}.$$

Now from (4.24)

$$Z_1^{(k+1)}(\mathbf{x}, \mathbf{y}; \boldsymbol{\omega}) = Z_1^{(k)}(x_1\mathbf{e}, y_1\mathbf{e}; \boldsymbol{\rho}) \sim r_1^2(1 - z_1).$$

We also have from a special case of (4.25) together with Lemma 4.7

$$\mathbf{V}^{(k)} \left(LZ^{(k)}(x_1\mathbf{e}, y_1\mathbf{e}; r_1, K_1^{(k)}); I \right) \sim \mathbf{V}^{(k)}(\Delta^{(k-1)}; I) = \Delta^{(k)},$$

so by (4.26)

$$LZ^{(k+1)}(\mathbf{x}, \mathbf{y}; r_1, K_1^{(k+1)}) \sim \mathbf{Z}^{(k)}(\Delta^{(k)}, \mathbf{Z}^{(k)}(L\mathbf{x}, L\mathbf{y}; K^{(k)}); K^{(k)}) \sim \Delta^{(k)}.$$

Thus (4.23) holds. □

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